THE SELF-REFERENTIAL IMPARTIAL GAME MINNIE

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Abstract

In self-referential subtraction games, the moves are determined by the heap sizes and, consequently, there is no decomposition into disjunctive sums. These games tend to be difficult to study and have proved to have interesting strategies, when (and if) they are solved. WYTHOFF'S NIM, EUCLID, and MAX NIM, for example, are basically solved but hard questions about them still remain. We introduce the subtraction game MINNIE, in which a player can remove any number of tokens up to the size of the smallest heap, and give outcomes for play on 1,2, and 3 heaps. For more heaps, we conjecture that any $\mathcal{P}$-position must have a 'small' heap. Note that amongst self-referential subtraction games, only GREEDY NIM has had its $\mathcal{P}$ positions found for positions with more than 3 heaps.

1. Introduction

This paper considers combinatorial games in which there are two players who move alternately and who have perfect information. Also, there are no chance devices and the game must end within a finite number of moves. This paper is self-contained but see [2, 3] for more background.

Impartial subtraction games consist of a multi-set of non-negative integers, $H = \{h_1, h_2, \ldots, h_m\}$, and a subtraction set, $S_H = \{s_1, s_2, \ldots, s_n\}$, of positive integers. The set H is referred to as the *position.* A *move* is to choose some i and j and replace h_i by $h_i - s_j$, provided $h_i - s_j \geq 0$. A position that results from a move in H is referred to as an *option* of H. The set H is usually referred to as a collection of heaps of tokens and, as such, a heap can never be negative. A move is never allowed on an empty heap. Duchêne and Rigo put these under the umbrella of *invariant* games [8]. Impartial subtraction games are among the first combinatorial

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games to be analyzed and there are still many open questions. For more on these games, see [2, 3] and also [12], Problem A1, for some open questions.

There are impartial subtraction games, however, where $S = S_H$ and is a function of H and thus may change after a move. Also, moves may affect more than one heap. These have been dubbed *self-referential* subtraction games. The literature on self-referential games show that they have interesting but complicated structures.

We briefly survey these later in this section.

In this paper, we introduce MINNIE. Here, $H = \{h_1, h_2, \ldots, h_m\}$ is a multi-set of positive integers. For notational convenience, we assume $0 < h_1 \leq h_2 \leq \cdots \leq h_m$. Note that if a heap is reduced to 0, then it is removed from H. The subtraction set $S_H = \{i : i = 1, 2, \ldots, h_1\}$. That is, a player may remove no more than the number of tokens in the smallest (non-empty) heap from any heap. We will denote a MINNIE position with heap sizes $a_1, a_2, \ldots, a_n$ as $[a_1, a_2, \ldots, a_n]$. Since S_H is easily calculated from H, we omit it.

In Figure 1, we see an example of MINNIE played on two heaps of sizes 7 and 9.

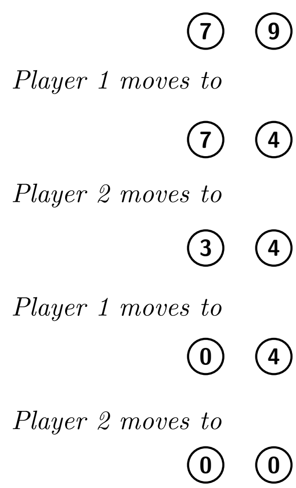

Figure 1: MINNIE on heaps of sizes 7 and 9.

Recall that for an impartial game there are two possible outcomes: an $\mathcal{N}$-position is one in which the next player can force a win and a $\mathcal{P}$-position is one in which the previous player can force a win (that is, the second player to play can force a win). In a given game, let $\mathcal{N}$ be the set of all $\mathcal{N}$-positions and $\mathcal{P}$ be the set of all $\mathcal{P}$-positions.

Theorem 1 (Partition Theorem for Impartial Games, Theorem 2.13, [2]). *Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets A and B with the properties: (i) every option of a position in A is in*

B; and (ii) every position in B has at least one option in A. Then A is the set of $\mathcal{P}$-positions and B is the set of $\mathcal{N}$-positions.

Outcome classes can be refined. The Sprague-Grundy Theorem, discovered independently by R.P. Sprague and P.M. Grundy, states that every impartial game is equivalent to an instance of NIM played on a single stack of some size (see [2] for more on Sprague-Grundy Theory). The *nim-value* of a position G is the size of the equivalent nim heap, written $\mathcal{G}(G)$. Thus, a $\mathcal{P}$-position corresponds to a 0 nim-heap and an $\mathcal{N}$-position to a non-empty nim-heap.

The *disjunctive sum* of games G and H, written $G + H$ is the game where a player chooses one of G and H and makes a move. The nim-sum of non-negative numbers m and n, written $m \bigoplus n$ is the XOR of the two numbers [2]. The Sprague-Grundy theory also says $\mathcal{G}(G+H) = \mathcal{G}(G) \bigoplus \mathcal{G}(H)$. This result allows us to analyze invariant impartial subtraction games by analyzing just single heaps. However, this is not possible for self-referential impartial subtraction games since the moves in $G + H$ do not correspond to the moves in G and H separately. Nim-values are considered in Section 3, in which the main conjecture about MINNIE is given.

The authors know of six self-referential subtraction games that have been studied, and only three have a detailed analysis.

EUCLID [6, 4] (and others), is a game based on the Euclidean algorithm[3]. It is played with two positive integers (a, b), $a \leq b$, and a move is to take a multiple of the smaller from the larger and the winner is the player who moves to (d, d) where d is the greatest common divisor of a and b. In our terminology, $H = \{a, b\}$ and $S_H = \{ia : i = 1, 2, \dots\}$. (See [11] for a partizan version.)

Theorem 2 ([**6**])**.** *In* EUCLID*, $[a, b] \in \mathcal{P}$ if and only if $\frac{b}{a} < \tau$ where τ is the golden ratio.*

See Section 3 for further details on EUCLID. EUCLID (and also WYTHOFF'S NIM [15] which doesn't decompose via the disjunctive sum) has been generalized to three and more heaps [5, 7, 14] but no good analysis yet exists. The other well-studied self-referential game is MAX-NIM [10]. The interesting structure occurs in the nim-values and these are discussed in Section 3.

Greedy NIM [2] is a variant of NIM in which players can only remove chips from the largest stack on their respective turns. While Greedy NIM is solved for $\mathcal{P}$-positions, there are no solutions for the nim-values.

Theorem 3 ([**1**])**.** *The $\mathcal{P}$-positions of Greedy* NIM *occur precisely in those games where there are an even number of heaps with an equal largest amount of chips.*

The next two games were defined and investigated at the Games-at-Dal Workshop in 2014.

[2] Or, write in binary and add without carrying

[3] There are two versions of the game, the second has the winner who reduces the position to $(d, 0)$ The winning strategies are related.

In SUSEN, $H = \{h_1, h_2, \ldots, h_m\}$ is, again, a multi-set of non-negative integers, and $S_H = \{h_1, h_2, \ldots, h_m\}$. In JENNIFER, SUSEN's contrarian cousin, $S_H = \mathbb{N}^{>0} \setminus \{h_1, h_2, \ldots, h_m\}$. We will denote a SUSEN position with heap sizes $a_1, a_2, \ldots, a_n$, as $S[a_1, a_2, \ldots, a_n]$.

At the time of writing, results for SUSEN are only known for three or fewer heaps[4].

All games of SUSEN with one heap are in $\mathcal{N}$ since the only legal move is for the next player to remove the entire heap and win.

Define $L(0,0) = 0$, $L(a,0) = 1$, $L(0,b) = 1$, $L(a,b) = L(b,a)$ for all $a, b > 0$, and $L(a,b) = 1 + L(a-b, b)$ where $a \geq b > 0$. This function measures the maximum number of moves of the game.

Theorem 4 ([**9**])**.** *Suppose $a \geq b$. $S[a,b] \in \mathcal{P}$ if and only if $L(a,b)$ is even.*

Theorem 5 ([**9**])**.** *For $a, b, c > 0$, the SUSEN position [a,b,c] is in $\mathcal{P}$ if and only if $[a,b]$, $[a,c]$, and $[b,c]$ are all in $\mathcal{N}$.*

For JENNIFER on one heap, the winning strategy is to always remove all but one token. Several participants from Games-at-Dal 2014 claim to have solved the two-heap game but no one has written it down.

In the next section, we solve MINNIE for outcomes on 1, 2, and 3 heaps in Lemma 6, Theorem 7, and Theorem 8, respectively. See Table 6 for the nim-values of two-heap MINNIE. These results and further computer evidence suggest that the absence of small heaps gives the next player many options and hence the advantage. We conclude with a conjecture as to how small, 'small' must be.

2. MINNIE

Since the case of MINNIE on three heaps is a complicated one, we will consider the problem on one and two heaps first. However, the problem on one heap is quite simple.

2.1. MINNIE on One and Two Heaps

Recall that, in MINNIE, the heap sizes are listed in non-decreasing order.

Lemma 6. *If a is a positive integer, then $[a] \in \mathcal{N}$.*

Proof. This game is the same as one heap NIM. □

Since MINNIE on one heap is the same as NIM on one heap, we get that the nim-value of $[a]$ is a.

[4]Thanks to Alex Fink, Eric Duchêne and Urban Larsson for written notes and photographs of the chalkboards

12	·	·	·	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·	·
11	·	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
10	·	·	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·
9	·	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
8	·	·	·	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·	·
7	·	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
6	·	·	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·
5	·	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
4	·	·	·	·	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·
3	·	$\mathcal{P}$	·	·	·	·	·	·	$\mathcal{P}$	·	·	·	$\mathcal{P}$	·	·	·	$\mathcal{P}$
2	·	·	$\mathcal{P}$	·	·	·	$\mathcal{P}$	·	·	·	$\mathcal{P}$	·	·	·	$\mathcal{P}$	·	·
1	·	$\mathcal{P}$	·	$\mathcal{P}$	·	$\mathcal{P}$	·	$\mathcal{P}$	·	$\mathcal{P}$	·	$\mathcal{P}$	·	$\mathcal{P}$	·	$\mathcal{P}$	·
0	$\mathcal{P}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 1: Partial Table of 2-heap MINNIE outcomes with dots representing N-positions.

Theorem 7. *Let a, b be positive integers. $[a, b] \in \mathcal{P}$ if and only if*

$$[a,b] = \begin{cases} [1, 2i+1], i \geq 0 \\ [2, 4i+2], i \geq 0 \\ [3, 4i], i \geq 2 \\ [4, 4]. \end{cases}$$

Proof. For the purposes of this proof, we will make use of the Partition Theorem for Impartial Games, Theorem 1. Let $\mathcal{A}$ be the set of MINNIE positions $\{[1, 2i+1], i > 0\} \cup \{[2, 4i+2], i > 0\} \cup \{[3, 4i], i > 2\} \cup \{[4, 4]\} \cup \{[]\}$.

Let $\mathcal{B}$ be the set $M_2/\mathcal{A}$, where M_2 is the set of the MINNIE positions with two or less heaps. Thus, $\mathcal{B} = \{[1, 2i], i > 1\} \cup \{[2, 4i], i > 1\} \cup \{[2, 4i+1], i > 1\} \cup \{[2, 4i+3], i > 0\} \cup \{[3, 4i+1], i > 1\} \cup \{[3, 4i+2], i > 1\} \cup \{[3, 4i+3], i > 0]\} \cup \{[3, 4]\} \cup (\{[n, m], n, m > 4\}/\{[4, 4]\}) \cup \{[n], n > 0\}$.

We define $\rightarrow$ to be the action of moving from a position to an option. It is easy to check that a position in $\mathcal{A}$ has no options in $\mathcal{A}$. We now show that all positions of $\mathcal{B}$ have at least one option in $\mathcal{A}$:

$[1, 2i] \rightarrow [1, 2i-1]$
$[2, 4i+1] \rightarrow [1, 4i+1]$ and $[2, 4i+3] \rightarrow [1, 4i+3]$
$[2, 4i] \rightarrow [2, 4i-2]$
$[3, 4i+1] \rightarrow [1, 4i+1]$ and $[3, 4i+3] \rightarrow [1, 4i+3]$
$[3, 4i+2] \rightarrow [2, 4i+2]$
$[3, 4] \rightarrow [1, 3]$
$[n, m] \rightarrow [2, m]$, if $m \equiv 2 \pmod 4$
$[n, m] \rightarrow [3, m]$, if $m \equiv 0 \pmod 4$
$[n, m] \rightarrow [1, m]$, otherwise
$[n] \rightarrow [\,]$. □

2.2. MINNIE on Three Heaps

For three heaps, we begin with some tables to show that the P-positions are not so easily categorized as they are in the two-heap game. We will represent P-positions by the letter $\mathcal{P}$ and N-positions by a dot.

10	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$
9	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$
8	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$
7	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$
6	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$
5	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$
4	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$
3	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$
2	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$
1	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$
	1	2	3	4	5	6	7	8	9	10

Table 2: Table of 3-heap MINNIE outcomes with third heap having one token.

10	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$
9	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$
8	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$
7	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$
6	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$
5	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$
4	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$
3	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
2	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$
1	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$
	1	2	3	4	5	6	7	8	9	10

Table 3: Table of 3-heap MINNIE outcomes with third heap having two tokens.

10	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$
9	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$
8	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$
7	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$
6	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$
5	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
4	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$
3	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$
2	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
1	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\mathcal{P}$	$\cdot$	$\cdot$	$\cdot$	$\mathcal{P}$
	1	2	3	4	5	6	7	8	9	10

Table 4: Table of 3-heap MINNIE outcomes with third heap having three tokens.

Theorem 8. *Suppose* MINNIE *is being played on three heaps of positive sizes ℓ, m, and n. Then $[\ell, m, n] \in \mathcal{P}$ if and only if*

$$[\ell, m, n] = \begin{cases} ①[1, m, n], m + n \equiv 1 \ (mod\ 2), m \neq 3 \\ ②[1, 3, 4] \\ ③[1, 3, n], n \equiv 2 \ (mod\ 4) \\ ④[2, m, n], m \neq 3,\ m + n \equiv 2 \ (mod\ 4) \\ ⑤[3, 3, n], n \equiv 3 \ (mod\ 4) \\ ⑥[3, m, n], m > 3,\ m \neq 5,\ n > 4,\ and\ m + n \equiv 0 \ (mod\ 4). \\ ⑦[5, 5, n], n \equiv 3 \ (mod\ 8). \end{cases}$$

Proof. For the purposes of this proof, we will make use of the Partition Theorem for Impartial Games, Theorem 1. We classify the positions by type. We define our set A to contain all P-positions on one and two heaps as well as all positions on three heaps of the 7 types from the statement of the theorem. The set B will be all other three heap positions.

It is easy to check that no position in A has an option in A and this we leave to the reader. We will show that all positions in B have at least one option in A.

We define ⓡ $\rightarrow$ ⓢ to be a move from a position of type ⓡ to a position of type ⓢ. We also define $\not\rightarrow$ to signify an illegal move, or a move to a position that is not an option of the current game. The positions in B each belong to one of the following cases.

$$\begin{cases} (8)[1, m, n], m + n \equiv 0 \ (\text{mod } 2), m \neq 3 \\ (9)[1, 3, n], n \neq 4, n \not\equiv 2 \ (\text{mod } 4) \\ (10)[2, m, n], m \neq 3, m + n \not\equiv 2 \ (\text{mod } 4) \\ (11)[2, 3, n] \\ (12)[3, 3, n], n \not\equiv 3 \ (\text{mod } 4) \\ (13)[3, 5, n], m > 3 \\ (14)[3, 4, 4] \\ (15)[3, m, n], m > 3, m + n \not\equiv 0 \ (\text{mod } 4) \\ (16)[4, m, n], m \neq 5 \\ (17)[4, 5, n] \\ (18)[5, 5, n], n \not\equiv 3 \ (\text{mod } 8) \\ (19)[5, m, n], m > 5 \\ (20)[\ell, m, n], \ell > 5 \end{cases}$$

The following table shows, for each position in set B, a corresponding option in set A. One case does not fit into our table. The position $[5, 5, n]$ in which $n \equiv 7(8)$

and $n > 7$ has an option in A: $[5, 5, n-4]$.

From	To P-position $(1, x, y)$	$(2, x, y)$	$(3, x, y)$	$(0, x, y)$
(8) $[1, m, n]$	$[1, m, n-1]$			
(9) $[1, 3, n]$	$n \equiv 1(2) \Rightarrow [1, 2, n]$			$n \equiv 0(4) \Rightarrow [0, 3, n]$
(10) $[2, m, n]$	$m + n \equiv 1(2) \Rightarrow [1, m, n]$	$m + n \equiv 0(4) \Rightarrow [2, m, n-2]$		
(11) $[2, 3, n]$	$n \equiv 1(2) \Rightarrow [1, 2, n]$ $n = 4 \Rightarrow [1, 3, n]$ $n \equiv 2(4) \Rightarrow [1, 3, n]$			$n = 4k, k > 1, \Rightarrow [0, 3, n]$
(12) $[3, 3, n]$	$n = 4 \Rightarrow [1, 3, 4]$ $n \equiv 2(4) \Rightarrow [1, 3, n]$		$n \equiv 1(4) \Rightarrow [3, 3, n-2]$	$n = 4k, k > 1 \Rightarrow [0, 3, 4]$
(13) $[3, 5, n]$	$n \equiv 0(2) \Rightarrow [1, 5, n]$	$n \equiv 1(4) \Rightarrow [2, 5, n]$	$n \equiv 3(4) \Rightarrow [3, 3, n]$	
(14) $[3, 4, 4]$				$[0, 4, 4]$
(15) $[3, m, n]$	$m + n \equiv 1(2) \Rightarrow [1, m, n]$	$m + n \equiv 2(4) \Rightarrow [2, m, n]$		
(16) $[4, m, n]$	$m + n \equiv 1(2) \Rightarrow [1, m, n]$	$m + n \equiv 2(4) \Rightarrow [2, m, n]$	$m + n \equiv 0(4), n > 4, \Rightarrow [3, m, n]$	$[4, 4, 4] \Rightarrow [0, 4, 4]$
(17) $[4, 5, n]$	$n \equiv 0(2) \Rightarrow [1, 5, n]$ $n \equiv 3(4) \Rightarrow [1, 4, n]$	$n \equiv 1(4) \Rightarrow [2, 5, n]$		
(18) $[5, 5, n]$	$n \equiv 0(2) \Rightarrow [1, 5, n]$ $n \equiv 1(4) \Rightarrow [2, 5, n]$	$[5, 5, 7] \Rightarrow [2, 5, 5]$		
(19) $[5, m, n]$	$m + n \equiv 1(2) \Rightarrow [1, m, n]$	$m + n \equiv 2(4) \Rightarrow [2, m, n]$	$m + n \equiv 0(4) \Rightarrow [3, m, n]$	
(20) $[\ell, m, n]$	$m + n \equiv 1(2) \Rightarrow [1, m, n]$	$m + n \equiv 2(4) \Rightarrow [2, m, n]$	$m + n \equiv 0(4) \Rightarrow [3, m, n]$	

Table 5: Table showing how each position in B has an option in A

Therefore, by Theorem 1, our conclusion follows. □

3. Nim-Values and Open Questions

The *minimum excluded value* or *mex* of a subset of the nonnegative integers is the least nonnegative integer to not be contained within the subset. According to Sprague-Grundy Theory, the nim-value of a position A is equal to the mex of the set of nim-values of options of A. In this way, nim-values can always be calculated inductively.

The Sprague-Grundy Theorem also gives an algorithm for finding the nim-value of a position even in the case of a disjunctive sum. Note that Player One has the obvious winning strategy in any game of NIM consisting of only one stack of removing the entire stack. So, any game with nim value $n > 0$ is in $\mathcal{N}$. However, conversely, any game with nim-value $n = 0$ is in $\mathcal{P}$.

Theorem 9. *In* EUCLID*, the nim-value of a game played with numbers x and y is equal to* $\lfloor |\frac{x}{y} - \frac{y}{x}| \rfloor$.

The nim-values of EUCLID were given by Nivasch [13].

Let f be a function taking non-negative integers to non-negative integers with the properties that $f(0) = 0$, $f(n) \leq f(n+1)$ and $f(n) \leq n$. Call this a *rule function*. The subtraction game MAX-NIM$_f$ has $H = \{a\}$ and $S_H = \{i : i = 1, 2, \ldots, f(a)\}$. This class of games was studied by Levine [10]. For example, if $f(n) = \lfloor \frac{n}{2} \rfloor$ then (i) the heap sizes in $\mathcal{P}$ are those of size 2^n for some n, and (ii) the sequence of nim-values of heap sizes 0, 1, 2, ... starts

$$\mathbf{0}, 0, \mathbf{1}, 0, \mathbf{2}, 1, \mathbf{3}, 0, \mathbf{4}, 2, \mathbf{5}, 1, \mathbf{6}, 3, \mathbf{7}, 0, \mathbf{8}, 4, \mathbf{9}, 2, \mathbf{10}, \ldots.$$

$$0, 0, 1, 0, 2, 1, 3, 0, 4, 2, \ldots$$

Note how the sequence remains when the first occurrence of each number (the boldface numbers) is removed. Levine called such a sequence *fractal*.

Levine actually proved the following theorem.

Theorem 10 (**[10]**)**.** *Let $(g_n)_{n \geq 0}$ be an infinite sequence. The following are equivalent:*

1. *g is a fractal sequence;*
2. *g is the nim-sequence for* MAX-NIM$_f$ *for some rule function f.*

We draw this conjecture from our previous theorems and also some computer data on the four, and five heap games.

Question 11. What is the explicit formula for nim-values in two-heap MINNIE? Is it true that each row and column is eventually periodic?

12	12	1	3	0	2	4	5	10	11	9	6	8	7	15	13	12	14
11	11	0	2	1	4	9	10	7	3	13	5	6	8	12	11	14	17
10	10	1	0	2	7	3	9	11	8	12	4	5	6	14	10	13	15
9	9	0	1	3	6	8	7	5	4	2	12	13	9	11	6	8	7
8	8	1	3	0	5	7	6	9	2	4	8	3	11	10	7	5	9
7	7	0	2	1	3	6	8	4	9	5	11	7	10	6	12	4	8
6	6	1	0	2	4	5	3	8	6	7	9	10	5	4	8	3	2
5	5	0	1	3	2	4	5	6	7	8	3	9	4	2	5	7	6
4	4	1	3	5	0	2	4	3	5	6	7	4	2	5	3	6	4
3	3	0	2	1	5	3	2	1	0	3	2	1	0	3	2	1	0
2	2	1	0	2	3	1	0	2	3	1	0	2	3	1	0	2	3
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 6: Partial Table of 2-heap MINNIE nim-values.

We know the outcomes for MINNIE on 1,2, and 3 heaps, but what about for an arbitrarily large number of heaps?

Conjecture 1. Suppose MINNIE is being played on k heaps, the smallest of which having greater than 3 tokens and the second smallest having greater than 5 tokens. Then, Player One has a winning strategy.

We draw this conjecture from our previous theorems and also some computer data on the four, and five heap games.

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