APPEARANCE OF BALANCING AND RELATED NUMBER SEQUENCES IN STEADY STATE PROBABILITIES OF SOME MARKOV CHAINS

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Abstract

The balancing numbers x and the Lucas-balancing numbers y are solutions of the Diophantine equation $8x^2 + 1 = y^2$, and both types of numbers satisfy a common second order recurrence relation. These numbers can be seen as numerators and denominators in the steady state probabilities of a class of Markov chains.

1. Introduction

A natural number B is a *balancing number* with *balancer* R if the pair (B, R) is a solution of the Diophantine equation $1+2+\cdots+(B-1) = (B+1)+\cdots+(B+R)$ [1]. If B is a balancing number, then $8B^2 + 1$ is a perfect square and $C = \sqrt{8B^2 + 1}$ is called a *Lucas-balancing number* ([6], [9]). The n-th balancing number is denoted by B_n and the balancing numbers satisfy the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ with initial terms $B_0 = 0, B_1 = 1$. Similarly, the n-th Lucas-balancing number is denoted by C_n and the Lucas-balancing numbers satisfy a recurrence relation identical to that of balancing numbers with initial terms $C_0 = 1,\ C_1 = 3$ (see [9]). Panda and Rout [8] studied a class of binary recurrences $x_{n+1} = Ax_n - x_{n-1}$ with initial terms $x_0 = 0,\ x_1 = 1$. These sequences are known as *balancing-like sequences* [9]. Many properties of these sequences resemble the corresponding properties of balancing numbers. The sequences of natural numbers, the even indexed Fibonacci numbers and the balancing numbers are particular cases of balancing-like sequences corresponding to $A = 2, 3$ and 6, respectively.

A *cobalancing number* b is a natural number that satisfies the equation $1 + 2 + \cdots + b = (b+1) + \cdots + (b+r)$ for some natural number r (see [5]). If b is a cobalancing number, then $8b^2+8b+1$ is a perfect square and $\sqrt{8b^2+8b+1}$ is called a Lucas-cobalancing number. The n-th cobalancing number is denoted by b_n and the cobalancing numbers satisfy $b_{n+1} = 6b_n - b_{n-1} + 2$ with initial terms $b_0 = 0,\ b_1 = 0$. The n-th *Lucas-cobalancing number* is denoted by c_n and the recurrence relation for the Lucas-cobalancing numbers is the same as that of balancing numbers with initial terms $c_0 = -1,\ c_1 = 1$ [9]. A very important relationship between balancing and cobalancing numbers is that the sum of first $n-1$ balancing numbers is equal to half of the n^{th} cobalancing number, that is, $\sum_{i=1}^{n-1} B_i = b_n/2$ [[5], Theorem 4.1].

The *Pell sequence* is defined by the recurrence relation $P_{n+1} = 2P_n + P_{n-1}$, $P_0 = 0,\ P_1 = 1$. The Pell sequence is related to the balancing sequence by several means. For example, $P_{2n} = 2B_n,\ P_{2n+1} = B_{n+1} - B_n,\ n = 1, 2, \ldots$. For many other relations among Pell and balancing numbers, the readers are advised to refer to [7].

It is well known that if n is large, then the $(n+1)^{st}$ Fibonacci number F_{n+1} is approximately $\phi = \dfrac{1+\sqrt{5}}{2}$ times the n^{th} Fibonacci number F_n and ϕ is known as the *golden ratio* [2]. So far as the balancing numbers are concerned, the $(n+1)^{st}$ balancing number B_{n+1} is approximately $3+2\sqrt{2}$ times the n^{th} balancing number B_n (see [1]) and the approximation is very sharp. Furthermore, $3+2\sqrt{2} = (1+\sqrt{2})^2$, and $1+\sqrt{2}$ is known as the *silver ratio* [11]. It is interesting to note that for large n, P_{n+1} is approximately equal to $(1+\sqrt{2})P_n$.

Hlynka and Sajobi [3] established the presence of Fibonacci numbers in numerators and denominators of the steady state probabilities of a particular class of Markov chains. Motivated by their work, we construct a class of Markov chains whose steady state probabilities involve balancing, Lucas-balancing or balancing-like numbers.

2. Balancing Numbers in Steady State Probabilities

A discrete time *Markov chain* is a stochastic process $\{X_k\}$, where k runs over nonnegative integers, such that $Pr\{X_{k+1} = j | X_k = i, X_{k-1} = l, ..., X_0 = r\} = Pr\{X_{k+1} = j | X_k = i\}$. In other words, the Markovian property asserts that any probability related to the future behavior of the process depends on the present state of the process and not on the past states. The set of all possible values that a Markov chain is allowed to take is known as its state space. The *transition probability* $p_{ij} = Pr\{X_{k+1} = j | X_k = i\}$ not only depends on the initial state i and final state j, but it also depends on the time of transition k. When this probability depends only on i and j and not on k for all possible states i and j and time k, then the Markov chain $\{X_k\}$ is said to have *stationary transition probabilities*. In this case, $p_{ij}=Pr\{X_{k+1} = j|X_k = i\}$ is the probability of passing from state i to state j in one transition and the matrix $\mathbf{P} = (p_{ij})$ is known as the transition probability matrix [10]. Throughout this paper, unless explicitly mentioned, by a Markov chain, we mean a discrete time Markov chain with stationary transition probabilities having the set of nonnegative integers as its state space.

Let $\{X_k\}$ be a Markov chain. The probability $p_{ij}^{(n)} = Pr\{X_{k+n} = j|X_k = i\}$ is called an n-step transition probability. $p_{ij}^{(n)}$ is the probability of passing from state i to state j in n transitions and is the ij^{th} entry of the matrix $\mathbf{P}^n$(the n^{th} power of the matrix $\mathbf{P}$). As $n \to \infty$, $p_{ij}^{(n)}$ becomes independent of the initial state i and for $j = 0, 1, \dots$, the numbers $\pi_j = \lim_{n\to\infty} p_{ij}^{(n)}$ are known as the *steady state probabilities* of the Markov chain $\{X_k\}$. The vector $\vec{\pi} = (\pi_0, \pi_1, \dots)$ satisfies the relationships $\vec{\pi} = \vec{\pi}\mathbf{P}$ and $\pi_0 + \pi_1 + \cdots = 1$ ([4], [10]).

Consider a village which can accommodate not more than a population of size $n-1$. Let X_k be the population of the village at time k. The population increases by one for each birth, decreases by one for each death and becomes zero when the entire population is migrated to a different destination. Then, $\{X_k\}_{k=0}^{\infty}$ can be viewed as a Markov chain and assume that, it has the transition probabilities $p_{i,i+1} = p_{i,i-1} = \dfrac{1}{6}$ if $1 \le i \le n-2$, $p_{01} = \dfrac{1}{6}$, $p_{00} = p_{10} = p_{n-1,0} = \dfrac{5}{6}$, $p_{i0} = \dfrac{2}{3}$ if $2 \le i \le n-2$, and $p_{ij} = 0$ otherwise. Thus, the transition probability matrix is given by

$$\mathbf{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots & 0 & 0 \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots & 0 & 0 \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 2/3 & 0 & 0 & 0 & 0 & \ddots & 0 & 1/6 \\ 5/6 & 0 & 0 & 0 & 0 & \ddots & 1/6 & 0 \end{bmatrix}. \tag{1}$$

The following theorem shows that the steady state probabilities corresponding to $\mathbf{P}$ are functions of balancing and cobalancing numbers.

Theorem 2.1. *The steady state probability vector corresponding to the transition probability matrix* $\mathbf{P}$ *given in* (1) *is* $\vec{\pi} = \left(\frac{2B_n}{b_{n+1}}, \frac{2B_{n-1}}{b_{n+1}}, \dots, \frac{2B_1}{b_{n+1}}\right)$.

Proof. Using the relation $\vec{\pi} = \vec{\pi}\mathbf{P}$, we get the following system of equations (written in reverse order):

$$\pi_{n-1} = \frac{1}{6}\pi_{n-2},\ \pi_i = \frac{1}{6}\pi_{i-1} + \frac{1}{6}\pi_{i+1},\ i = 1, 2, \dots, n-2,$$

$$\pi_0 = \frac{5}{6}\pi_0 + \frac{5}{6}\pi_1 + \frac{2}{3}(\pi_2 + \cdots + \pi_{n-2}) + \frac{5}{6}\pi_{n-1}.$$

These equations are known as the *balance equations*. On rearranging the first $n-1$ equations, we get

$$\pi_{n-2} = 6\pi_{n-1},\ \pi_{i-1} = 6\pi_i - \pi_{i+1},\ i = 1, 2, \ldots, n-2. \tag{2}$$

Setting $\pi_{n-1} = k$, we can rewrite the system of equations in (2) as

$$\pi_{n-1} = k = kB_1,\ \pi_{n-2} = 6k = kB_2,\ \pi_{i-1} = 6\pi_i - \pi_{i+1},\ i = 1, 2, \ldots, n-2.$$

We claim that $\pi_i = kB_{n-i}$ for $i = 0, 1, \ldots, n-1$. Since $\pi_{n-1} = k = kB_1$, $\pi_{n-2} = 6k = kB_2$, the assertion is true for $i = 0, 1$. Assume that the assertion is true for $i = j \leq n-1$. Using $\pi_{j-1} = 6\pi_j - \pi_{j+1}$ from (2), we get

$$\pi_{j+1} = 6kB_{n-j} - kB_{n-j+1} = k(6B_{n-j} - B_{n-j+1}) = kB_{n-(j+1)},$$

and hence, by mathematical induction, the assertion is true for $i = j+1 \leq n-1$. Furthermore, $\pi_0 + \pi_1 + \cdots + \pi_{n-1} = 1$ implies that $k = \frac{1}{\sum_{l=1}^{n} B_l}$. Hence,

$$\pi_i = \frac{2B_{n-i}}{\sum_{l=1}^{n} B_l},\ i = 0, 1, \ldots, n-1. \tag{3}$$

Since $\sum_{l=1}^{n} B_l = \frac{b_{n+1}}{2}$ ([5], Theorem 4.1), the proof is complete. □

Remark. Since $2B_i = P_{2i}$, (3) can be rewritten as $\pi_i = \frac{2P_{2(n-i)}}{\sum_{l=1}^{n} P_{2l}}$, $i = 0, 1, \ldots, n-1$.

We noticed that the steady state probability π_j is the limiting value of p_{ij}^n as $n \to \infty$, and hence the role of the initial state i is lost. In other words, $\pi_j = \lim_{n\to\infty} p_{ij}^n$ for $i = 0, 1, \ldots$. In the following theorem, we will show that a class of transition probability matrices, that includes $\mathbf{P}$ defined in (1) as a member, results in the same steady state probabilities.

Theorem 2.2. *Let $\{X_k\}_{k=0}^{\infty}$ be a Markov chain with state space $\{0, 1, \ldots, n-1\}$, and transition probability matrix*

$$\boldsymbol{P}(q) = \begin{bmatrix} 1-q & q & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5q & 1-6q & q & 0 & 0 & \cdots & 0 & 0 \\ 4q & q & 1-6q & q & 0 & \cdots & 0 & 0 \\ 4q & 0 & q & 1-6q & q & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 4q & 0 & 0 & 0 & 0 & \ddots & 1-6q & q \\ 5q & 0 & 0 & 0 & 0 & \ddots & q & 1-6q \end{bmatrix}, \tag{4}$$

where $0 < q \leq 1/6$*. Then, for each such* q*,* (4) *has the same steady state probabilities as that of* (1)*.*

Proof. Let $0 < q \leq 1/6$. Then, for each such q, the steady state probability vector corresponding to the transition probability matrix $\boldsymbol{P}(q)$ can be obtained from $\vec{\pi} = \vec{\pi}\mathbf{P}$. This leads to the system of equations (written in reverse order)

$$\pi_{n-1} = q\pi_{n-2} + (1-6q)\pi_{n-1},\ \pi_i = q\pi_{i-1} + (1-6q)\pi_i + q\pi_{i+1},\ i = 1, 2, \cdots, n-2 \quad (5)$$

and

$$\pi_0 = (1-q)\pi_0 + 5q\pi_1 + 4q(\pi_2 + \cdots + \pi_{n-2}) + 5q\pi_{n-1}. \quad (6)$$

On simplifying the equations in (5), we get

$$\pi_{n-2} = 6\pi_{n-1},\ \pi_{i-1} = 6\pi_i - \pi_{i+1},\ i = 1, 2, \ldots, n-2. \quad (7)$$

Since the system of equations in (7) is essentially the same as the system of equations in (2), the conclusion of the theorem follows. □

In Theorem 2.1, we notice that the steady state probabilities of $\mathbf{P}$, given in (1), are expressible in terms of balancing and cobalancing numbers. In the following theorem, we consider a transition probability matrix, not much different from the one used in Theorem 2.1, such that each of its steady state probabilities is expressible as a function of three balancing numbers.

Theorem 2.3. *The steady state probability vector corresponding to the transition probability matrix*

$$\boldsymbol{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots & 0 & 0 \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots & 0 & 0 \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 2/3 & 0 & 0 & 0 & 0 & \ddots & 0 & 1/6 \\ 2/3 & 0 & 0 & 0 & 0 & \ddots & 1/6 & 1/6 \end{bmatrix} \quad (8)$$

($p_{i,i+1} = p_{i,i-1} = \dfrac{1}{6}$ *if* $1 \leq i \leq n-1$*,* $p_{01} = \dfrac{1}{6}$*,* $p_{00} = p_{10} = \dfrac{5}{6}$*,* $p_{i0} = \dfrac{2}{3}$ *if* $2 \leq i \leq n-1$ *and* $p_{ij} = 0$ *otherwise) is given by* $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{n-1})$*, where* $\pi_i = \dfrac{B_{n+i} - B_{n+i-1}}{B_n}$*,* $i = 0, 1, \ldots, n-1$*.*

Proof. The balance equations for finding the steady state probabilities corresponding to the transition probability matrix $\boldsymbol{P}$ in (8) are given by

$$\pi_{n-1} = \frac{1}{6}\pi_{n-2} + \frac{1}{6}\pi_{n-1},\ \pi_i = \frac{1}{6}\pi_{i-1} + \frac{1}{6}\pi_{i+1},\ i = 1, 2, \ldots, n-2.$$

On rearrangement, we get

$$\pi_{n-2} = 5\pi_{n-1},\ \pi_{i-1} = 6\pi_i - \pi_{i+1},\ i = 1, 2, \ldots, n-2.$$

Letting $\pi_{n-1} = k$, we see that $\pi_{n-2} = 5k$. Observe that

$$\pi_{n-i} = k\,(B_i - B_{i-1}) \tag{9}$$

is true for $i = 1, 2$. Assuming that (9) is true for $i = r$, the balance equation $\pi_{n-r-1} = 6\pi_{n-r} - \pi_{n-r+1}$ implies that

$$\pi_{n-r-1} = 6k(6B_r - B_{r-1}) - k(B_{r-1} - B_{r-2}) = k(B_{r+1} - B_r),$$

and hence, (9) is true for $i = r + 1$, and by mathematical induction, (9) holds for $i = 0, 1, \ldots, n-1$. Since $\pi_0 + \pi_1 + \cdots + \pi_{n-1} = 1$, the conclusion of the theorem follows. □

Remark. Since $P_{2n} = 2B_n$ and $P_{2n-1} = B_n - B_{n-1}$, we can express the steady state probabilities in the previous theorem as $\pi_i = \dfrac{2P_{2(n-i)-1}}{P_{2n}},\ i = 0, 1, \cdots, n-1$.

3. Lucas Balancing Numbers in Steady State Probabilities

In the previous section, we established the presence of balancing numbers in the steady state probabilities of some Markov chains. In the present section, we introduce some finite state Markov chains having rational functions of Lucas-balancing numbers as steady state probabilities.

To start with, let $\{X_k\}_{k=0}^{\infty}$ be a Markov chain having the state space $\{0, 1, \cdots, n-1\}$ and transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots & 0 & 0 \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots & 0 & 0 \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 2/3 & 0 & 0 & 0 & 0 & \ddots & 0 & 1/6 \\ 1/3 & 0 & 0 & 0 & 0 & \ddots & 1/6 & 1/2 \end{bmatrix}. \tag{10}$$

(More specifically, $\mathbf{P} = (p_{ij})$, $p_{i,i+1} = p_{i,i-1} = \dfrac{1}{6}$ if $1 \le i \le n-2$, $p_{01} = \dfrac{1}{6}$, $p_{00} = p_{10} = \dfrac{5}{6}$, $p_{i0} = \dfrac{2}{3}$ if $2 \le i \le n-2$, $p_{n-1,0} = \dfrac{1}{3}$, $p_{n-1,n-1} = \dfrac{1}{2}$, and $p_{ij} = 0$

otherwise.) The following theorem shows that the steady state probabilities, corresponding to **P** given in (10), are similar to those of the matrix used in Theorem 2.1. The only difference is that the balancing numbers appearing in the steady state probabilities are replaced by corresponding Lucas-balancing numbers.

Theorem 3.1. *The steady state probability vector, corresponding to the transition probability matrix* **P** *given in* (10)*, is* $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{n-1})$*, where* $\pi_i = \frac{C_{n-i}}{\sum_{i=1}^{n} C_i}$ *for* $i = 0, 1, \ldots, n-1$*, and* C_i *is the* i^{th} *Lucas-balancing number.*

Proof. The balance equations relating the steady state probabilities associated with **P** are given by

$$\pi_{n-1} = \frac{1}{6}\pi_{n-2} + \frac{1}{2}\pi_{n-1},\ \pi_i = \frac{1}{6}\pi_{i-1} + \frac{1}{6}\pi_{i+1},\ \ i = 1, 2, \ldots, n-2.$$

On rearrangement, we get

$$\pi_{n-2} = 3\pi_{n-1},\ \pi_{i-1} = 6\pi_i - \pi_{i+1},\ \ i = 1, 2, \ldots, n-2.$$

Letting $\pi_{n-1} = k$, we see that $\pi_{n-2} = 3k$. Observe that $\pi_{n-i} = kC_{n-i}$ is true for $i = 1, 2$. Assuming the assertion to be true for $i = r$, the balance equation $\pi_{n-r-1} = 6\pi_{n-r} - \pi_{n-r+1}$ implies that

$$\pi_{n-r-1} = 6kC_r - kC_{r-1} = kC_{r+1},$$

and hence, the assertion is true for $i = r + 1$. Since $\pi_0 + \pi_1 + \cdots + \pi_{n-1} = 1$, the conclusion of the theorem follows. □

The $n \times n$ transition probability matrix **P** defined in (10) results in the steady state probabilities $\pi_i = \frac{C_{n-i}}{\sum_{i=1}^{n} C_i}$, $i = 0, 1, \ldots, n-1$. In the following theorem, we will see that a class of transition probability matrices in which **P** defined in (10) is a member, results in the same steady state probabilities.

Theorem 3.2. *Let* $\{X_k\}_{k=0}^{\infty}$ *be a Markov chain having the state space* $\{0, 1, \ldots, n-1\}$ *and transition probability matrix*

$$\boldsymbol{P}(q) = \begin{bmatrix} 1-q & q & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5q & 1-6q & q & 0 & 0 & \cdots & 0 & 0 \\ 4q & q & 1-6q & q & 0 & \cdots & 0 & 0 \\ 4q & 0 & q & 1-6q & q & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 4q & 0 & 0 & 0 & 0 & \ddots & 1-6q & q \\ 2q & 0 & 0 & 0 & 0 & \ddots & q & 1-3q \end{bmatrix}, \quad (11)$$

where q is a real number and $0 < q \leq 1/6$. Then, for each such q, $\boldsymbol{P}(q)$ results in the same steady state probabilities as that of the transition probability matrix used in Theorem 3.1.

Proof. The proof of the above theorem is similar to that of Theorem 2.2, and hence, it is omitted. □

Lucas-cobalancing numbers are associated with cobalancing numbers in the same manner the Lucas-balancing numbers are associated with balancing numbers. However, as discussed in Section 1, the recurrence relation for Lucas-cobalancing numbers is identical to that of balancing numbers. In the following theorem, we consider a finite state Markov chain, whose steady state probabilities are functions of Lucas-cobalancing numbers. Since the proof is similar to that of Theorem 3.1, we prefer to omit it.

Theorem 3.3. *Let $\{X_k\}_{k=0}^{\infty}$ be a Markov chain having the state space $\{0, 1, \ldots, n-1\}$, and transition probability matrix*

$$\boldsymbol{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots & 0 & 0 \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots & 0 & 0 \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ 2/3 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\ 16/21 & 0 & 0 & 0 & 0 & \ddots & 0 & 1/14 \\ 1/3 & 0 & 0 & 0 & 0 & \ddots & 1/6 & 1/2 \end{bmatrix} \tag{12}$$

($p_{i,i+1} = p_{i,i-1} = \dfrac{1}{6}$, if $1 \leq i \leq n-2$, $p_{01} = \dfrac{1}{6}$, $p_{00} = P_{10} = \dfrac{5}{6}$, $p_{i0} = \dfrac{2}{3}$, if $2 \leq i \leq n-3$, $p_{n-2,0} = \dfrac{16}{21}$, $p_{n-1,0} = \dfrac{1}{3}$, $p_{n-2,n-1} = \dfrac{1}{14}$ $p_{n-1,n-1} = \dfrac{1}{2}$, and $p_{ij} = 0$ otherwise). Then, the steady state probability vector corresponding to the transition probability matrix $\boldsymbol{P}$ is given by $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{n-1})$, where $\pi_i = \dfrac{c_{n-i}}{\sum_{i=1}^{n} c_i}$, $i = 0, 1, \ldots, n-1$, where c_n denotes the n^{th} Lucas-cobalancing number.

4. Silver Ratio in Steady State Probabilities of a Markov Chain With Infinite State Space

In the last two sections, we studied some finite state Markov chains whose steady state probabilities are functions of balancing, cobalancing or Lucas-balancing numbers. In this section, we study the steady state probabilities of a Markov chain having the infinite state space $\{0, 1, \cdots\}$ and transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 5/6 & 1/6 & 0 & 0 & 0 & \cdots \\ 5/6 & 0 & 1/6 & 0 & 0 & \cdots \\ 2/3 & 1/6 & 0 & 1/6 & 0 & \cdots \\ 2/3 & 0 & 1/6 & 0 & 1/6 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{13}$$

More specifically, $\mathbf{P}= (p_{ij})$ with $p_{i,i+1} = \dfrac{1}{6}$ if $i \geq 1$, $p_{10} = \dfrac{5}{6}$, $p_{i0} = \dfrac{2}{3}$ if $i \geq 2$, and $p_{ij} = 0$ otherwise. Observe that the transition probability matrix in (13) is the limiting form of the matrix in (1) as $n \to \infty$.

In the following theorem, using the limiting behavior of steady state probabilities corresponding to the matrix $\mathbf{P}$ used in Theorem 2.3, we will show that the steady state probabilities corresponding to $\mathbf{P}$ in (13) are functions of the silver ratio.

Theorem 4.1. *The steady state probability vector corresponding to the transition probability matrix* $\mathbf{P}$ *given in* (13) *is* $\vec{\pi} = (\pi_0, \pi_1, \ldots)$, *where* $\pi_i = \beta^i - \beta^{i+1}$, $i = 0, 1, \ldots$, *and* $\beta = 3 - 2\sqrt{2}$.

Proof. Using the identity $\vec{\pi} = \vec{\pi}\mathbf{P}$, the balance equations for the calculation of steady state probabilities are given by

$$\pi_0 = \frac{5}{6}\pi_0 + \frac{5}{6}\pi_1 + \frac{2}{3}(\pi_2 + \pi_3 + \cdots),\ \pi_i = \frac{1}{6}\pi_{i-1} + \frac{1}{6}\pi_{i+1} \text{ for } i \geq 1.$$

On simplification, we get $\pi_1 = 5\pi_0 - 4$, $\pi_2 = 29\pi_0 - 24$, $\pi_3 = 169\pi_0 - 140$, and using mathematical induction, one can see that

$$\pi_i = (B_{i+1} - B_i)\pi_0 - 4B_i,\ i = 1, 2, \ldots \tag{14}$$

Solving the infinite system (14) is not easy. Instead, we observe that transition probability matrix $\mathbf{P}$ given in (13) is a limiting case of the $n\times n$ transition probability matrix $\mathbf{P}$ that appear in Theorem 2.3. In the proof of Theorem 2.3, we noticed that $\pi_0 = \frac{B_n - B_{n-1}}{B_n}$. Hence, in this case, we have

$$\pi_0 = lim_{n\to\infty}\frac{B_n - B_{n-1}}{B_n} = 1 - lim_{n\to\infty}\frac{B_{n-1}}{B_n} = 1 - (3 - 2\sqrt{2}) = 1 - \beta,$$

where $\beta = 3 - 2\sqrt{2}$. Similarly,

$$\pi_1 = lim_{n\to\infty}\frac{B_{n-1}-B_{n-2}}{B_n}=lim_{n\to\infty}\left(\frac{B_{n-1}}{B_n}-\frac{B_{n-2}}{B_{n-1}}.\frac{B_{n-1}}{B_n}\right)=\beta-\beta^2,$$

and using mathematical induction, it is easy to see that

$$\pi_i = lim_{n\to\infty}\frac{B_{n-i-1}-B_{n-i}}{B_n}=\beta^i-\beta^{i+1},\ i=0,1,\ldots.$$

□

It is easy to verify that, if the transition probability matrix **P** used in Theorem 4.1 is replaced by

$$\mathbf{P}=\begin{bmatrix} 1-q & q & 0 & 0 & 0 & \cdots \\ 5q & 1-6q & q & 0 & 0 & \cdots \\ 4q & q & 1-6q & q & 0 & \cdots \\ 4q & 0 & q & 1-6q & q & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{15}$$

where $0 < q \leq 1/6$, the conclusion of Theorem 4.1 remains unchanged.

Remark. Observe that $\beta = 3-2\sqrt{2} = (1+\sqrt{2})^{-2}$, and hence, the steady state probabilities obtained in Theorem 4.1, are functions of the silver ratio.

The following corollary, which is a consequence of Theorem 4.1, establishes an interesting relationship between consecutive balancing numbers and powers of β.

Corollary 4.2. *For* $n = 1, 2, \ldots,\ \beta^{n+1} = \beta B_{n+1} - B_n$.

Proof. In view of Theorem 4.1, $\pi_0 = 1-\beta$ and $\beta^i - \beta^{i+1} = \beta^i(1-\beta)$. Also, by virtue of (14),

$$\pi_i = (B_{i+1}-B_i)\pi_0 - 4B_i,\ i = 1, 2, \ldots.$$

Substituting the value of π_0 in the last equation, we get

$$\beta^i(1-\beta) = (B_{i+1}-B_i)(1-\beta) - 4B_i.$$

Thus,

$$\beta^i = (B_{i+1}-B_i) - \frac{4B_i}{1-\beta} = B_{i+1} - (3+2\sqrt{2})B_i = B_{i+1} - \frac{B_i}{\beta}$$

from which the desired identity follows. □

Remark. It is easy to see that the identity appearing in Corollary 4.2 is equivalent to $\alpha^n B_{n+1} - \alpha^{n+1}B_n = 1$, where $\alpha = 3+2\sqrt{2}$.

5. Balancing-like Numbers in the Steady State Probabilities of Markov Chains

In Section 1, we have seen that if $A > 2$ is a fixed positive integer, then the sequence $\{x_n\}_{n=1}^{\infty}$, defined recursively by $x_{n+1} = Ax_n - x_{n-1}$ with initial terms $x_0 = 0,\ x_1 = 1$, is called a balancing-like sequence and this sequence serves as a generalization of the balancing sequence. In this section, we consider a finite state Markov chain whose steady state probabilities involve balancing-like numbers.

We consider a Markov chain $\{X_k\}$ with the state space $\{0, 1, \ldots, n-1\}$ and transition probability matrix

$$\mathbf{P}(A) = \begin{bmatrix} 1-\frac{1}{A} & \frac{1}{A} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1-\frac{1}{A} & 0 & \frac{1}{A} & 0 & 0 & \cdots & 0 & 0 \\ 1-\frac{2}{A} & \frac{1}{A} & 0 & \frac{1}{A} & 0 & \cdots & 0 & 0 \\ 1-\frac{2}{A} & 0 & \frac{1}{A} & 0 & \frac{1}{A} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 1-\frac{2}{A} & 0 & 0 & 0 & 0 & \ddots & 0 & \frac{1}{A} \\ 1-\frac{1}{A} & 0 & 0 & 0 & 0 & \ddots & \frac{1}{A} & 0 \end{bmatrix}. \tag{16}$$

To be more specific, $p_{i,i+1} = p_{i,i-1} = \frac{1}{A}$ if $1 \le i \le n-2,\ p_{01} = \frac{1}{A},\ p_{00} = p_{10} = p_{n-1,0} = 1 - \frac{1}{A},\ p_{i0} = 1 - \frac{2}{A}$ if $2 \le i \le n-2$, and $p_{ij} = 0$ otherwise. In the following theorem, we will show that the steady state probabilities corresponding to $\mathbf{P}(A)$, are rational functions of the first n balancing-like numbers.

Theorem 5.1. *The steady state probability vector corresponding to the $n \times n$ transition probability matrix $\mathbf{P}(A)$ is $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{n-1})$, where $\pi_i = \dfrac{x_{n-i}}{\sum_{i=1}^{n} x_i}$, $i = 0, 1, \ldots, n-1$.*

Proof. The proof of this theorem is similar to that of Theorem 2.1, and hence it is omitted. □

It is important to note that if q is any real number with $0 < q \le \frac{1}{A}$, then the transition probability matrix

$$\mathbf{P}_q(A) = \begin{bmatrix} 1-q & q & 0 & 0 & 0 & \cdots & 0 & 0 \\ (A-1)q & 1-Aq & q & 0 & 0 & \cdots & 0 & 0 \\ (A-2)q & q & 1-Aq & q & 0 & \cdots & 0 & 0 \\ (A-2)q & 0 & q & 1-Aq & q & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ (A-2)q & 0 & 0 & 0 & 0 & \ddots & 1-Aq & q \\ (A-1)q & 0 & 0 & 0 & 0 & \ddots & q & 1-Aq \end{bmatrix}$$

results in the same steady state probabilities as that of $\mathbf{P}(A)$.

6. Conclusion

In this work, we established the appearance of balancing and related numbers in steady state probabilities of some Markov chains. We also noticed that, in many instances, a class of transition probability matrices give rise to the same steady state probabilities. Using the balance equations, we also derived an identity relating the balancing numbers and the silver ratio. Some problems in this area are still open. We encourage the readers to study the appearance of other number sequences in the steady state probabilities of some specially constructed transition probability matrices. In this process, they may possibly be able to explore some interesting identities using the balance equations.

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