ANALYTIC CONTINUATION OF APOSTOL-VU MULTIPLE BALANCING ZETA FUNCTIONS

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Abstract

In this note we introduce Apostol-Vu multiple balancing zeta functions as

$$\zeta_{AVB}(s_1, s_2, \ldots, s_r; s_{r+1}) = \sum_{1 \le m_1 < \cdots < m_r < \infty} \frac{1}{B_{m_1}^{s_1} B_{m_2}^{s_2} \cdots B_{m_r}^{s_r} B_{m_1+m_2+\cdots+m_r}^{s_{r+1}}},$$

where s_i $(i = 1, 2, \ldots, r+1)$ are complex variables. We study the analytic continuation of Apostol-Vu multiple balancing zeta functions of three variables and calculate a complete list of the residues corresponding to their respective poles. Further we examine the values of these zeta functions at negative integers.

1. Introduction

The Euler-Zagier multiple zeta function of depth r is defined by

$$\zeta(s_1, s_2, \ldots, s_r) = \sum_{1 \le m_1 < m_2 \cdots < m_r < \infty} \frac{1}{{m_1}^{s_1} {m_2}^{s_2} \cdots {m_r}^{s_r}},$$

where s_i $(i = 1, 2, \ldots, r)$ are complex variables [13]. Zhao [14] gave an analytic continuation of $\zeta(s_1, s_2, \ldots, s_r)$ as a function of $s_i (i = 1, 2, \ldots, r)$ to $\mathbb{C}^r$ using Gelfand and Shilov's generalized functions. The series

$$\sum_{m=1}^{\infty} \sum_{n<m} \frac{1}{m^{s_1} n^{s_2} (m+n)} \tag{1.1}$$

was first introduced by Apostol and Vu [2] and they obtained partial results on its analytic continuation. The meromorphic continuation of (1.1), and the more general series

$$\sum_{m=1}^{\infty}\sum_{n<m}\frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}} \tag{1.2}$$

to the whole space, was first proved in [6]. In [7], Matsumoto generalized (1.2) and introduced the Apostol-Vu r-ple zeta function

$$\zeta_{AV,r}(s_1,s_2,\ldots,s_r;s_{r+1}) = \sum_{1\le m_1<m_2<\cdots<m_r<\infty}\frac{1}{m_1^{s_1}m_2^{s_2}\cdots m_r^{s_r}(m_1+m_2+\cdots+m_r)^{s_{r+1}}}. \tag{1.3}$$

He proved that the series (1.3) is convergent absolutely when $\mathrm{Re}(s_i)>1$ $(1\le i\le r)$, $\mathrm{Re}(s_{r+1})>0$, and can be extended meromorphically to the whole space $\mathbb{C}^{r+1}$.

Let $\{F_n\}_{n\ge 0}$ be the sequence of Fibonacci numbers which is recursively defined by

$$F_0=0, F_1=1, F_{n+1}=F_n+F_{n-1}, \text{ for } n\ge 1.$$

The closed form expression for $\{F_n\}_{n\ge 0}$ is $F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, where $\alpha=\frac{1+\sqrt{5}}{2}$ and $\beta=\frac{1-\sqrt{5}}{2}$ are the zeros of the Fibonacci characteristic equation $x^2-x-1=0$. Navas [8] introduced the Fibonacci zeta function $\zeta_F(s)=\sum_{n=1}^{\infty}F_n^{-s}$, where F_n denotes the n-th Fibonacci number, and proved that $\zeta_F(s)$ is meromorphically continued to the whole complex plane $\mathbb{C}$. For $s=1$, $\zeta_F(1)=\sum_{n=1}^{\infty}F_n^{-1}$ is the sum of the reciprocal of Fibonacci constants which is an irrational number [1]. Duverney et al. [4] proved that $\zeta_F(2m)$ for $m=1,2,3,\ldots$ are all transcendental numbers. Elsner et al. [5] proved that $\zeta_F(2)$, $\zeta_F(4)$ and $\zeta_F(6)$ are algebraically independent. By applying the theory of modular forms, Ram Murty [9] also showed that $\zeta_F(2m)$ are transcendental numbers for $m\ge 1$. Recently, Rout and Meher [11] defined the multiple Fibonacci zeta function of depth r as:

$$\zeta_F(s_1,s_2,\ldots,s_r)=\sum_{1\le m_1<m_2<\cdots<m_r<\infty}\frac{1}{F_{m_1}^{s_1}F_{m_2}^{s_2}\cdots F_{m_r}^{s_r}}.$$

They studied the analytic continuation of $\zeta_F(s_1,s_2,\ldots,s_r)$ of depth 2, and found a complete list of poles and their corresponding residues. In [11], they also examined the arithmetic nature of multiple Fibonacci zeta functions at negative integer arguments.

Before going to the study of analytic continuation of Apostol-Vu multiple balancing zeta functions, our foremost task is to discuss the theory of balancing numbers and balancing zeta function. A natural number m is said to be a balancing number if it is the solution of a simple Diophantine equation $1+2+\cdots+(m-1)=(m+1)+(m+2)+\cdots+(m+r)$, where r is a balancer corresponding to m [3]. Let $\{B_m\}_{m\ge 0}$ be the balancing sequence which is recursively defined as $B_0=0$, $B_1=1$ and $B_m=6B_{m-1}-B_{m-2}$ for $m\ge 2$. The closed form expression for

$\{B_m\}_{m\geq 0}$ is $B_m = \dfrac{\alpha^m - \beta^m}{\alpha - \beta}$, where $\alpha = 3 + 2\sqrt{2}$ and $\beta = \alpha^{-1} = 3 - 2\sqrt{2}$ are the roots of the balancing characteristic equation $x^2 - 6x + 1 = 0$ [10]. Rout and Panda [12] considered balancing zeta function $\zeta_B(s) = \sum_{m=1}^{\infty} B_m^{-s}, \mathrm{Re}(s) > 0$ for $s \in \mathbb{C}$, where B_m denotes the m-th balancing number, and derived that $\zeta_B(s)$ can be meromorphically continued to the whole complex plane.

In the present study we introduce the Apostol-Vu multiple balancing zeta function and investigate its analytic continuation of three variable along with poles and their corresponding residues. We also evaluate the values of Apostol-Vu multiple balancing zeta functions at negative integers.

2. Analytic Continuation of Apostol-Vu Multiple Balancing Zeta Functions

In this section we define the Apostol-Vu multiple balancing zeta function and study its analytic continuation of three variables. *The Apostol-Vu multiple balancing zeta function* is defined by

$$\zeta_{AVB}(s_1, s_2, \ldots, s_r; s_{r+1}) = \sum_{1\leq m_1 < \cdots < m_r < \infty} \frac{1}{B_{m_1}^{s_1} B_{m_2}^{s_2} \cdots B_{m_r}^{s_r} B_{m_1+m_2+\cdots+m_r}^{s_{r+1}}}, \tag{2.1}$$

where s_i $(i = 1, 2, \ldots, r+1)$ are complex variables.

Proposition 1. *The series* $\zeta_{AVB}(s_1, s_2, s_3) = \sum_{1\leq m_1<m_2<\infty} \dfrac{1}{B_{m_1}^{s_1} B_{m_2}^{s_2} B_{m_1+m_2}^{s_3}}$ *converges absolutely in the domain*

$$D_3 = \{(s_1, s_2, s_3) \in \mathbb{C}^3 | Re(s_i) > 0, i = 1, 2, 3\}.$$

Proof. Let $s_j = \sigma_j + iy_j \in \mathbb{C}$ and $\sigma_j = \mathrm{Re}(s_j) > 0$, $j = 1, 2, 3$. From (2.1), we can write

$$\begin{aligned} \zeta_{AVB}(s_1, s_2, s_3) &= \sum_{1\leq m_1<m_2<\infty} \frac{1}{B_{m_1}^{s_1} B_{m_2}^{s_2} B_{m_1+m_2}^{s_3}} \\ &= \sum_{m_1=1}^{\infty} \frac{1}{B_{m_1}^{s_1}} \sum_{m_2=1}^{\infty} \frac{1}{B_{m_1+m_2}^{s_2} B_{2m_1+m_2}^{s_3}}. \end{aligned} \tag{2.2}$$

Now,

$$\begin{aligned}\left|\frac{1}{B_{m_1}^{s_1}}\right| =&(\alpha-\beta)^{\sigma_1}\left|\frac{1}{\left(\alpha^{m_1}-\beta^{m_1}\right)^{s_1}}\right|\\ \leq&(\alpha-\beta)^{\sigma_1}\frac{1}{\alpha^{m_1\sigma_1}\left(1-(|\frac{\beta}{\alpha}|)^{m_1}\right)^{\sigma_1}}\\ \leq&\left(\frac{\alpha-\beta}{1-|\beta/\alpha|}\right)^{\sigma_1}\frac{1}{\alpha^{\sigma_1 m_1}}\\ =&\Lambda_{\sigma_1}\frac{(\alpha-\beta)^{\sigma_1}}{\alpha^{\sigma_1 m_1}},\end{aligned} \tag{2.3}$$

where $\Lambda_{\sigma_1}=\frac{1}{(1-|\beta/\alpha|)^{\sigma_1}}$. Similarly,

$$\left|\frac{1}{B_{m_1+m_2}^{s_2}}\right| \leq \Lambda_{\sigma_2}\frac{(\alpha-\beta)^{\sigma_2}}{a^{\sigma_2(m_1+m_2)}} \tag{2.4}$$

and

$$\left|\frac{1}{B_{2m_1+m_2}^{s_3}}\right| \leq \Lambda_{\sigma_3}\frac{(\alpha-\beta)^{\sigma_3}}{a^{\sigma_3(2m_1+m_2)}}, \tag{2.5}$$

where $\Lambda_{\sigma_j}=\frac{1}{(1-|\beta/\alpha|)^{\sigma_j}}$ for $j=2,3$. By virtue of (2.2), (2.3), (2.4) and (2.5),

$$\begin{aligned}&\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left|\frac{1}{B_{m_1}^{s_1}B_{m_1+m_2}^{s_2}B_{2m_1+m_2}^{s_3}}\right|\\ \leq\quad&\sum_{m_1=1}^{\infty}\left|\frac{1}{B_{m_1}^{s_1}}\right|\sum_{m_2=1}^{\infty}\left|\frac{1}{B_{m_1+m_2}^{s_2}B_{2m_1+m_2}^{s_3}}\right|\\ \leq\quad&\Lambda_{\sigma_1}\Lambda_{\sigma_2}\Lambda_{\sigma_3}(\alpha-\beta)^{\sigma_1+\sigma_2+\sigma_3}\sum_{m_1=1}^{\infty}\frac{1}{\alpha^{\sigma_1 m_1}}\sum_{m_2=1}^{\infty}\frac{1}{\alpha^{\sigma_2\left(m_1+m_2\right)}\times\alpha^{\sigma_3\left(2m_1+m_2\right)}}\\ =\quad&\Lambda(\alpha-\beta)^{\sigma_1+\sigma_2+\sigma_3}\sum_{m_1=1}^{\infty}\frac{1}{\alpha^{\left(\sigma_1+\sigma_2+2\sigma_3\right)m_1}}\sum_{m_2=1}^{\infty}\frac{1}{\alpha^{\left(\sigma_2+\sigma_3\right)m_2}}\\ =\quad&\Lambda(\alpha-\beta)^{\sigma_1+\sigma_2+\sigma_3}\frac{1}{\left(\alpha^{\sigma_1+\sigma_2+2\sigma_3}-1\right)}\times\frac{1}{\left(\alpha^{\sigma_2+\sigma_3}-1\right)}\\ <\quad&\infty,\end{aligned}$$

as $\alpha>1$ where $\Lambda=\Lambda_{\sigma_1}\Lambda_{\sigma_2}\Lambda_{\sigma_3}$. Therefore, the series (2.2) converges absolutely in the domain D_3, which ends the proof. □

Theorem 1. *The series $\zeta_{AVB}(s_1,s_2,s_3)$ can be analytically continued to a meromorphic function on $\mathbb{C}^3$. It has possible simple poles on the hyperplanes*

$$s_1+s_2+2s_3=-2(k+l+2t)+\frac{2\pi ia}{\log\alpha}\ \textit{with}\ k,l,t\in\mathbb{Z}_{\geq 0}, a\in\mathbb{Z},$$

and

$$s_2 + s_3 = -2(l+t) + \frac{2\pi i b}{\log \alpha} \text{ with } l, t \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}.$$

Proof. For any $s = \sigma + iy \in \mathbb{C}$, we have

$$\begin{aligned} B_m^s = \left(\frac{\alpha^m - \beta^m}{4\sqrt{2}}\right)^s &= (4\sqrt{2})^{-s}\alpha^{ms}\left(1 - \left(\frac{\beta}{\alpha}\right)^m\right)^s \\ &= 2^{\frac{-5s}{2}}\alpha^{ms}\left(1 - \frac{1}{\alpha^{2m}}\right)^s \\ &= 2^{\frac{-5s}{2}}\sum_{k=0}^{\infty}\binom{s}{k}\alpha^{m(s-2k)}. \end{aligned} \tag{2.6}$$

The above series converges since $\alpha > 1$. By virtue of (2.2) and (2.6), we get

$$\begin{aligned} &\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{1}{B_{m_1}^{s_1}B_{m_1+m_2}^{s_2}B_{2m_1+m_2}^{s_3}} \\ = &\sum_{m_1,m_2=1}^{\infty} 2^{\frac{5s_1}{2}}\sum_{k=0}^{\infty}\binom{-s_1}{k}\alpha^{-m_1\left(s_1+2k\right)} \times 2^{\frac{5s_2}{2}}\sum_{l=0}^{\infty}\binom{-s_2}{l}\alpha^{-\left(m_1+m_2\right)\left(s_2+2l\right)} \\ &\times 2^{\frac{5s_3}{2}}\sum_{t=0}^{\infty}\binom{-s_3}{t}\alpha^{-\left(2m_1+m_2\right)\left(s_3+2t\right)}. \end{aligned} \tag{2.7}$$

We know that $\left|\binom{-s}{k}\right| \leq (-1)^k\binom{-|s|}{k}$. Thus we have

$$\begin{aligned} &\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left|\frac{1}{B_{m_1}^{s_1}B_{m_1+m_2}^{s_2}B_{2m_1+m_2}^{s_3}}\right| \\ = &\sum_{m_1,m_2=1}^{\infty}\left|2^{\frac{5s_1}{2}}\sum_{k=0}^{\infty}\binom{-s_1}{k}\alpha^{-m_1\left(s_1+2k\right)} \times 2^{\frac{5s_2}{2}}\right. \\ &\left.\times \sum_{l=0}^{\infty}\binom{-s_2}{l}\alpha^{-\left(m_1+m_2\right)\left(s_2+2l\right)}2^{\frac{5s_3}{2}}\sum_{t=0}^{\infty}\binom{-s_3}{t}\alpha^{-\left(2m_1+m_2\right)\left(s_3+2t\right)}\right| \\ \leq &\sum_{m_1,m_2=1}^{\infty} 2^{\frac{5(\sigma_1+\sigma_2+\sigma_3)}{2}}\alpha^{-m_1\sigma_1}\sum_{k=0}^{\infty}\binom{-|s_1|}{k}(-1)^k\alpha^{-2m_1k} \\ &\times \alpha^{-\left(m_1+m_2\right)\sigma_2}\sum_{l=0}^{\infty}\binom{-|s_2|}{l}(-1)^l\alpha^{-2\left(m_1+m_2\right)l} \\ &\times \alpha^{-\left(2m_1+m_2\right)\sigma_3}\sum_{t=0}^{\infty}\binom{-|s_3|}{t}(-1)^t\alpha^{-2\left(2m_1+m_2\right)t} \end{aligned}$$

$$\begin{aligned}
&= \sum_{m_1,m_2=1}^{\infty} 2^{\frac{5(\sigma_1+\sigma_2+\sigma_3)}{2}} \alpha^{-m_1\sigma_1}\Big(1-\alpha^{-2m_1}\Big)^{-|s_1|} \times \alpha^{-\big(m_1+m_2\big)\sigma_2} \\
&\quad \times \Big(1-\alpha^{-2\big(m_1+m_2\big)}\Big)^{-|s_2|} \alpha^{-\big(2m_1+m_2\big)\sigma_3}\Big(1-\alpha^{-2\big(2m_1+m_2\big)}\Big)^{-|s_3|} \\
&\le \sum_{m_1,m_2=1}^{\infty} 2^{\frac{5(\sigma_1+\sigma_2+\sigma_3)}{2}} \alpha^{-m_1\sigma_1}\Big(1-\alpha^{-2}\Big)^{-|s_1|} \times \alpha^{-\big(m_1+m_2\big)\sigma_2}\Big(1-\alpha^{-4}\Big)^{-|s_2|} \\
&\quad \times \alpha^{-\big(2m_1+m_2\big)\sigma_3}\Big(1-\alpha^{-6}\Big)^{-|s_3|} \\
&= 2^{\frac{5(\sigma_1+\sigma_2+\sigma_3)}{2}}\Big(1-\alpha^{-2}\Big)^{-|s_1|}\Big(1-\alpha^{-4}\Big)^{-|s_2|}\Big(1-\alpha^{-6}\Big)^{-|s_3|} \\
&\quad \times \sum_{m_1=1}^{\infty} \alpha^{-\big(\sigma_1+\sigma_2+2\sigma_3\big)m_1} \sum_{m_2=1}^{\infty} \alpha^{-\big(\sigma_2+\sigma_3\big)m_2} \\
&< \infty.
\end{aligned}$$

Thus, the above series absolutely converges for a fixed point (s_1, s_2, s_3) in D_3. Then by interchanging the order of summation in (2.7), we get

$$\begin{aligned}
\zeta_{AVB}(s_1,s_2,s_3) &= 2^{\frac{5(s_1+s_2+s_3)}{2}} \sum_{k=0}^{\infty}\binom{-s_1}{k}\sum_{l=0}^{\infty}\binom{-s_2}{l}\sum_{t=0}^{\infty}\binom{-s_3}{t} \\
&\quad \sum_{m_1=1}^{\infty}\Big(\alpha^{-(s_1+s_2+2s_3+2k+2l+4t)}\Big)^{m_1}\sum_{m_2=1}^{\infty}\Big(\alpha^{-(s_2+s_3+2l+2t)}\Big)^{m_2} \\
&= 2^{\frac{5(s_1+s_2+s_3)}{2}} \sum_{k=0}^{\infty}\binom{-s_1}{k}\sum_{l=0}^{\infty}\binom{-s_2}{l}\sum_{t=0}^{\infty}\binom{-s_3}{t} \\
&\quad \frac{\alpha^{-(s_1+s_2+2s_3+2k+2l+4t)}}{\Big(1-\alpha^{-(s_1+s_2+2s_3+2k+2l+4t)}\Big)} \times \frac{\alpha^{-(s_2+s_3+2l+2t)}}{\Big(1-\alpha^{-(s_2+s_3+2l+2t)}\Big)} \\
&= 2^{\frac{5(s_1+s_2+s_3)}{2}} \sum_{k=0}^{\infty}\binom{-s_1}{k}\sum_{l=0}^{\infty}\binom{-s_2}{l}\sum_{t=0}^{\infty}\binom{-s_3}{t} \\
&\quad \frac{1}{\Big(\alpha^{s_1+s_2+2s_3+2k+2l+4t}-1\Big)} \times \frac{1}{\Big(\alpha^{s_2+s_3+2l+2t}-1\Big)}. \qquad (2.8)
\end{aligned}$$

For any $s_1, s_2, s_3 \in \mathbb{C}$, $\left|\alpha^{s_1+s_2+2s_3+2k+2l+4t}-1\right| > \alpha^{\sigma_1+\sigma_2+2\sigma_3+k+l+2t}$ for $k \ge k_0,\ l \ge l_0,\ t \ge t_0$ and $\left|\alpha^{s_2+s_3+2l+2t}-1\right| > \alpha^{\sigma_2+\sigma_3+l+t}$ for $l \ge l_1,\ t \ge t_1$, where $k_0 = k_0(\sigma_1,\sigma_2,\sigma_3,\alpha) \gg 0$, $l_0 = l_0(\sigma_1,\sigma_2,\sigma_3,\alpha) \gg 0$, $t_0 = t_0(\sigma_1,\sigma_2,\sigma_3,\alpha) \gg 0$, $l_1 = l_1(\sigma_2,\sigma_3,\alpha) \gg 0$, and $t_1 = t_1(\sigma_2,\sigma_3,\alpha) \gg 0$. Assume that $l^* = \max\{l_0, l_1\}$ and $t^* = \max\{t_0, t_1\}$.

Now,

$$\begin{aligned}
&\sum_{k>k_0}^{\infty}\sum_{l>l^*}^{\infty}\sum_{t>t^*}^{\infty}\left|\binom{-s_1}{k}\binom{-s_2}{l}\binom{-s_3}{t}\frac{1}{\left(\alpha^{s_1+s_2+2s_3+2k+2l+4t}-1\right)}\right.\\
&\left.\times\frac{1}{\left(\alpha^{s_2+s_3+2l+2t}-1\right)}\right|\\
\le\ &\sum_{\substack{k>k_0,\ l>l^*,\\ t>t^*}}^{\infty}\binom{-|s_1|}{k}(-1)^k\binom{-|s_2|}{l}(-1)^l\binom{-|s_3|}{t}(-1)^t\frac{1}{\alpha^{\sigma_1+\sigma_2+2\sigma_3+k+l+2t}}\\
&\times\frac{1}{\alpha^{\sigma_2+\sigma_3+l+t}}\\
\le\ &\alpha^{-(\sigma_1+2\sigma_2+3\sigma_3)}\sum_{\substack{k>k_0,\ l>l^*,\\ t>t^*}}^{\infty}\binom{-|s_1|}{k}(-1)^k\alpha^{-k}\binom{-|s_2|}{l}(-1)^l\alpha^{-2l}\\
&\times\binom{-|s_3|}{t}(-1)^t\alpha^{-3t}\\
\le\ &\alpha^{-(\sigma_1+2\sigma_2+3\sigma_3)}\left(1-\alpha^{-1}\right)^{-|s_1|}\left(1-\alpha^{-2}\right)^{-|s_2|}\left(1-\alpha^{-3}\right)^{-|s_3|}.
\end{aligned}$$

The above bound is uniform where (s_1, s_2, s_3) varies over compact subsets of $\mathbb{C}^3$. Thus the series (2.8) converges uniformly and absolutely on the compact subsets of $\mathbb{C}^3$ not containing any poles. Indeed, the series (2.8) determines the holomorphic function on $\mathbb{C}^3$ except for the poles derived from the functions $\alpha^{s_1+s_2+2s_3+2k+2l+4t} - 1 = 0$ and $\alpha^{s_2+s_3+2l+2t} - 1 = 0$. Therefore, $\zeta_{AVB}(s_1, s_2, s_3)$ can be analytically continued to a meromorphic function on $\mathbb{C}^3$ and its simple poles are at

$$s_1+s_2+2s_3 = -2(k+l+2t)+\frac{2\pi i a}{\log\alpha} \text{ with } k,l,t\in\mathbb{Z}_{\ge 0}, a\in\mathbb{Z}, \tag{2.9}$$

and

$$s_2+s_3 = -2(l+t)+\frac{2\pi i b}{\log\alpha}\text{with } l,t\in\mathbb{Z}_{\ge 0}, b\in\mathbb{Z}. \tag{2.10}$$

This completes the proof. □

3. Residues of Apostol-Vu Multiple Balancing Zeta Functions at Poles

In this section, we find the residues of $\zeta_{AVB}(s_1, s_2, s_3)$ at the simple poles derived from Theorem 1. We define the residue of the Apostol-Vu multiple balancing zeta functions $\zeta_{AVB}(s_1, s_2, s_3)$ along the hyperplane (2.9) to be the restriction of the meromorphic function

$$\left(s_1 + s_2 + 2s_3 + 2(k + l + 2t) - \frac{2\pi i a}{\log \alpha}\right)\zeta_{AVB}(s_1, s_2, s_3)$$

to the hyperplane (2.9). Similarly, define the residue of $\zeta_{AVB}(s_1, s_2, s_3)$ along the hyperplane (2.10) to be the restriction of the meromorphic function

$$\left(s_2 + s_3 + 2(l + t) - \frac{2\pi i b}{\log \alpha}\right)\zeta_{AVB}(s_1, s_2, s_3)$$

to the hyperplane (2.10).

Theorem 2. *The residue of* $\zeta_{AVB}(s_1, s_2, s_3)$ *at* $s_1 + s_2 + 2s_3 = s_{k',l',t'} = -2(k' + l' + 2t') + \frac{2\pi i a}{\log \alpha}$ *with* $k', l', t' \in \mathbb{Z}_{\geq 0}, a \in \mathbb{Z}$ *is*

$$\frac{2^{\frac{5s_{k',l',t'}}{2}}}{\log \alpha} \sum_{\substack{t,l,k \geq 0 \\ k+l+2t = k'+l'+2t'}} 2^{-\frac{5s_3}{2}} \binom{-s_3}{t}\binom{-s_2}{l}\binom{-s_1}{k} \frac{1}{\left(\alpha^{s_2+s_3+2l+2t} - 1\right)}.$$

Proof. Let $k', l', t' \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}$. The function $\alpha^{s_1+s_2+2s_3+2k'+2l'+4t'} - 1$ is an analytic function which has simple zeros at

$$s_1 + s_2 + 2s_3 = s_{k',l',t'} = -2(k' + l' + 2t') + \frac{2\pi i a}{\log \alpha}.$$

Now, $\lim_{\substack{s_1+s_2+2s_3 \\ \to s_{k',l',t'}}} \frac{s_1 + s_2 + 2s_3 - s_{k',l',t'}}{\alpha^{s_1+s_2+2s_3+2k'+2l'+4t'} - 1} = \frac{1}{\log \alpha}$. The residue of $\zeta_{AVB}(s_1, s_2, s_3)$ along $s_1 + s_2 + 2s_3 = s_{k',l',t'}$ is

$$\begin{aligned}
& \mathrm{Res}_{\substack{s_1+s_2+2s_3 \\ \to s_{k',l',t'}}} \zeta_{AVB}(s_1, s_2, s_3) \\
= \; & \lim_{\substack{s_1+s_2+2s_3 \\ \to s_{k',l',t'}}} \left(s_1 + s_2 + 2s_3 - s_{k',l',t'}\right) 2^{\frac{5(s_1+s_2+s_3)}{2}} \sum_{t=0}^{\infty} \binom{-s_3}{t} \\
& \sum_{l=0}^{\infty} \binom{-s_2}{l} \sum_{k=0}^{\infty} \binom{-s_1}{k} \frac{1}{\left(\alpha^{s_1+s_2+2s_3+2k+2l+4t} - 1\right)} \\
& \times \frac{1}{\left(\alpha^{s_2+s_3+2l+2t} - 1\right)}
\end{aligned}$$

$$= 2^{-\frac{5s_3}{2}} \sum_{t=0}^{\infty} \binom{-s_3}{t} \sum_{l=0}^{\infty} \binom{-s_2}{l} \sum_{k=0}^{\infty} \binom{-s_1}{k} \frac{1}{\left(\alpha^{s_2+s_3+2l+2t}-1\right)}\Big|_{s_1+s_2+2s_3=s_{k',l',t'}}$$

$$\times \lim_{\substack{s_1+s_2+2s_3\\ \to s_{k',l',t'}}} 2^{\frac{5(s_1+s_2+2s_3)}{2}} \frac{s_1+s_2+2s_3-s_{k',l',t'}}{\alpha^{s_1+s_2+2s_3+2k+2l+4t}-1}$$

$$= \frac{2^{\frac{5s_{k',l',t'}}{2}}}{\log\alpha} \sum_{\substack{t,l,k\geq 0\\ k+l+2t=k'+l'+2t'}} 2^{-\frac{5s_3}{2}} \binom{-s_3}{t}\binom{-s_2}{l}\binom{-s_1}{k} \frac{1}{\left(\alpha^{s_2+s_3+2l+2t}-1\right)}.$$

This finishes the proof. ☐

Theorem 3. *Let* $l', t' \in \mathbb{Z}_{\geq 0}$ *and* $b \in \mathbb{Z}$*. Then the residue of* $\zeta_{AVB}(s_1, s_2, s_3)$ *at* $s_2 + s_3 = s_{l',t'} = -2(l'+t') + \dfrac{2\pi i b}{\log\alpha}$ *is*

$$\frac{2^{\frac{5s_{l',t'}}{2}}}{\log\alpha} \sum_{\substack{t,l,k\geq 0\\ l+t=l'+t'}} 2^{\frac{5s_1}{2}} \binom{-s_3}{t}\binom{-s_2}{l}\binom{-s_1}{k} \frac{1}{\left(\alpha^{s_1+s_2+2s_3+2k+2l+4t}-1\right)}.$$

Proof. The function $\alpha^{s_2+s_3+2l'+2t'}-1$ is an analytic function which has simple zeros at

$$s_2+s_3 = s_{l',t'} = -2(l'+t') + \frac{2\pi i b}{\log\alpha} \text{ with } l', t' \in \mathbb{Z}_{\geq 0} \text{ and } b \in \mathbb{Z}.$$

Now,

$$\lim_{s_2+s_3\to s_{l',t'}} \frac{s_2+s_3-s_{l',t'}}{\alpha^{s_2+s_3++2l'+2t'}-1} = \operatorname{Res}_{\substack{s_2+s_3\\ =s_{l',t'}}} \frac{1}{\left(\alpha^{s_2+s_3+2l'+2t'}-1\right)} = \frac{1}{\log\alpha}.$$

The residue of $\zeta_{AVB}(s_1, s_2, s_3)$ along $s_2 + s_3 = s_{l',t'}$ is

$$\operatorname{Res}_{\substack{s_2+s_3\\ \to s_{l',t'}}} \zeta_{AVB}(s_1,s_2,s_3)$$

$$= \lim_{\substack{s_2+s_3\\ \to s_{l',t'}}} \left(s_2+s_3-s_{l',t'}\right) 2^{\frac{5(s_1+s_2+s_3)}{2}} \sum_{t=0}^{\infty}\binom{-s_3}{t}\sum_{l=0}^{\infty}\binom{-s_2}{l}\sum_{k=0}^{\infty}\binom{-s_1}{k}$$

$$\frac{1}{\left(\alpha^{s_1+s_2+2s_3+2k+2l+4t}-1\right)} \frac{1}{\left(\alpha^{s_2+s_3+2l+2t}-1\right)}$$

$$= 2^{\frac{5s_1}{2}} \sum_{t=0}^{\infty}\binom{-s_3}{t}\sum_{l=0}^{\infty}\binom{-s_2}{l}\sum_{k=0}^{\infty}\binom{-s_1}{k} \frac{1}{\left(\alpha^{s_1+s_2+2s_3+2k+2l+4t}-1\right)}\Big|_{s_2+s_3=s_{l',t'}}$$

$$\times \lim_{\substack{s_2+s_3\\ \to s_{l',t'}}} 2^{\frac{5(s_2+s_3)}{2}} \frac{s_2+s_3-s_{l',t'}}{\alpha^{s_2+s_3+2l+2t}-1}$$

$$= \frac{2^{\frac{5s_{l',t'}}{2}}}{\log\alpha} \sum_{\substack{t,l,k\geq 0 \\ l+t=l'+t'}} 2^{\frac{5s_1}{2}} \binom{-s_3}{t}\binom{-s_2}{l}\binom{-s_1}{k} \frac{1}{\left(\alpha^{s_1+s_2+2s_3+2k+2l+4t}-1\right)}.$$

This ends the proof. □

4. Values of Apostol-Vu Multiple Balancing Zeta Functions at Negative Integers

In this section we consider the values of $\zeta_{AVB}(s_1,s_2,s_3)$ at negative integers. First we give a sufficient condition for $\zeta_{AVB}(s_1,s_2,s_3)$ to be holomorphic at $(s_1,s_2,s_3) = (-r_1,-r_2,-r_3)$ where $r_i \in \mathbb{N}$ for $i = 1,2,3$.

Lemma 1. *Let $(r_1,r_2,r_3) \in \mathbb{N}^3$. Then $\zeta_{AVB}(s_1,s_2,s_3)$ is holomorphic at $(s_1,s_2,s_3) = (-r_1,-r_2,-r_3)$ if and only if $r_1+r_2+2r_3 \not\equiv 0 \pmod 2$, $r_2+r_3 \not\equiv 0 \pmod 2$.*

Proof. The infinite series (2.8) is holomorphic except the poles derived from

$$\left(\alpha^{s_1+s_2+2s_3+2k+2l+4t}-1\right) \times \left(\alpha^{s_2+s_3+2l+2t}-1\right) = 0$$

This is true if and only if, one of the following equations holds:

$$s_1+s_2+2s_3 = -2(k+l+2t), s_2+s_3 = -2(l+t),$$

for $(k,l,t) \in \mathbb{N}^3$, and the result follows. □

Theorem 4. *For positive integers $r_i, i = 1,2,3$, $\zeta_{AVB}(-r_1,-r_2,-r_3) \in \mathbb{Q}$ if even number of r_i in (r_1,r_2,r_3) are odd and $\zeta_{AVB}(-r_1,-r_2,-r_3) \in \sqrt{2}\mathbb{Q}$ if odd number of r_i in (r_1,r_2,r_3) are odd except for singularities.*

Proof. For positive integers r_1,r_2,r_3, from (2.8) we have

$$\begin{aligned}\zeta_{AVB}(-r_1,-r_2,-r_3) &= 2^{\frac{-5(r_1+r_2+r_3)}{2}} \sum_{k=0}^{\infty}\binom{r_1}{k}\sum_{l=0}^{\infty}\binom{r_2}{l}\sum_{t=0}^{\infty}\binom{r_3}{t} \\ &\quad \times \frac{1}{(\alpha^{-r_1-r_2-2r_3+2k+2l+4t}-1)} \times \frac{1}{(\alpha^{-r_2-r_3+2l+2t}-1)}.\end{aligned}$$

For $k > r_1$, $l > r_2$ and $t > r_3$, $\binom{r_1}{k}$, $\binom{r_2}{l}$ and $\binom{r_3}{t}$ vanish respectively. Thus this is a finite sum belonging to $\mathbb{Q}(\sqrt{2})$. Therefore,

$$\begin{aligned}\zeta_{AVB}(-r_1,-r_2,-r_3) &= 2^{\frac{-5(r_1+r_2+r_3)}{2}} \sum_{k=0}^{r_1}\binom{r_1}{k}\sum_{l=0}^{r_2}\binom{r_2}{l}\sum_{t=0}^{r_3}\binom{r_3}{t} \\ &\quad \times \frac{1}{(\alpha^{-r_1-r_2-2r_3+2k+2l+4t}-1)} \times \frac{1}{(\alpha^{-r_2-r_3+2l+2t}-1)}. \end{aligned} \tag{4.1}$$

Let

$$\sigma_{l,t} = \binom{r_2}{l}\binom{r_3}{t}\frac{1}{(\alpha^{-r_2-r_3+2l+2t}-1)} \text{ and } \theta_{k,l,t} = \binom{r_1}{k}\frac{1}{(\alpha^{-r_1-r_2-2r_3+2k+2l+4t}-1)}.$$

Using these identities, from (4.1), we have

$$\begin{aligned}
&\zeta_{AVB}(-r_1,-r_2,-r_3)\\
= \;& 5^{\frac{-(r_1+r_2+r_3)}{2}} \sum_{t=0}^{r_3}\sum_{l=0}^{r_2}\sum_{k=0}^{r_1} \sigma_{l,t}\theta_{k,l,t}\\
= \;& \frac{2^{\frac{-5(r_1+r_2+r_3)}{2}}}{2} \sum_{t=0}^{r_3}\Bigg[\sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{l,t}\theta_{k,l,t} + \sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{l,r_3-t}\theta_{k,l,r_3-t}\Bigg]\\
= \;& \frac{2^{\frac{-5(r_1+r_2+r_3)}{2}}}{4} \sum_{t=0}^{r_3}\Bigg[\sum_{l=0}^{r_2}\Bigg(\sum_{k=0}^{r_1}\sigma_{l,t}\theta_{k,l,t} + \sum_{k=0}^{r_1}\sigma_{r_2-l,t}\theta_{k,r_2-l,t}\\
&+\sum_{k=0}^{r_1}\sigma_{l,r_3-t}\theta_{k,l,r_3-t} + \sum_{k=0}^{r_1}\sigma_{r_2-l,r_3-t}\theta_{k,r_2-l,r_3-t}\Bigg)\Bigg]\\
= \;& \frac{2^{\frac{-5(r_1+r_2+r_3)}{2}}}{8} \sum_{t=0}^{r_3}\Bigg[\sum_{l=0}^{r_2}\Bigg(\Big(\sum_{k=0}^{r_1}\sigma_{l,t}\theta_{k,l,t} + \sum_{k=0}^{r_1}\sigma_{l,t}\theta_{r_1-k,l,t}\Big) + \Big(\sum_{k=0}^{r_1}\sigma_{r_2-l,t}\theta_{k,r_2-l,t}\\
&+\sum_{k=0}^{r_1}\sigma_{r_2-l,t}\theta_{r_1-k,r_2-l,t}\Big) + \Big(\sum_{k=0}^{r_1}\sigma_{l,r_3-t}\theta_{k,l,r_3-t} + \sum_{k=0}^{r_1}\sigma_{l,r_3-t}\theta_{r_1-k,l,r_3-t}\Big)\\
&+\Big(\sum_{k=0}^{r_1}\sigma_{r_2-l,r_3-t}\theta_{k,r_2-l,r_3-t} + \sum_{k=0}^{r_1}\sigma_{r_2-l,r_3-t}\theta_{r_1-k,r_2-l,r_3-t}\Big)\Bigg)\Bigg]. \qquad (4.2)
\end{aligned}$$

Let us denote:

$P = \sum_{t=0}^{r_3}\sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{l,t}\theta_{k,l,t}$, $Q = \sum_{t=0}^{r_3}\sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{l,t}\theta_{r_1-k,l,t}$, $R = \sum_{t=0}^{r_3}\sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{r_2-l,t}\theta_{k,r_2-l,t}$,
$S = \sum_{t=0}^{r_3}\sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{r_2-l,t}\theta_{r_1-k,r_2-l,t}$, $T = \sum_{t=0}^{r_3}\sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{l,r_3-t}\theta_{k,l,r_3-t}$,
$U = \sum_{t=0}^{r_3}\sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{l,r_3-t}\theta_{r_1-k,l,r_3-t}$, $V = \sum_{t=0}^{r_3}\sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{r_2-l,r_3-t}\theta_{k,r_2-l,r_3-t}$
and $W = \sum_{t=0}^{r_3}\sum_{l=0}^{r_2}\sum_{k=0}^{r_1}\sigma_{r_2-l,r_3-t}\theta_{r_1-k,r_2-l,r_3-t}$. Let $X = P+Q+R+S+T+U+V+W$

and $\Phi \neq \mathrm{Id}$ be an automorphism of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ and hence $\Phi(\alpha) = \beta$. Now,

$$\begin{aligned}
&\sigma_{r_2-l,r_3-t}\theta_{r_1-k,r_2-l,r_3-t}\\
&= \binom{r_2}{r_2-l}\binom{r_3}{r_3-t}\frac{1}{\left(\alpha^{-r_2-r_3+2(r_2-l)+2(r_3-t)}-1\right)}\\
&\times\binom{r_1}{r_1-k}\frac{1}{\left(\alpha^{-r_1-r_2-2r_3+2(r_1-k)+2(r_2-l)+4(r_3-t)}-1\right)}\\
= \;& \binom{r_3}{t}\binom{r_2}{l}\frac{1}{\left(\alpha^{r_2+r_3-2(l+t)}-1\right)}\binom{r_1}{k}\frac{1}{\left(\alpha^{r_1+r_2+2r_3-2(k+l+2t)}-1\right)}\\
= \;& \binom{r_3}{t}\binom{r_2}{l}\frac{1}{\left(\beta^{-r_2-r_3+2l+2t}-1\right)}\binom{r_1}{k}\frac{1}{\left(\beta^{-r_1-r_2-2r_3+2k+2l+4t}-1\right)}. \quad (4.3)
\end{aligned}$$

Similarly, one can deduce that

$$\begin{aligned}
&\sigma_{r_2-l,r_3-t}\theta_{k,r_2-l,r_3-t}\\
&= \binom{r_3}{t}\binom{r_2}{l}\frac{1}{\left(\beta^{-r_2-r_3+2(l+t)}-1\right)}\binom{r_1}{k}\frac{1}{\left(\beta^{r_1-r_2-2r_3-2(k-l-2t)}-1\right)}, \quad (4.4)
\end{aligned}$$

$$\begin{aligned}
&\sigma_{l,r_3-t}\theta_{r_1-k,l,r_3-t}\\
&= \binom{r_3}{t}\binom{r_2}{l}\frac{1}{\left(\beta^{r_2-r_3-2(l-t)}-1\right)}\binom{r_1}{k}\frac{1}{\left(\beta^{-r_1+r_2-2r_3+2k-2l+4t}-1\right)} \quad (4.5)
\end{aligned}$$

and

$$\begin{aligned}
&\sigma_{l,r_3-t}\theta_{k,l,r_3-t}\\
&= \binom{r_3}{t}\binom{r_2}{l}\frac{1}{\left(\beta^{r_2-r_3-2(l-t)}-1\right)}\binom{r_1}{k}\frac{1}{\left(\beta^{r_1+r_2-2r_3-2(k+l-2t)}-1\right)}. \quad (4.6)
\end{aligned}$$

Let $\mathcal{N}_o$ and $\mathcal{N}_e$ denote the set of odd and even positive integers respectively. We consider two cases.

`Case 1.` $r_1, r_2, r_3 \in \mathcal{N}_e$ (or) $r_1, r_2 \in \mathcal{N}_o$ *and* $r_3 \in \mathcal{N}_e$ (or) $r_1, r_3 \in \mathcal{N}_o$ *and* $r_2 \in \mathcal{N}_e$ (or) $r_2, r_3 \in \mathcal{N}_o$ *and* $r_1 \in \mathcal{N}_e$. In this case, $(r_1 + r_2 + r_3)/2$ is an integer which implies $2^{\frac{5(r_1+r_2+r_3)}{2}} \in \mathbb{Q}$. Then from (4.3), (4.4), (4.5), (4.6), we obtain that $\Phi(P) = W$, $\Phi(Q) = V$, $\Phi(R) = U$ and $\Phi(S) = T$, and hence from (4.2), $X \in \mathbb{Q}$. Therefore, $\zeta_{AVB}(-r_1, -r_2, -r_3) \in \mathbb{Q}$.

`Case 2.` $r_1, r_2, r_3 \in \mathcal{N}_o$ (or) $r_1, r_2 \in \mathcal{N}_e$ *and* $r_3 \in \mathcal{N}_o$ (or) $r_1, r_3 \in \mathcal{N}_e$ *and* $r_2 \in \mathcal{N}_o$ (or) $r_2, r_3 \in \mathcal{N}_e$ *and* $r_1 \in \mathcal{N}_o$. In this case, $(r_1 + r_2 + r_3)/2$ is not an integer, and hence $2^{\frac{5(r_1+r_2+r_3)}{2}} \in \sqrt{2}\mathbb{Q}$. Then from (4.3), (4.4), (4.5) and (4.6), we also obtain that $\Phi(P) = W$, $\Phi(Q) = V$, $\Phi(R) = U$ and $\Phi(S) = T$. Thus from (4.2), we have $X \in \mathbb{Q}$ which implies $\zeta_{AVB}(-r_1, -r_2, -r_3) \in \sqrt{2}\mathbb{Q}$. This completes the proof. □

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