



AN EXTENSION OF THE NORMAL PLAY CONVENTION TO N -PLAYER COMBINATORIAL GAMES

Mark Spindler

The Johns Hopkins Center for Talented Youth, Baltimore, Maryland

mark.edward.spindler@gmail.com

Received: 3/3/19, Revised: 2/4/20, Accepted: 3/17/20, Published: 4/27/20

Abstract

We examine short combinatorial games for three or more players under a new play convention in which a player who cannot move on their turn is the unique loser. We show that many theorems of impartial and partizan two-player games under normal play have natural analogues in this setting. For impartial games with three players, we investigate the possible outcomes of a sum in detail, and determine the outcomes and structure of three-player NIM.

1. Introduction

The vast majority of prior work in combinatorial game theory has focused on two-player games. In this paper, we investigate a novel play convention for N -player games. When no move is available, the next player is declared the unique loser of the game, and the other players all win equally. This convention was independently conceived by Salt in [7]. In one sense, this complicates matters, since more than one player may have a winning strategy. However, as compared to previous work on multiplayer games, this convention leads to results that better parallel those for two-player games under the normal play convention. Our work and objects of study fit into the general framework for combinatorial game theory laid out in [9].

A summary of much of the prior work done on combinatorial games with three or more players can be found in I.4 of [8]. This paper is closely connected with the related work of Propp, Cincotti, and Doles.

In [6], Propp considers three-player impartial games with a certain play convention: in analogy with two-player normal play, when no move is available, the previous player is declared the unique winner of the game. This yields a recursive characterization of outcome classes \mathcal{N} , \mathcal{O} , and \mathcal{P} in which the test for \mathcal{O} contains a proviso reminiscent of the *misère* play convention for two-player games.

In [1], Cincotti considers three-player partizan games under a convention which reduces to Propp's in the impartial case. When a player cannot move on their turn,

they are eliminated. The other players continue to play from that position. The last player remaining is the unique winner. Cincotti extends this convention to N -player games in [2].

In [3], Doles considers three-player partizan games with a convention similar to Propp's. The last player to move wins first place. The player before that wins second place. And the player who cannot move on their turn loses and is ranked in third place. Under this convention, an assumption that players always play rationally leads to some player having a winning strategy.

In this paper, we consider the play convention in which the first player without an available move is the unique loser of the game. In Section 2, we examine three-player impartial games and give special attention to three-player NIM, closely paralleling the work in [6]. We examine how the outcome of a disjunctive sum can vary depending on the outcomes of the summands, discuss reverting games, construct some undetermined games which make all sums undetermined, and derive the finite quotient describing play in three-player NIM. In Section 3, we generalize much of the content of the previous section to N players. We do not completely characterize the outcomes of a sum or of NIM, but some partial results are obtained, including the outcomes of all NIM positions with two heaps. In Section 4, we consider partizan games with N players. We define a preorder based on favorability to a particular player in disjunctive sums, characterize all comparisons with the empty game 0 (and show that 0 is not equal to any other partizan game), show that the simplification theorems regarding dominated and reversible options have natural analogues for N players, and examine some particular games related to the two-player integer games. We conclude in Section 5 with a collection of open questions.

2. Three-Player Impartial Games

2.1. Preliminaries

In this subsection, we introduce the games under discussion, the notation and terminology we use for three-player impartial games, and some initial observations.

2.1.1. Notation

Unless otherwise specified, all games in this paper are assumed to be short games; they have finitely many distinct subpositions, and admit no infinite runs.

As is common, an impartial game is usually considered to be the set of its options, $+$ denotes the disjunctive sum of games, $n \cdot G$ denotes the sum of n copies of G , G' and G^* typically denote an option of a game G , etc. We use \cong to indicate that two games are isomorphic (i.e. the game trees are isomorphic). Additionally, we also

make use of structural induction in proofs throughout, often without comment.

Many impartial games in this paper are built out of nim-heaps. A nim-heap of size n is denoted by $*n$. Equivalently, $*n \cong \{ *0, *1, \dots, *(n-1) \}$. As special cases, 0 denotes the game with no options $\{ \}$, and $*$ denotes $*1$ (so that $*$ \cong $\{0\}$).

We define the *normal* play convention for three-player games to be the one in which the next player to move is the unique loser of the empty game 0 . The other two players both win equally.

Definition 2.1. The (normal-play) *outcome* $o(G)$ of a game G is a subset of $\{\mathbf{N}, \mathbf{O}, \mathbf{P}\}$ whose members are determined recursively as follows.

- $\mathbf{N} \in o(G)$ exactly when there exists an option G^* with $\mathbf{P} \in o(G^*)$.
- $\mathbf{O} \in o(G)$ exactly when every option G' satisfies $\mathbf{N} \in o(G')$.
- $\mathbf{P} \in o(G)$ exactly when every option G' satisfies $\mathbf{O} \in o(G')$.

By an induction argument, the elements of $o(G)$ tell us who has a winning strategy in G . \mathbf{N} represents the next player to move, \mathbf{P} , the previous player, and \mathbf{O} , the other player. When discussing a single position, we often use “Next”, “Other”, and “Previous” to refer to the players.

For brevity, we use $\emptyset, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{PN}, \mathcal{NO}, \mathcal{OP}$, and \mathcal{NOP} to denote the corresponding subsets of $\{\mathbf{N}, \mathbf{O}, \mathbf{P}\}$. For example, $\mathcal{NO} = \{\mathbf{N}, \mathbf{O}\}$.

Unlike in the two-player case, the outcome of a game need not correspond to the set of winners when all players play perfectly. (This is also noted in [3] and [6].)

Proposition 2.2. *In $G \cong \{0, *2\}$, Next can only guarantee that they win by winning with Previous. However, Next and Other can collude to ensure that Previous loses.*

Proof. If Next plays perfectly, then they must move to 0 . Otherwise, Other could choose to move to $*$ on their turn. However, if Next moves to $*2$, then Other can choose to move to 0 to force (the original) Previous to lose. Thus, $o(G) = \mathcal{N}$ despite the fact that both Next and Previous would win if Next were to play perfectly. \square

2.1.2. Observations

Proposition 2.3. *$o(G)$ is always a proper subset of \mathcal{NOP} .*

Proof. Let G be a game with $o(G) = \mathcal{NOP}$. Then, by the recursive Definition 2.1, we can choose G^* with $o(G^*) = \mathcal{NOP}$. Since games do not admit any infinite runs, no such G can exist. \square

Proposition 2.4. *$\mathbf{N} \in o(G)$ exactly when $\mathbf{O} \in o(\{G\})$, $\mathbf{O} \in o(G)$ exactly when $\mathbf{P} \in o(\{G\})$, and $\mathbf{P} \in o(G)$ exactly when $\mathbf{N} \in o(\{G\})$.*

Proof. Immediate from the recursive definitions of Definition 2.1. \square

Proposition 2.5. *All seven proper subsets of \mathcal{NCP} are possible outcomes.*

Proof. These examples are readily verified: $o(0) = \mathcal{OP}$, $o(*) = \mathcal{PN}$, $o(\{*\}) = \mathcal{NO}$, $o(*2) = \mathcal{N}$, $o(\{*2\}) = \mathcal{O}$, $o(\{\{*2\}\}) = \mathcal{P}$, and $o(\{*2, \{*2\}\}) = \emptyset$. \square

Definition 2.6. A game G is said to be *undetermined* if $o(G) = \emptyset$.

Proposition 2.7. *If G has an undetermined option G^* and has no option G' with $\mathbf{P} \in o(G')$, then G is undetermined.*

Proof. The existence of G^* means that $\mathbf{O}, \mathbf{P} \notin o(G)$, and the other condition forces $\mathbf{N} \notin o(G)$. \square

2.2. General Results

In this subsection, we investigate the outcomes of sums of games, and derive an inequation and some equations for games.

2.2.1. Outcomes of Sums

In the two-player theory of impartial games under normal play, if we put aside the more-precise Sprague–Grundy Theory, there are two main facts worth noting about the outcomes of sums as compared to the outcomes of the summands: (1) the outcome of $G + G$, and (2) the possible outcomes of $G + H$ —if $o(H) = \mathcal{P}$ then $o(G + H) = o(G)$, but if $o(G) = o(H) = \mathcal{N}$ then $o(G + H)$ can be either \mathcal{P} or \mathcal{N} . We examine three-player analogues of these results.

Theorem 2.8 (Next Generation). *If $\mathbf{N} \notin o(H)$, then $o(G + H) \subseteq o(G)$.*

In other words, adding on a component to a game cannot introduce a winning strategy for a new player unless Next has a winning strategy in the new component. Since $o(0) = \mathcal{OP}$, this is similar to the two-player result that adding a \mathcal{P} -position to a game does not change its outcome.

Proof. Let G, H be games, and suppose $\mathbf{N} \notin o(H)$.

If $\mathbf{P} \notin o(G)$, then we can choose an option G^* with $\mathbf{O} \notin o(G^*)$. But then $\mathbf{O} \notin o(G^* + H)$ by inductive hypothesis, so that $\mathbf{P} \notin o(G + H)$.

If $\mathbf{O} \notin o(G)$, then we can choose an option G^* with $\mathbf{N} \notin o(G^*)$. But then $\mathbf{N} \notin o(G^* + H)$ by induction, so that $\mathbf{O} \notin o(G + H)$.

If $\mathbf{N} \notin o(G)$, then $\mathbf{P} \notin o(G')$ for all options G' . Similarly, $\mathbf{P} \notin o(H')$ for all options H' . Therefore, by induction, we have both $\mathbf{P} \notin o(G' + H)$ and $\mathbf{P} \notin o(G + H')$ for arbitrary options. As such, $\mathbf{N} \notin o(G + H)$ by definition. \square

Corollary 2.9. *If G and H are undetermined, then $G + H$ is undetermined.*

Proof. By Theorem 2.8 (Next Generation), $\mathbf{N} \notin o(H)$ yields $o(G + H) \subseteq o(G) = \emptyset$. □

Theorem 2.8 (Next Generation) has an important consequence for sums of two games.

Theorem 2.10 (Other Procreation). *If $\mathbf{O} \notin o(G), o(H)$, then $\mathbf{P} \notin o(G + H)$.*

In other words, a sum of two games cannot produce a winning strategy for Previous unless Other has a winning strategy in one of the summands.

Proof. Since $\mathbf{O} \notin o(G), o(H)$, we can choose options G^* and H^* such that $\mathbf{N} \notin o(G^*), o(H^*)$. By Theorem 2.8 (Next Generation), $\mathbf{N} \notin o(G^* + H^*)$, so that $\mathbf{O} \notin o(G + H^*)$ and $\mathbf{P} \notin o(G + H)$. □

With Theorem 2.10 (Other Procreation) in hand, we can strengthen Corollary 2.9.

Corollary 2.11. *If $o(G), o(H) \subseteq \mathcal{P}$, then $G + H$ is undetermined.*

Proof. By Theorem 2.8 (Next Generation), $o(G + H) \subseteq o(G) \subseteq \mathcal{P}$. But by Theorem 2.10 (Other Procreation), $\mathbf{P} \notin o(G + H)$. Therefore, $o(G + H) = \emptyset$. □

Corollary 2.12. *If G is undetermined, then $\mathbf{P} \notin o(G + H)$ for any H .*

Proof. Suppose (G, H) is a counterexample. By Theorem 2.8 (Next Generation), $\mathbf{N} \in o(H)$ as otherwise $o(G + H) \subseteq o(G) = \emptyset$. Also, since $\mathbf{N} \notin o(G)$, we have $o(G + H) \subseteq o(H)$ so that $\mathbf{P} \in o(H)$ as well. Since $o(H) \neq \mathcal{NOP}$, we have $\mathbf{O} \notin o(H)$, and the contradiction follows immediately from Theorem 2.10 (Other Procreation). □

With two players, $G + H$ can have any normal-play outcome not disallowed by the two-player version of Theorem 2.8 (Next Generation) (e.g. if $o(G) = o(H) = \mathcal{N}$ then $o(G + H)$ can be either \mathcal{P} or \mathcal{N}). This result has a three-player analogue.

Theorem 2.13. *All outcomes of a sum of two games not immediately disallowed by Theorem 2.8 (Next Generation) or Theorem 2.10 (Other Procreation) are possible.*

Proof. For a list of representative examples, see Table 9 in the Appendix. For instance, $o(*2) = \mathcal{N}$, $o(*) = \mathcal{PN}$, and $o(*2 + *) = \mathcal{NO}$. □

The possible outcomes of sums are summarized in Table 1, where entries indicate that any *subset* (other than \mathcal{NOP}) is possible. For example, if $o(G) = \mathcal{O}$ and $o(H) = \mathcal{PN}$, then $o(G + H)$ may be any of $\emptyset, \mathcal{N}, \mathcal{P}, \mathcal{PN}$.

As in [6], we also examine sums of a game with itself. For example, if $o(G) = \emptyset$ or $o(G) = \mathcal{P}$, then $o(G + G) = \emptyset$ by Corollary 2.11. And if $o(G) = \mathcal{O}$, then $o(G + G) = \mathcal{O}$ or $o(G + G) = \emptyset$, by Theorem 2.8 (Next Generation). In fact, there are no new obstructions for $o(G + G)$.

$+$	\emptyset	\mathcal{N}	\mathcal{O}	\mathcal{P}	\mathcal{NO}	\mathcal{OP}	\mathcal{PN}
\emptyset	\emptyset	\mathcal{N}	\emptyset	\emptyset	\mathcal{NO}	\emptyset	\mathcal{N}
\mathcal{N}	\mathcal{N}	\mathcal{NO}	\mathcal{N}	\mathcal{N}	\mathcal{NOP}	\mathcal{N}	\mathcal{NO}
\mathcal{O}	\emptyset	\mathcal{N}	\mathcal{O}	\emptyset	\mathcal{NO}	\mathcal{O}	\mathcal{PN}
\mathcal{P}	\emptyset	\mathcal{N}	\emptyset	\emptyset	\mathcal{NO}	\mathcal{P}	\mathcal{N}
\mathcal{NO}	\mathcal{NO}	\mathcal{NOP}	\mathcal{NO}	\mathcal{NO}	\mathcal{NOP}	\mathcal{NO}	\mathcal{NOP}
\mathcal{OP}	\emptyset	\mathcal{N}	\mathcal{O}	\mathcal{P}	\mathcal{NO}	\mathcal{OP}	\mathcal{PN}
\mathcal{PN}	\mathcal{N}	\mathcal{NO}	\mathcal{PN}	\mathcal{N}	\mathcal{NOP}	\mathcal{PN}	\mathcal{NO}

Table 1: The three-player pairwise addition table

Theorem 2.14. $G + G$ can have any outcome not disallowed by Theorem 2.8 (Next Generation) and Theorem 2.10 (Other Procreation).

Proof. For examples of all 23 cases, see Table 7 in the Appendix. For instance, $o(*2) = \mathcal{N}$ and $o(*2 + *2) = \mathcal{O}$. \square

The possible outcomes of the sum of a game with itself are summarized in Table 2.

$o(G)$	$o(G + G)$
\emptyset	$= \emptyset$
\mathcal{N}	$\subseteq \mathcal{NO}$
\mathcal{O}	$\subseteq \mathcal{O}$
\mathcal{P}	$= \emptyset$
\mathcal{NO}	$\subseteq \mathcal{NOP}$
\mathcal{OP}	$\subseteq \mathcal{OP}$
\mathcal{PN}	$\subseteq \mathcal{NO}$

Table 2: The three-player doubling table

In the impartial two-player setting, the Tweedledum–Tweedledee strategy yields $o(G + G) = \mathcal{P}$. A similar result holds for three players.

Proposition 2.15. For all games G , $\mathbf{N} \notin o(G + G + G)$.

Proof. The other two players can guarantee Next’s loss by mirroring their moves in the other two components. \square

Combined with Theorem 2.8 (Next Generation) and Theorem 2.10 (Other Procreation), this theorem reduces the possibilities for $o(G + G + G)$ to those listed in Table 3.

$o(G)$	$o(G + G + G)$
\emptyset	$= \emptyset$
\mathcal{N}	$\subseteq \mathcal{OP}$
\mathcal{O}	$\subseteq \mathcal{O}$
\mathcal{P}	$= \emptyset$
\mathcal{NO}	$\subseteq \mathcal{OP}$
\mathcal{OP}	$\subseteq \mathcal{OP}$
\mathcal{PN}	$\subseteq \mathcal{OP}$

Table 3: The three-player trebling table

Proposition 2.16. *All cases in Table 3 are possible, except perhaps for $o(G) = \mathcal{N}$, where it is not known whether $\mathbf{P} \in o(G + G + G)$ occurs.*

Proof. See Table 8 in the Appendix for example games covering the known cases. For instance, $o(*) = \mathcal{PN}$, but $o(* + * + *) = \mathcal{OP}$. □

2.2.2. Equality of Impartial Games

As in the two-player case, we are interested in when games are similar enough to be replaced in disjunctive sums without affecting the outcome. As is standard, we define $=$ by the fundamental equivalence (see [8]).

Definition 2.17. (Impartial Equality) We say G equals H and write $G = H$ if $o(G + X) = o(H + X)$ for all impartial games X .

We first examine reversible moves, as they are key to understanding equality classes of two-player games, both in normal and misère play. Our presentation is based on the similar Definition V.1.3 in [8].

Definition 2.18. Let G, H be games. We say that H is *revertible to G* if both of the following hold.

- For every option G' of G , H has a corresponding option \widehat{G}' equal to G' .
- For every option H' of H not equal to some G' , H' has a second option H'^{**} equal to G .

In two-player settings, if H is revertible to G (with a slight modification in misère play), then $H = G$. With three players under normal play, only one inclusion holds in general (see Proposition 2.21).

Theorem 2.19. *If H is revertible to G , then $o(H + X) \subseteq o(G + X)$ for all X .*

Proof. Suppose $\mathbf{N} \in o(H + X)$. If the winning move has the form $H + X^*$, then $\mathbf{P} \in o(H + X^*)$ so that $\mathbf{P} \in o(G + X^*)$ by induction. Hence, the same move would

win in $G + X$, and $\mathbf{N} \in o(G + X)$. Now suppose the winning move in $H + X$ is to something of the form $\widehat{G}^* + X$ with \widehat{G}^* equal to an option G^* of G . Then $\mathbf{P} \in o(\widehat{G}^* + X) = o(G^* + X)$, so that $\mathbf{N} \in o(G + X)$. Finally, if the winning move is to an option H^* not equal to an option of G , then the other two players can move to yield some $H^{***} + X$ where $H^{***} = G$. Note that $\mathbf{N} \in o(H^{***} + X)$ as we started with a winning move in $H + X$. From $H^{***} = G$ it follows that $\mathbf{N} \in o(G + X)$, as desired.

Suppose $\mathbf{O} \in o(H + X)$. Then $\mathbf{N} \in o(H + X')$ for all options X' . By induction, $\mathbf{N} \in o(G + X')$ for all X' , as well. Since $\mathbf{O} \in o(H + X)$, we also have $\mathbf{N} \in o(\widehat{G}' + X) = o(G' + X)$ for all options G' , as well. We have shown $\mathbf{N} \in o((G + X)')$ for all options, so $\mathbf{O} \in o(G + X)$.

The argument for $\mathbf{P} \in o(H + X)$ is exactly analogous to that for \mathbf{O} . □

The phrasing of Corollary 2.20 is from the Replacement Lemma V.1.2 in [8].

Corollary 2.20. *Suppose that $G \cong \{G'_i : i \in I\}$, $H \cong \{H'_i : i \in I\}$, and $G'_i = H'_i$ for all i . Then $G = H$.*

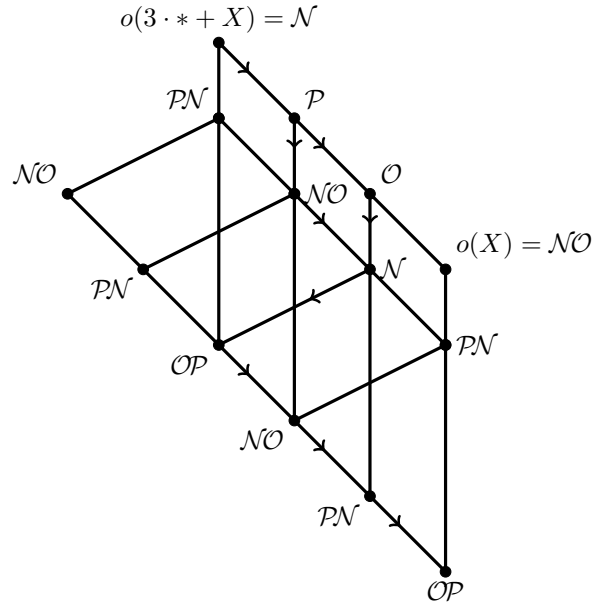
Proof. By symmetry, it suffices to show that $o(H + X) \subseteq o(G + X)$ for all X . But that follows immediately from Theorem 2.19 since H is revertible to G . □

Unfortunately, the reverse inclusion for Theorem 2.19 does not always hold in this three-player setting. For instance, it is not true that $3 \cdot * = 0$, even though in the two-player setting, $2 \cdot * = 0$ holds in the both the partizan normal-play and impartial misère-play contexts.

Proposition 2.21. $3 \cdot * \neq 0$.

Proof. It is useful to give the players consistent names throughout, even though which one is next to play varies; for this purpose, we call the first three players to move “Nicky”, “Olly”, and “Pat”.

Define $X \cong \{0, * + *\}$. In $0 + X$, Olly can win by making the move to 0. But in the sum $3 \cdot * + X$, Nicky and Pat can stop Olly from winning. First, Nicky moves in the $3 \cdot *$ component to $* + *$. Then, no matter which move Olly makes, it’s possible for Pat to move to a position isomorphic to $\{3 \cdot *, *, \{0, * + *\}\}$, from which Nicky can move to $3 \cdot *$. The lines of play are illustrated in the game graph below. □



Definition 2.22. A game G is said to be *absorbing* if $G + H$ is undetermined for any game H .

Proposition 2.23. If G and H are absorbing games, then $G = H$.

Proof. Note that $o(G + X) = \emptyset = o(H + X)$ for all X . □

Therefore, there is at most one equality class of absorbing games.

Proposition 2.24. If G is an absorbing game, then $G + H = G$ for any H .

Proof. Note that $o((G + H) + X) = o(G + (H + X)) = \emptyset = o(G + X)$. □

Proposition 2.24 justifies the name “absorbing”.

Theorem 2.25 (Absorbing Game Construction). *Suppose that G has at least one option, and all options G' are undetermined ($o(G') = \emptyset$). Then G is an absorbing game.*

Proof. Let G be as in the statement. Since $o(G') = \emptyset$ for all options G' , $\mathbf{P} \notin o(G' + X)$ by Corollary 2.12. By induction, $o(G + X') = \emptyset$ for all X' . If $X \not\cong 0$, then $o(G + X) = \emptyset$ by Proposition 2.7 since $o(G + X^*) = \emptyset$ for some option X^* . If $X \cong 0$, then since G has at least one option G^* with $o(G^*) = \emptyset$, $o(G + X) = o(G) = \emptyset$ by Proposition 2.7. □

Corollary 2.26. An absorbing game exists.

Proof. By Proposition 2.5, $o(\{ *2, \{ *2 \} \}) = \emptyset$. Therefore, by Theorem 2.25 (Absorbing Game Construction), $\{ \{ *2, \{ *2 \} \} \}$ is an absorbing game. \square

The NIM position $3 \cdot *3$ is also absorbing (see Lemma 2.37). By Proposition 2.23, all such games are equal.

Proposition 2.27. *If G, H, J are undetermined, then all options of $G + H + J$ are undetermined.*

Proof. Let G, H, J be undetermined. Suppose, for the sake of contradiction, that $G + H + J$ has an option that is not undetermined. Without loss of generality, suppose J^* is an option of J satisfying $o(G + H + J^*) \neq \emptyset$. Since G is undetermined, Corollary 2.12 yields $\mathbf{P} \notin o(G + (H + J^*))$. Therefore, either $\mathbf{N} \in o(G + H + J^*)$ or $\mathbf{O} \in o(G + H + J^*)$.

Suppose $\mathbf{N} \in o(G + H + J^*)$. Then either $\mathbf{P} \in o(G + H + J^{**})$ for some J^{**} or (without loss of generality) $\mathbf{P} \in o(G + H^* + J^*)$ for some H^* . But both are impossible by Corollary 2.12.

Now suppose $\mathbf{O} \in o(G + H + J^*)$. Since $\mathbf{N} \notin o(J)$, $J^* \not\cong 0$. Hence, we can choose a second option J^{**} . Then $\mathbf{N} \in o(G + H + J^{**})$. By the argument from the previous paragraph, Corollary 2.12 yields a contradiction in this case, too. \square

Corollary 2.28. *If G is undetermined, then $3 \cdot G$ is an absorbing game.*

Proof. Combine Proposition 2.27 with Theorem 2.25 (Absorbing Game Construction). \square

Note that we can find many equal absorbing games with Proposition 2.24 and Corollary 2.28. And by using Corollary 2.20, we can build many games equal to a given one that has an absorbing subposition.

2.3. Three-Player Nim

NIM is a classic ruleset in combinatorial game theory that can serve as a demonstrative case study. In this subsection, we calculate the outcomes of all NIM positions, as well as the quotient corresponding to playing only NIM. When playing NIM with three players under normal play, there are only finitely many different classes of positions to be concerned with, though more than in the play convention considered in [6].

In the language of [4], the positions of three-player NIM form a three-player analogue of a “universe”. But it is not “parental” as all NIM positions are impartial. And it is not “dense” even in an impartial sense, as no NIM position has outcome \mathcal{P} (see Theorem 2.40).

2.3.1. Outcomes of Nim Positions

We classify the outcomes of NIM positions. We start by noting the outcomes of 40 particular positions, and then use a few lemmas to justify that the apparent patterns extend to all positions. This approach is similar to applications of the Octal Periodicity Theorem (see Thm. IV.2.7 of [8]).

Lemma 2.29. *The outcomes of 40 NIM positions of the form $k \cdot * + G$ are recorded in Table 4.*

$G \setminus k$	0	1	2	3
0	\mathcal{OP}	\mathcal{PN}	\mathcal{NO}	\mathcal{OP}
*2	\mathcal{N}	\mathcal{NO}	\mathcal{PN}	\mathcal{N}
*3	\mathcal{N}	\mathcal{NO}	\mathcal{N}	\mathcal{N}
*4	\mathcal{N}	\mathcal{NO}	\mathcal{N}	\mathcal{N}
*2 + *2	\mathcal{O}	\mathcal{N}	\mathcal{NO}	\mathcal{O}
*2 + *3	\emptyset	\mathcal{N}	\mathcal{NO}	\emptyset
*2 + *4	\emptyset	\mathcal{N}	\mathcal{NO}	\emptyset
*2 + *2 + *2	\emptyset	\emptyset	\emptyset	\emptyset
*2 + *2 + *3	\emptyset	\emptyset	\emptyset	\emptyset
*2 + *2 + *4	\emptyset	\emptyset	\emptyset	\emptyset

Table 4: The three-player outcomes of 40 NIM positions

Proof. These outcomes can be calculated manually or with a computer program. \square

Definition 2.30. A game G is said to be *3-periodic* if the infinite sequence of outcomes $(o(G), o(* + G), o(2 \cdot * + G), \dots)$ has period 3.

Lemma 2.31 (Nim Periodicity). *Let G be a game. Suppose that all options G' are 3-periodic and that $o(G) = o(3 \cdot * + G)$. Then G is 3-periodic as well.*

Proof. Let $k \geq 4$. Then the options of $k \cdot * + G$ are $(k - 1) \cdot * + G$ and those of the form $k \cdot * + G'$ for options G' . These have the same outcomes as $(k - 4) \cdot * + G$ (by induction on k) and $(k - 3) \cdot * + G'$ (since G' is 3-periodic), respectively. But those are exactly all of the options of $(k - 3) \cdot * + G$. Since $k \cdot * + G$ has the same outcomes of options, $o(k \cdot * + G) = o((k - 3) \cdot * + G)$, as desired. \square

Lemma 2.32 (Nim Stability). *Let G be a game. Suppose that for all subpositions H of G (including G itself) $o(H + *4) = o(H + *3)$. Then $o(G + *m) = o(G + *3)$ for all $m \geq 3$.*

Proof. Let $m \geq 4$. Via induction on m , it suffices to verify that $o(G + *(m + 1)) = o(G + *m)$.

The options of $G + *(m + 1)$ are $G + *m$, those of the form $G + *\ell$ for $\ell < m$ (all of which are options of $G + *m$), and those of the form $G' + *(m + 1)$.

By induction on m , $o(G + *m) = o(G + *(m - 1))$. And by induction on G , $o(G' + *(m + 1)) = o(G' + *m)$. Trivially, $o(G + *\ell) = o(G + *\ell)$. In all cases, an option of $G + *(m + 1)$ has the same outcome as an option of $G + *m$.

Conversely, the options of $G + *m$ are those of the form $G + *\ell$ for $\ell < m$, and those of the form $G' + *m$. All of which have outcome equal to that of an option of $G + *(m + 1)$.

As outcomes of options are the same, $o(G + *(m + 1)) = o(G + *m)$. □

Lemma 2.33. *The patterns of outcomes for the NIM positions of the form $k \cdot * + G$ in Table 5 are accurate.*

Proof. Apply Lemma 2.31 (Nim Periodicity) to the upper 7 rows of Table 4, and then apply Lemma 2.32 (Nim Stability) to 0, followed by $*2$. □

$G \setminus k \pmod 3$	$\equiv 0$	$\equiv 1$	$\equiv 2$
0	\mathcal{OP}	\mathcal{PN}	\mathcal{NO}
$*2$	\mathcal{N}	\mathcal{NO}	\mathcal{PN}
$*m$	\mathcal{N}	\mathcal{NO}	\mathcal{N}
$*2 + *2$	\mathcal{O}	\mathcal{N}	\mathcal{NO}
$*2 + *m$	\emptyset	\mathcal{N}	\mathcal{NO}

Table 5: The outcomes of some NIM positions of the form $k \cdot * + G$ ($m \geq 3$)

Lemma 2.34. *For all $k \geq 0$, and $m_2 \geq m_1 \geq 3$, $o(k \cdot * + *m_1 + *m_2) = \emptyset$.*

Proof. Set $G \cong k \cdot * + *m_1 + *m_2$. If $k = 0$, then G has the undetermined option $*2 + *m_2$ by Lemma 2.33. If $k \geq 1$, then G has the undetermined option $(k - 1) \cdot * + *m_1 + *m_2$ by induction on k . By Proposition 2.7, it suffices to show that no option G' has $\mathbf{P} \in o(G')$. By induction and Lemma 2.33, the only subpositions with that property are of the form $j \cdot * + *2$ or $j \cdot *$, neither of which is possible for an option. □

Lemma 2.35. *For all $k \geq 0$ and $n \geq 2$, $o(k \cdot * + *2 + *2 + *n) = \emptyset$.*

Proof. Apply Lemma 2.31 (Nim Periodicity) to the lower 3 rows of Table 4, and then apply Lemma 2.32 (Nim Stability) to $*2 + *2$. □

Lemma 2.36. *For all $k \geq 0$, and $m_2 \geq m_1 \geq 3$, $o(k \cdot * + *2 + *m_1 + *m_2) = \emptyset$.*

Proof. By Lemma 2.34, the option $k \cdot * + *m_1 + *m_2$ is undetermined. The outcomes of options G' are listed by Lemmas 2.33 through 2.35. And the only subpositions

where Previous has a winning strategy are of the form $j \cdot * + *2$ or $j \cdot *$. By Proposition 2.7, $k \cdot * + *2 + *m_1 + *m_2$ is undetermined. \square

Lemma 2.37. *For all $m_1, m_2, m_3 \geq 3$, $*m_1 + *m_2 + *m_3$ is an absorbing game.*

Proof. By the Theorem 2.25 (Absorbing Game Construction) (Theorem 2.25), it suffices to show that all options have outcome \emptyset . If all heaps of an option have size at least 3, then the outcome is \emptyset by induction. If an option has reduced a heap to $*2$, then the outcome is \emptyset by Lemma 2.36. If the move reduces a heap to $*$ or eliminates a heap, then the outcome is \emptyset by Lemma 2.34. \square

Lemma 2.38. *For all $n_2 \geq n_1 \geq 2$, $*2 + *2 + *n_1 + *n_2$ is an absorbing game.*

Proof. By the Theorem 2.25 (Absorbing Game Construction), it suffices to show that all options have outcome \emptyset . If an option has four heaps of size at least 2 then the outcome is \emptyset by induction. Otherwise, the outcome is \emptyset by Lemma 2.35 or 2.36, as applicable. \square

Lemma 2.39. *Any NIM position that has at least three heaps of size at least 3, or at least four heaps of size at least 2 is an absorbing game.*

Proof. By Lemmas 2.37 and 2.38, any such position has an absorbing game as a summand. By Proposition 2.24, the entire position is an absorbing game. \square

Theorem 2.40. *The outcome of a NIM position is determined by the numbers of heaps of size 1, size 2, and size at least 3. Specifically, the outcome is \emptyset unless the position has the form $k \cdot * + *n$ or $k \cdot * + *2 + *n$, in which case the outcome is listed in Table 5.*

Proof. This follows immediately from Lemma 2.29 and Lemmas 2.33 through 2.39. \square

2.3.2. Nim Quotient

In direct analogy to misère quotients (as covered in V.4 of [8]), we define an equivalence relation on the set of NIM positions which is coarser than equality. While the two-player misère quotient of NIM is infinite, the three-player normal play quotient of NIM has size 16.

Definition 2.41. If G and H are NIM positions, then we write $G \equiv H$ if $o(G+X) = o(H+X)$ for all NIM positions X .

Proposition 2.42. *If G, H, J are NIM positions and $G \equiv H$, then $G+J \equiv H+J$.*

Proof. Since NIM positions are closed under addition, $J + X$ is a NIM position whenever X is, so that $o((G + J) + X) = o(G + (J + X)) = o(H + (J + X)) = o((H + J) + X)$, where the middle equation comes from $G \equiv H$. \square

Proposition 2.43. *If G, H, J, K are NIM positions and both $G \equiv H$ and $J \equiv K$ hold, then $G + J \equiv H + K$.*

Proof. By two applications of Proposition 2.42, $G + J \equiv H + J \cong J + H \equiv K + H \cong H + K$. \square

For convenience, let \mathcal{Q} be the set of equivalence classes of NIM positions under \equiv . Note that since 0 is a NIM position, Proposition 2.43 gives \mathcal{Q} the structure of a commutative monoid. As with the convention for misère quotients, let $\Phi(G)$ denote the equivalence class of a NIM position G . We follow convention to write elements of \mathcal{Q} with lowercase letters and the operation multiplicatively. For example, if $x = \Phi(G)$ and $y = \Phi(H)$, then $xy = \Phi(G + H)$; we also write $1 = \Phi(0)$.

Proposition 2.44. *For a NIM position G , $\Phi(G)$ determines $o(G)$.*

Proof. Since 0 is a NIM position, note that $G \equiv H$ implies $o(G) = o(G + 0) = o(H + 0) = o(H)$. \square

Since the restriction of o to the NIM positions factors through \mathcal{Q} , we adopt notation from V.6 of [8] and use Π to denote the corresponding map on \mathcal{Q} . For example, $\Pi(1) = \Pi(\Phi(0)) = \mathcal{OP}$.

Definition 2.45. The *Nim quotient* is the structure $(\mathcal{Q}, \cdot, \Pi)$.

For convenience, for the remainder of this subsection, define $a = \Phi(*)$, $b = \Phi(*2)$, and $c = \Phi(*3)$.

Theorem 2.46. *The Nim quotient is given by the commutative monoid with presentation $(\mathcal{Q}, \cdot) \cong \langle a, b, c \mid a^3 = 1, b^3 = b^2c = c^2 = c^3 = ac^2 \rangle$ (which has order 16) and the values of Π given in Table 6.*

x	1	a	a^2	b	ab	a^2b	c	ac	a^2c	b^2	ab^2	a^2b^2	bc	abc	a^2bc	c^2
$\Pi(x)$	\mathcal{OP}	\mathcal{PN}	\mathcal{NO}	\mathcal{N}	\mathcal{NO}	\mathcal{PN}	\mathcal{N}	\mathcal{NO}	\mathcal{N}	\mathcal{O}	\mathcal{N}	\mathcal{NO}	\emptyset	\mathcal{N}	\mathcal{NO}	\emptyset

Table 6: Outcomes of the three-player Nim quotient

To prove Theorem 2.46, we verify various facts individually as a series of lemmas.

Lemma 2.47. *The outcomes in Table 6 are accurate.*

Proof. All of the outcomes follow directly from Lemma 2.33, with the exception of $\Pi(c^2)$, which follows from Lemma 2.34. \square

Lemma 2.48. $a^3 = 1$, that is, $* + * + * \equiv 0$.

Proof. In Theorem 2.40, the outcomes only depend on the number of heaps of size 1, modulo 3. □

Lemma 2.49. *The 16 equivalence classes listed in Table 6 are distinct.*

Proof. Note that Π is well defined on equivalence classes, so the only pairs of classes that might be identical are those with the same outcome. We apply Lemma 2.48 repeatedly below:

- $a \neq a^2b$ because $\Pi(a^2) = \mathcal{NO}$ and $\Pi(a^3b) = \Pi(1b) = \Pi(b) = \mathcal{N}$;
- $a^2, ab, ac, a^2b^2, a^2bc$ are all distinct because the outcomes after multiplying by a are $\mathcal{OP}, \mathcal{PN}, \mathcal{N}, \mathcal{O}, \emptyset$, respectively;
- b, c, a^2c, ab^2, abc are all distinct because the outcomes after multiplying by a^2 are $\mathcal{PN}, \mathcal{N}, \mathcal{NO}, \mathcal{O}, \emptyset$, respectively;
- $bc \neq c^2$ because $\Pi(abc) = \mathcal{N}$ and $\Pi(ac^2) = \emptyset$ by Lemma 2.34.

□

It remains to prove enough other relations to conclude that Table 6 is complete.

Lemma 2.50. *For each $n \geq 3$, $c = \Phi(*n)$, that is, $*3 \equiv *n$.*

Proof. In Theorem 2.40, none of the outcome determinations depend on the exact size of a heap that has size at least 3. □

Lemma 2.51. *For any $x \in \mathcal{Q}$, $xb^3 = xb^2c = xc^2$.*

Proof. If X is a NIM position, then $o(3 \cdot *2 + X) = o(2 \cdot *2 + *3 + X) = o(2 \cdot *3 + X) = \emptyset$ by theorem 2.40. Thus, all representatives have outcome \emptyset . □

By Lemma 2.39, $*2 + *2 + *2 + *2$ and $*3 + *3 + *3$ are absorbing games. But note that Lemma 2.51 tells us that in the context of NIM positions only, $*2 + *2 + *2$ and $*3 + *3$ have a similar absorbing property.

Corollary 2.52. $b^3 = b^2c = c^2 = c^3 = ac^2$.

Lemma 2.53. *There are only 16 elements of \mathcal{Q} as listed in Table 6, and the presentation in the statement of Theorem 2.46 is accurate.*

Proof. By $a^3 = 1$ (from Lemma 2.48) and $b^4 = b(b^3) = b(b^2c) = (b^3)c = (c^2)c = b^3$ and $c^2 = c^3$ (from Corollary 2.52) and the commutativity of disjunctive sum, the only possible elements of \mathcal{Q} are of the form $a^x b^y c^z$ for $x, z < 3, y < 4$. However, if $z = 2$ or $y + z \geq 3$ then the element is equal to c^2 by Corollary 2.52. This leaves 15 other possibilities with $y + z < 3$ and $z < 2$, for a total of 16. □

With this, we have proven Theorem 2.46. For comparison, in Section 5 of [6], NIM is analyzed under Propp’s play convention in which Previous is the unique winner of 0. Rephrasing his results in our terminology, the three-player NIM quotient for that play convention has size 9, though he shows that there are infinitely many equality classes of NIM positions.

Note that in the case of NIM positions alone, we recover something like a more faithful version of Theorem 2.8 (Next Generation).

Corollary 2.54. *If G is a NIM position satisfying $o(G) = \mathcal{OP} = o(0)$, then $o(G + H) = o(H)$ for all NIM positions H .*

Proof. By Theorem 2.46, $\Phi(G) = 1$. Thus, $o(G + H) = \Pi(1\Phi(H)) = o(H)$. \square

3. N-Player Impartial Games

Many results from Section 2 can be generalized to N players. Throughout this entire section, N refers to the number of players ($N \geq 2$). We explicitly mention when assuming $N > 2$ is necessary.

Some results and proofs are nearly identical to those in the three-player case, but others become more complex, such as the construction of absorbing games for N players.

A game is still a finite set of options, and we continue to use \cong to indicate that two games are isomorphic (i.e. the game trees are isomorphic).

3.1. Preliminaries

In the impartial setting, we denote the various players as $\mathbf{N}, \mathbf{O}_1, \dots, \mathbf{O}_{N-2}, \mathbf{P}$. \mathbf{N} is the “Next” player, \mathbf{P} is the “Previous” player, and the rest are the “Other” players. At times, it is convenient to use \mathbf{O}_0 for \mathbf{N} and \mathbf{O}_{N-1} for \mathbf{P} .

Definition 3.1. The N -player (normal-play) *outcome* $o(G)$ of a game G is a subset of $\{\mathbf{N}, \dots, \mathbf{P}\}$ whose members are determined recursively as follows.

- $\mathbf{N} \in o(G)$ exactly when there exists an option G^* with $\mathbf{P} \in o(G^*)$.
- For $1 \leq i \leq N - 1$, $\mathbf{O}_i \in o(G)$ exactly when every option G' has $\mathbf{O}_{i-1} \in o(G')$.

Proposition 3.2. *The following analogues of the propositions in 2.1.2 all hold.*

1. $o(G)$ is always a proper subset of $\{\mathbf{N}, \mathbf{O}_1, \dots, \mathbf{O}_{N-2}, \mathbf{P}\}$.
2. For $0 \leq i \leq N - 2$, $\mathbf{O}_i \in o(G)$ exactly when $\mathbf{O}_{i+1} \in o(\{G\})$. And $\mathbf{P} \in o(G)$ exactly when $\mathbf{N} \in o(\{G\})$.

3. If G has an option G^* with $o(G^*) = \emptyset$ and has no option G' with $\mathbf{P} \in o(G')$, then $o(G) = \emptyset$.
4. If $N > 2$, then all $2^N - 1$ proper subsets are possible outcomes.

Proof. The proofs of claims 1, 2, and 3 are exactly analogous to the proofs in the three-player case in 2.1.2. For claim 4, we need a more general idea.

For any non-empty outcome set that includes \mathbf{N} , we can use a game with various options of the form $m \cdot *$. For example, if there are at least 6 players (if $N \geq 6$), then $\{*, 3 \cdot *, 4 \cdot *\}$ is a game in which Next chooses whether \mathbf{O}_2 , \mathbf{O}_4 , or \mathbf{O}_5 will lose; all other players are guaranteed to win.

For a non-empty outcome set that *does not* include \mathbf{N} , we can apply claim 2 to cycle the outcome set of a game whose outcome includes \mathbf{N} . For example, if $N = 6$, then $\{*, 3 \cdot *, 4 \cdot *\}$ has outcome $\{\mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_4\}$.

Finally, if $N > 2$, we construct a game G with $o(G) = \emptyset$. We can take $G \cong \{H, \{H\}\}$ where $H \cong \{0, \dots, (N - 2) \cdot *\}$. In G , Next chooses whether \mathbf{O}_1 or \mathbf{O}_2 selects any player (other than themselves) to lose the game. \square

3.2. Outcomes of Sums

Theorem 3.3 (*N-Player Next Generation*). *If $\mathbf{N} \notin o(H)$ then $o(G + H) \subseteq o(G)$.*

Note that $\mathbf{N} \notin o(H)$ is still equivalent to $o(H) \subseteq o(0)$ in the N -player setting. In fact, the proof for N players is nearly identical to the one for three players.

Proof. Let G, H be games, and suppose $\mathbf{N} \notin o(H)$.

If $\mathbf{O}_i \notin o(G)$ for some i with $1 \leq i \leq N - 1$, then we can choose an option G^* with $\mathbf{O}_{i-1} \notin o(G^*)$. But then $\mathbf{O}_{i-1} \notin o(G^* + H)$ by inductive hypothesis, so that $\mathbf{O}_i \notin o(G + H)$.

If $\mathbf{N} \notin o(G)$, then $\mathbf{P} \notin o(G')$ for all options G' . Similarly, $\mathbf{P} \notin o(H')$ for all options H' . Therefore, by induction, we have both $\mathbf{P} \notin o(G' + H)$ and $\mathbf{P} \notin o(G + H')$ for arbitrary options. As such, $\mathbf{N} \notin o(G + H)$ by definition. \square

We generalize Theorem 2.10 (Other Procreation) in two respects—more summands, and different players.

Theorem 3.4. *For $k \geq 1$ and $m \geq 0$, if $N > km$ and $\mathbf{O}_m \notin o(G_1), \dots, o(G_k)$, then $\mathbf{O}_{km} \notin o(G_1 + \dots + G_k)$.*

Proof. Note that the case of $k = 1$ is trivial, so we assume $k \geq 2$.

First, we consider the case $m = 0$, so that $\mathbf{O}_m = \mathbf{N}$. By repeatedly applying Theorem 3.3 (Next Generation), $\mathbf{N} \notin o(G_1 + \dots + G_k)$.

For $m > 0$, we induct on m . Since $\mathbf{O}_m \notin o(G_1), \dots, o(G_k)$, we can choose options G_i^* such that $\mathbf{O}_{m-1} \notin o(G_i^*)$ for all i . Then by induction, $\mathbf{O}_{km-k} \notin$

$o(G_1^* + \cdots + G_k^*)$, so that $\mathbf{O}_{km-(k-1)} \notin o(G_1 + G_2^* + \cdots + G_k^*)$ and similarly for each successive term until we reach $\mathbf{O}_{km} \notin o(G_1 + \cdots + G_k)$. \square

Proposition 2.15 about the outcome of $G + G + G$ generalizes straightforwardly.

Proposition 3.5. *For all games G , $\mathbf{N} \notin o(N \cdot G)$.*

Proof. All players can mirror the moves made by Next in the N components. \square

In the N -player setting, we can still examine reversible moves.

Definition 3.6. We say that H is *revertible to G* if both of the following hold.

- For every option G' of G , H has a corresponding option $\widehat{G'}$ equal to G' .
- For every option H' of H not equal to some G' , H' has an $(N - 1)^{\text{st}}$ option $H'^{*\dots*}$ equal to G .

Theorem 3.7. *If H is revertible to G then $o(H + X) \subseteq o(G + X)$ for all X .*

The proof is nearly identical to the argument for three players (see Theorem 2.19), but is written here in full, for convenience.

Proof. Suppose $\mathbf{N} \in o(H + X)$. If the winning move has the form $H + X^*$, then $\mathbf{P} \in o(H + X^*)$ so that $\mathbf{P} \in o(G + X^*)$ by induction. Hence, the same move would win in $G + X$, and $\mathbf{N} \in o(G + X)$. Now suppose the winning move in $H + X$ is to something of the form $\widehat{G^*} + X$ with $\widehat{G^*}$ equal to an option G^* of G . Then $\mathbf{P} \in o(\widehat{G^*} + X) = o(G^* + X)$, so that $\mathbf{N} \in o(G + X)$. Finally, if the winning move is to an option H^* not equal to an option of G , then the other $N - 1$ players can move to yield some $H'^{*\dots*} + X$ where $H'^{*\dots*} = G$. Note that $\mathbf{N} \in o(H'^{*\dots*} + X)$ as we started with a winning move in $H + X$. From $H'^{*\dots*} = G$ it follows that $\mathbf{N} \in o(G + X)$, as desired.

Suppose $\mathbf{O}_i \in o(H + X)$ for some i with $1 \leq i \leq N - 1$. Then $\mathbf{O}_{i-1} \in o(H + X')$ for all options X' . By induction, $\mathbf{O}_{i-1} \in o(G + X')$ for all X' as well. Since $\mathbf{O}_i \in o(H + X)$, we also have $\mathbf{O}_{i-1} \in o(\widehat{G'} + X) = o(G' + X)$ for all options G' as well. We have shown $\mathbf{O}_{i-1} \in o((G + X)')$ for all options, so $\mathbf{O}_i \in o(G + X)$. \square

As with three players, replacing options with equal games yields an equal game.

Corollary 3.8. *Suppose that $G \cong \{G'_i : i \in I\}$ and $H \cong \{H'_i : i \in I\}$ and that $G'_i = H'_i$ for all i . Then $G = H$.*

The proof is identical to that of the three-player case (see Corollary 2.20).

Proof. By symmetry, it suffices to show that $o(H + X) \subseteq o(G + X)$ for all X . But that follows immediately from Theorem 3.7 since H is revertible to G . \square

Proposition 2.21 about $3 \cdot *$ generalizes to the N -player setting.

Proposition 3.9. *If $N > 2$, then $N \cdot * \neq 0$.*

Proof. In $G \cong \{(N - 1) \cdot *, 0\}$, all players except \mathbf{O}_2 have winning strategies. There is only one decision; \mathbf{O}_1 can win by moving to 0, in which case \mathbf{O}_2 loses. \mathbf{O}_1 loses if they move to $(N - 1) \cdot *$ instead.

However, in $G + N \cdot *$, players \mathbf{N} , \mathbf{O}_2 and (if $N > 3$) \mathbf{O}_3 can cooperate to ensure that \mathbf{O}_1 loses. Next can move to $G + (N - 1) \cdot *$. Then, for either of \mathbf{O}_1 's moves, \mathbf{O}_2 can move to $G' + (N - 2) \cdot *$. Then \mathbf{O}_3 (or \mathbf{N} if $N = 3$) can move in G' to $(N - 1) \cdot *$, for a total position of $(2N - 3) \cdot *$. At this point, \mathbf{O}_4 is to move unless $N < 5$, in which case it is \mathbf{O}_{4-N} . In any case, after $2N - 3$ more moves, \mathbf{O}_1 loses. \square

3.3. Undetermined and Absorbing Games

Recall (see Definition 2.6) that a game G is said to be *undetermined* if $o(G) = \emptyset$.

Proposition 3.10. *A sum of undetermined games is undetermined.*

Proof. As with Corollary 2.9, this follows from Theorem 3.3 (Next Generation) (Theorem 3.3). \square

We can make this result more precise.

Definition 3.11. An undetermined game G is said to be *1-undetermined*. And, for $k > 1$, G is said to be *k-undetermined* if G is undetermined and has an option G^* that is $(k - 1)$ -undetermined.

In other words, G is k -undetermined if there is a directed path of length $k - 1$ starting at G through undetermined subpositions. Note that if G is k -undetermined, then it is also, *a fortiori*, j -undetermined for $1 \leq j < k$.

Proposition 3.12. *If G is k -undetermined and H is m -undetermined, then $G + H$ is $(k + m)$ -undetermined.*

Proof. Let G be k -undetermined and H be m -undetermined. By Proposition 3.10, $G + H$ is undetermined. It remains to show that $G + H$ has an option that is $(k + m - 1)$ -undetermined.

First, suppose that $k = m = 1$. Note that since $\mathbf{O}_1 \notin o(H)$, we can choose an option H^* with $\mathbf{N} \notin o(H^*)$. By Theorem 3.3 (Next Generation), we have $o(G + H^*) \subseteq o(G) = \emptyset$.

If $k > 1$, we can choose an undetermined option G^* of G . Then $G^* + H$ is $(k + m - 1)$ -undetermined by induction. Similarly if $m > 1$. \square

The concept of k -undetermined games allows us to generalize Corollary 2.12.

Lemma 3.13. *If $N > 2$ and G is $(N - 2)$ -undetermined, then $\mathbf{P} \notin o(G + H)$ for all H .*

Proof. Since G is $(N - 2)$ -undetermined, we can choose a sequence of $N - 2$ subpositions J_0, J_1, \dots, J_{N-3} where $J_0 \cong G$, each J_{i+1} is an option of J_i , and each J_i is undetermined. Note that $J_i + H$ is a subposition of $G + H$ reached after i moves.

Suppose, for the sake of contradiction, that $\mathbf{P} \in o(G + H)$ for some particular game H . Then $\mathbf{P} \in o(J_0 + H)$, and since J_0 is undetermined, Theorem 3.3 (Next Generation) tells us that $\mathbf{N} \in o(H)$ (as otherwise $o(J_0 + H) \subseteq o(J_0) = \emptyset$) and $\mathbf{P} \in o(H)$.

Since $\mathbf{P} = \mathbf{O}_{N-1}$, we have $\mathbf{O}_{N-2} \in o(J_1 + H)$ (if $N > 3$). In general, $\mathbf{O}_{N-1-j} \in o(J_j + H)$ for $0 \leq j \leq N - 3$.

Since each J_j is undetermined, $\mathbf{O}_{N-1-j} \in o(H)$ for $0 \leq j \leq N - 3$ as well, in addition to the $\mathbf{N} \in o(H)$ observed initially.

Finally, since J_{N-3} is undetermined, $\mathbf{O}_1 \notin o(J_{N-3})$. Thus, we can choose an option J_{N-3}^* such that $\mathbf{N} \notin o(J_{N-3}^*)$. But since $\mathbf{O}_2 \in o(J_{N-3} + H)$, we have $\mathbf{O}_1 \in o(J_{N-3}^* + H)$. This forces $\mathbf{O}_1 \in o(H)$.

Putting all of these conclusions about H together, we have shown that all players have winning strategies in H , which is impossible by claim 1 of Proposition 3.2. \square

Recall (see Definition 2.22) that a game G is said to be *absorbing* if $G + H$ is undetermined for all H .

We can use Lemma 3.13 to obtain an N -player analogue of Theorem 2.25 (Absorbing Game Construction).

Theorem 3.14 (N -Player Absorbing Game Construction). *Suppose that $N > 2$, G has at least one option, and all options G' are $(N - 2)$ -undetermined. Then for all games H , $G + H$ is undetermined.*

Proof. Note that G is undetermined from the outcomes of its options (cf. Corollary 2.11). We show by a finite induction on k that $\mathbf{O}_k \notin o(G + H)$ for any H .

First, we handle the base case $k = 0$. Suppose, for the sake of contradiction, that $\mathbf{N} \in o(G + H)$ for some game H . Then either $\mathbf{P} \in o(G^* + H)$ for some G^* or $\mathbf{P} \in o(G + H^*)$ for some H^* . Both cases are impossible by Lemma 3.13; the former is impossible since each G^* is $(N - 2)$ -undetermined, and the latter is impossible since G is $(N - 1)$ -undetermined and hence $(N - 2)$ -undetermined.

Now suppose $\mathbf{O}_k \in o(G + H)$ for some k with $1 \leq k \leq N - 1$. Then since G is undetermined, $\mathbf{N} \in o(H)$ by Theorem 3.3 (Next Generation). In particular, H has an option H^* and thus $\mathbf{O}_{k-1} \in o(G + H^*)$, which is impossible by induction. \square

Note that Propositions 2.24 and 2.23 hold with the same proofs, so there is a unique equality class of absorbing games which absorbs summands to produce undetermined games.

We use Lemma 3.13 to generalize Proposition 2.27.

Proposition 3.15. *If $N > 2$ and G is $(N - 2)$ -undetermined, then all $(N - 2)^{\text{nd}}$ options of $N \cdot G$ are undetermined.*

Proof. Let G be $(N - 2)$ -undetermined, and let H be an $(N - 2)^{\text{nd}}$ option of $N \cdot G$. We aim to show that H is undetermined.

Note that H is a sum of games with at least two terms of G . First, we will construct J , an $(N - 2)^{\text{nd}}$ option of H with two terms of G .

If H only has two terms of G , then H must have the form $G_1 + \dots + G_{N-2} + 2 \cdot G$ for some (first) options G_1, \dots, G_{N-2} of G . Since G is undetermined, $\mathbf{N} \notin o(G)$, so that each option of G must have an option of its own. Therefore, we can take J to be something of the form $G_1^* + \dots + G_{N-2}^* + 2 \cdot G$.

Otherwise, H has at least three terms of G . Write it in the form $\tilde{H} + 3 \cdot G$ for some game \tilde{H} . Note that since G is $(N - 2)$ -undetermined, G has an $(N - 2)^{\text{nd}}$ option K (which need not be undetermined). Thus, we can take J to be $\tilde{H} + K + 2 \cdot G$.

With J in hand, we can show that H is undetermined. Since G is $(N - 2)$ -undetermined, Lemma 3.13 yields $\mathbf{P} \notin o(H)$. And since any option of H would still have at least one term of G , we have $\mathbf{P} \notin o(H')$ for all H' . Thus, $\mathbf{N} \notin o(H)$ as well.

To handle the other players, let $j \in \{1, \dots, N - 2\}$ and choose \tilde{J} , a j^{th} option of H with J as a subposition. Since J has at least two terms of G , \tilde{J} does as well, so that any option \tilde{J}' has one term of G . Since G is $(N - 2)$ -undetermined, Lemma 3.13 yields $\mathbf{P} \notin o(\tilde{J}')$. Thus, we have $\mathbf{N} \notin o(\tilde{J})$ and $\mathbf{O}_j \notin o(H)$.

Since we have eliminated all possible elements of $o(H)$, H is undetermined. Since H was arbitrary, all $(N - 2)^{\text{nd}}$ options of $N \cdot G$ are undetermined. \square

We will not use this fact, but the above proof of Proposition 3.15 would have worked the same way for any sum of N games, each of which is $(N - 2)$ -undetermined.

Corollary 3.16. *If $N > 2$ and G is $(N - 2)$ -undetermined, then $N \cdot G$ is an absorbing game.*

Proof. By Proposition 3.15, all $(N - 2)^{\text{nd}}$ options of $N \cdot G$ are undetermined. Then, *a fortiori*, all options of $N \cdot G$ are $(N - 2)$ -undetermined. Thus, $N \cdot G$ is an absorbing game by Theorem 3.14 (Absorbing Game Construction). \square

Corollary 3.17. *If $N > 2$ and G is undetermined, then $(N^2 - 2N) \cdot G$ is an absorbing game.*

Proof. Note that G is at least 1-undetermined, so that $(N - 2) \cdot G$ is $(N - 2)$ -undetermined by Proposition 3.12. Then apply Corollary 3.16. \square

3.4. N -player Nim

In this subsection, we apply generalizations of Lemmas 2.31 (periodicity) and 2.32 (stability) lemmas to obtain some results about NIM that hold for general N .

In this subsection only, it is particularly helpful at times to write complements of outcomes. For example, if $N = 5$, then $\{\mathbf{O}_2, \mathbf{O}_3\} = \{\mathbf{N}, \mathbf{O}_1, \mathbf{P}\}$.

3.4.1. Periodicity

Definition 3.18. A game G is said to be N -periodic if the infinite sequence of outcomes $(o(G), o(* + G), o(2 \cdot * + G), \dots)$ has period N .

Proposition 3.19 (N -Player Nim Periodicity). *Let G be a game. Suppose that all options G' are N -periodic, and that $o(G) = o(N \cdot * + G)$. Then G is N -periodic.*

The proof is almost identical to that of three-player Lemma 2.31 (Nim Periodicity).

Proof. Let $k \geq N + 1$. Then the options of $k \cdot * + G$ are $(k - 1) \cdot * + G$ and those of the form $k \cdot * + G'$ for options G' . These have the same outcomes as $(k - N - 1) \cdot * + G$ (by induction on k) and $(k - N) \cdot * + G'$ (since G' is N -periodic), respectively. But those are exactly all of the options of $(k - N) \cdot * + G$. Since $k \cdot * + G$ has the same outcomes of options, $o(k \cdot * + G) = o((k - N) \cdot * + G)$, as desired. \square

Proposition 3.20. $*2$ is N -periodic.

Proof. First, note that 0 is N -periodic since $o(N \cdot *) = o(0) = \overline{\{\mathbf{N}\}}$ as there is only one line of play (or see claim 2 of Proposition 3.2). In general, $o(i \cdot *) = \overline{\{\mathbf{O}_i\}}$ for $i < N$.

Since 0 is N -periodic, $*$ is N -periodic as well, so that all options of $*2$ are N -periodic. Therefore, by Lemma 2.31 (Nim Periodicity), it suffices to show that $o(*2 + N \cdot *) = o(*2)$.

We show by induction on i that for $0 \leq i \leq N - 3$, $o(*2 + i \cdot *) = \overline{\{\mathbf{O}_{i+1}, \mathbf{O}_{i+2}\}}$. Note that if $N = 2$, then this is vacuously true. If $i = 0$, then Next can choose which of \mathbf{O}_1 and \mathbf{O}_2 will lose; $o(*2) = \overline{\{\mathbf{O}_1, \mathbf{O}_2\}}$.

For $i > 0$, note that Next can move to $*2 + (i - 1) \cdot *$, which has outcome $\overline{\{\mathbf{O}_i, \mathbf{O}_{i+1}\}}$, by induction. Since $i \leq N - 3$, we have $i + 2 \leq N - 1$, so that $\mathbf{O}_{i+1}, \mathbf{O}_{i+2} \notin o(*2 + i \cdot *)$. The other options of $*2 + i \cdot *$ are $i \cdot *$ and $(i + 1) \cdot *$, which have outcomes $\overline{\{\mathbf{O}_i\}}$ and $\overline{\{\mathbf{O}_{i+1}\}}$, respectively. Aside from the three players Next, \mathbf{O}_{i+1} , and \mathbf{O}_{i+2} , all other players have winning strategies. And since $i \neq N - 1$, the move to $i \cdot *$ is a winning move (i.e. $\mathbf{P} \in o(i \cdot *)$), and so $\mathbf{N} \in o(*2 + i \cdot *)$. Thus, $o(*2 + i \cdot *) = \overline{\{\mathbf{O}_{i+1}, \mathbf{O}_{i+2}\}}$, as desired.

Similarly, $o(*2 + (N - 2) \cdot *) = \overline{\{\mathbf{P}\}}$ since the options have outcomes $\overline{\{\mathbf{O}_{N-2}\}}$ (which includes \mathbf{P}), $\overline{\{\mathbf{P}\}}$, and (if $N > 2$) $\overline{\{\mathbf{O}_{N-2}, \mathbf{P}\}}$, respectively.

Then $o(*2 + (N - 1) \cdot *) = \overline{\{\mathbf{O}_1\}}$ since the options have outcomes $\overline{\{\mathbf{P}\}}$, $\overline{\{\mathbf{N}\}}$ (which includes \mathbf{P}), and $\overline{\{\mathbf{P}\}}$, respectively.

Finally, we consider $o(*2 + N \cdot *)$. The options have outcomes $\overline{\{\mathbf{N}\}}$, $\overline{\{\mathbf{O}_1\}}$, and $\overline{\{\mathbf{O}_1\}}$, respectively. Thus, if $N > 2$, $o(*2 + N \cdot *) = \overline{\{\mathbf{O}_1, \mathbf{O}_2\}} = o(*2)$. (If $N = 2$, $o(*2 + N \cdot *) = \overline{\{\mathbf{O}_1\}} = \overline{\{\mathbf{N}\}} = o(*2)$.) \square

Corollary 3.21. *The periodic sequence of outcomes $o(*2 + i \cdot *)$ is*

$$\left(\overline{\{\mathbf{O}_1, \mathbf{O}_2\}}, \overline{\{\mathbf{O}_2, \mathbf{O}_3\}}, \dots, \overline{\{\mathbf{O}_{N-2}, \mathbf{P}\}}, \overline{\{\mathbf{P}\}}, \overline{\{\mathbf{O}_1\}}, \dots \right).$$

Proof. This follows from the proof of Proposition 3.20 above. \square

3.4.2. Stability

Proposition 3.22 (*N-player Nim Stability*). *Let G be a game. Suppose that for all subpositions H of G (including G itself) $o(H + *(N + 1)) = o(H + *N)$. Then $o(G + *m) = o(G + *N)$ for all $m \geq N$.*

Proof. The proof is identical to that of Lemma 2.32 (Three-Player Nim Stability), except that 4 should be replaced with $k + 1$. \square

In fact, the proof would work with $*N$ replaced with any $*k$. However, when G is a “small” NIM position, it seems that we first observe stability at $*N$.

We can use Lemma 3.22 (Nim Stability) to calculate the outcomes of all NIM positions with at most two heaps, assuming $N > 2$. The results are collected in Theorem 3.27 and can be summarized as follows: Players other than Next and \mathbf{O}_1 do not have a winning strategy unless the game is guaranteed to end before their turn. Ignoring the case of the empty game 0, Next fails to have a winning strategy exactly when there are two heaps of size at least $N - 1$, and \mathbf{O}_1 fails to have a winning strategy when there is only one heap or if the sizes of the two heaps are at least $N - 1$ and N , respectively.

Proposition 3.23. *For $i \geq 1$, $o(*i) = \overline{\{\mathbf{O}_1, \dots, \mathbf{O}_{\min\{i, N-1\}}\}}$.*

Proof. We use induction on i for $1 \leq i < N$.

If $i = 1$, then there is only one line of play, and the outcome is certainly $\overline{\{\mathbf{O}_1\}}$. For $1 < i < N$, the options are 0 and all $*j$ with $1 \leq j \leq i - 1$. The former has outcome $\overline{\{\mathbf{N}\}}$, and the latter have outcomes of the form $\overline{\{\mathbf{O}_1, \dots, \mathbf{O}_j\}}$. Since $\mathbf{P} \in o(0)$, $\mathbf{N} \in o(*i)$. Since $\mathbf{N} \notin o(0)$, $\mathbf{O}_1 \notin o(*i)$. And since $\mathbf{O}_1, \dots, \mathbf{O}_{i-1} \notin o(*(i - 1))$, $\mathbf{O}_2, \dots, \mathbf{O}_i \notin o(*i)$. Thus, $o(*i) = \overline{\{\mathbf{O}_1, \dots, \mathbf{O}_{\min\{i, N-1\}}\}}$.

Similarly, $o(*N) = \overline{\{\mathbf{O}_1, \dots, \mathbf{O}_{N-1}\}} = \overline{\{\mathbf{N}\}}$ by the same argument as for $*(N - 1)$ except that it doesn't matter that $\mathbf{P} \notin o(*(N - 1))$. By the same argument, or by the argument in the proof of Lemma 3.22 (Nim Stability), this extends to all larger heaps as well. \square

Lemma 3.24. *If $1 \leq i \leq \min\{j, N-2\}$ then $o(*i+*j) = \overline{\{\mathbf{O}_2, \dots, \mathbf{O}_{\min\{i+j, N-1\}}\}}$.*

Proof. Note that the hypothesis forces $N > 2$. First, $o(*+*) = \overline{\{\mathbf{O}_2\}}$ since there is only one line of play.

Now suppose $i + j > 2$. Then the options of $*i + *j$ are $*i, *j$, and all smaller sums of two nonempty heaps of the forms $(*i)' + *j$ and $*i + (*j)'$.

Note that $\mathbf{O}_1 \notin o(*i)$ by Proposition 3.23, so that $\mathbf{O}_2 \notin o(*i + *j)$.

Since $i + j > 2$, either $1 \leq i \leq j - 1$ or $1 \leq j - 1 \leq i$. Thus, by induction, $o(*i + *(j - 1)) = \overline{\{\mathbf{O}_2, \dots, \mathbf{O}_{\min\{i+j-1, N-1\}}\}}$. Hence, $\mathbf{O}_3, \dots, \mathbf{O}_{\min\{i+j, N-1\}} \notin o(*i + *j)$.

Therefore, $o(*i + *j) \subseteq \overline{\{\mathbf{O}_2, \dots, \mathbf{O}_{\min\{i+j, N-1\}}\}}$.

Since $i \leq N - 2$, $\mathbf{P} = \mathbf{O}_{N-1} \in o(*i)$ by Proposition 3.23, so that $\mathbf{N} \in o(*i + *j)$.

Finally, note that \mathbf{N} and (if $i + j \leq N - 1$) $\mathbf{O}_{\min\{i+j, N-1\}}, \dots, \mathbf{O}_{N-1}$ are included in the outcome of each option (either by Proposition 3.23 or induction, as applicable). Thus, \mathbf{O}_1 and (if $i + j < N - 1$) $\mathbf{O}_{\min\{i+j+1, N-1\}}, \dots, \mathbf{O}_{N-1}$ are elements of $o(*i + *j)$. In other words, $\overline{\{\mathbf{O}_2, \dots, \mathbf{O}_{\min\{i+j, N-1\}}\}} \subseteq o(*i + *j)$.

Since we have shown both inclusions, $o(*i + *j) = \overline{\{\mathbf{O}_2, \dots, \mathbf{O}_{\min\{i+j, N-1\}}\}}$. \square

Lemma 3.25. $o(2 \cdot *(N - 1)) = \{\mathbf{O}_1\}$.

Proof. If $N = 2$, then $o(2 \cdot *) = \{\mathbf{P}\} = \{\mathbf{O}_1\}$ since there is only one line of play.

Now assume $N > 2$. Then the options of $2 \cdot *(N - 1)$ are $*(N - 1)$ (with outcome $\{\mathbf{N}\}$ by Proposition 3.23), and those with two heaps (all of which have outcome $\overline{\{\mathbf{O}_2 \cdots \mathbf{O}_{N-1}\}} = \{\mathbf{N}, \mathbf{O}_1\}$ by Lemma 3.24). Since no option's outcome includes \mathbf{P} (as $N > 2$) and the only element of an outcome shared by all options is \mathbf{N} , $o(2 \cdot *(N - 1)) = \{\mathbf{O}_1\}$. \square

Recall (see Definition 2.6) that a game G is said to be *undetermined* if $o(G) = \emptyset$.

Lemma 3.26. *If $N > 2$, then for $N - 1 \leq i \leq N + 1$ and $N \leq j \leq N + 1$, $*i + *j$ is undetermined.*

Proof. Firstly, we consider $*(N - 1) + *N$. It has two single-heap options each with outcome $\{\mathbf{N}\}$ by Proposition 3.23, the option $2 \cdot *(N - 1)$ with outcome $\{\mathbf{O}_1\}$ by Lemma 3.25, and various options (since $N > 2$) covered by Lemma 3.24 with outcome $\{\mathbf{N}, \mathbf{O}_1\}$. Since $N > 2$, $\mathbf{O}_1 \neq \mathbf{P}$, so that $o(*(N - 1) + *N) = \emptyset$.

Similarly, $*N + *N$ and $*(N - 1) + *(N + 1)$ have all the same sorts of options except that $*N + *N$ lacks “ $2 \cdot *(N - 1)$ ” and both have a new move to the undetermined $*(N - 1) + *N$. Thus, $o(*N + *N) = o(*(N - 1) + *(N + 1)) = \emptyset$.

$*N + *(N + 1)$ is similar to $*N + *N$, but adds the undetermined options $*N + *N$ and $*(N - 1) + *(N + 1)$. And $*(N + 1) + *(N + 1)$ is similar to $*N + *(N + 1)$, but adds the undetermined option $*N + *(N + 1)$. \square

Theorem 3.27. *If $N > 2$, then for $i \leq j$,*

$$o(*i + *j) = \begin{cases} \overline{\{\mathbf{N}\}} & \text{if } i = j = 0 \\ \overline{\{\mathbf{O}_1, \dots, \mathbf{O}_{\min\{j, N-1\}}\}} & \text{if } i = 0 < j \\ \overline{\{\mathbf{O}_2, \dots, \mathbf{O}_{\min\{i+j, N-1\}}\}} & \text{if } 1 \leq i \leq N - 2 . \\ \{\mathbf{O}_1\} & \text{if } i = j = N - 1 \\ \emptyset & \text{otherwise} \end{cases}$$

Proof. The first four cases follow immediately from Definition 3.1, Proposition 3.23, Lemma 3.24, and Lemma 3.25, respectively.

The final claim uses Lemma 3.22 (Nim Stability) repeatedly.

First, note that $o(0 + *N) = o(0 + *(N + 1))$ by Proposition 3.23. Similarly, by Lemma 3.24, $o(*i + *N) = o(*i + *(N + 1))$ for $1 \leq i \leq N - 2$. Also, by Lemma 3.26, $o(*(N - 1) + *N) = \emptyset = o(*(N - 1) + *(N + 1))$. With all of these results together, we apply Lemma 3.22 (Nim Stability) to $G = *(N - 1)$ to find that $o(*(N - 1) + *m) = \emptyset$ for all $m \geq N$. It remains to check that any sum of two heaps of size at least N is undetermined.

Since $o(*N + *N) = \emptyset = o(*N + *(N + 1))$ by Lemma 3.26, another application of Lemma 3.22 (Nim Stability) with $G = *N$ yields $o(*N + *m) = \emptyset$ for all $m \geq N$. And since $o(*N + *(N + 1)) = \emptyset = o(*(N + 1) + *(N + 1))$, another application with $G = *(N + 1)$ yields $o(*(N + 1) + *m) = \emptyset$ for all $m \geq N$.

For $j \geq 1$, $*(N + j) + *N$ and $*(N + j) + *(N + 1)$ are both undetermined. Therefore, for $j \geq 1$, we can recursively apply Lemma 3.22 (Nim Stability) to show that $*(N + j) + *m$ is undetermined for $m \geq N$. □

Recall (see Definition 2.22) that a game G is said to be *absorbing* if $G + H$ is undetermined for all H . We can use the results of Subsection 3.3 to build absorbing games from undetermined NIM positions.

Corollary 3.28. *If $N > 2$, then some NIM positions are absorbing games.*

Proof. By Theorem 3.27, $*(N - 1) + *(2N - 3)$ is $(N - 2)$ -undetermined. Thus, by Corollary 3.16, $N \cdot (*(N - 1) + *(2N - 3))$ is an absorbing game. Alternatively, by Lemma 3.26 (or Theorem 3.27), $*(N - 1) + *N$ is undetermined. Thus, by Corollary 3.17, $(N^2 - 2N) \cdot (*(N - 1) + *N)$ is absorbing. Alternatively, let k be the least integer greater than or equal to $N/2 - 1$. By Lemma 3.26 (or Theorem 3.27), $2 \cdot *N$ is 2-undetermined. By Proposition 3.12, $2k \cdot *N$ is $(N - 2)$ -undetermined. By Corollary 3.16, $2kN \cdot *N$ is an absorbing game. Thus, $(N^2 - N) \cdot *N$ is absorbing, too. □

4. Partizan Games

In this section, we discuss the extension of normal play to the N -player partizan setting. As in Section 3, throughout this entire section, N refers to the number of players ($N \geq 2$). And we explicitly mention when assuming $N > 2$ is necessary.

We begin by setting up notation and machinery to discuss and compare partizan games. In the following subsection, we prove some general results about comparing games. Finally, we consider some specific games generalizing integers in the two-player case.

4.1. Preliminaries

Throughout this section, a (partizan) game is now an ordered N -tuple of finite sets of partizan games. They are still “short” in that they are finite and nonloopy, so that there is a bound on the length of a run. We continue to use \cong to indicate that two games are isomorphic (i.e. their game trees with labeled edges are isomorphic). In general, our definitions and notation parallel the two-player standard as in Section II.1 of [8]. Similar decisions were made in [1], [2], and [3].

The N players are named “Left”, “Center₁”, “Center₂”, . . . , “Center _{$N-2$} ”, and “Right”. They each make moves in a cyclic fashion, with “Left” moving after “Right” and before “Center₁”, etc.

We write some games with an extension of the bar notation commonly used for two players. For example, $G \cong \{ \mathcal{G}^L \mid \mathcal{G}^{C_1} \mid \mathcal{G}^{C_2} \mid \dots \mid \mathcal{G}^{C_{N-2}} \mid \mathcal{G}^R \}$ or

$$G \cong \left\{ G_1^L, \dots, G_{m_0}^L \mid G_1^{C_1}, \dots, G_{m_1}^{C_1} \mid \dots \mid G_1^R, \dots, G_{m_{N-1}}^R \right\}.$$

When convenient, we use “ C_0 ” in place of L and “ C_{N-1} ” in place of R . In [1] and [3], which only consider three-player games, our Center₁ is called “Center” and our C_1 is “ C ”.

If it is Left’s turn to move in G , then they choose one of the *Left options* in \mathcal{G}^L to move to. Analogously for the other players.

If there is no option available to a player on their turn, we declare them the unique loser, and the other $N - 1$ players all win equally. As in the impartial case, we call this convention *normal play*.

As it should not cause undue confusion, we use 0 in partizan contexts to denote a game with no options $0 \cong \{ \mid \dots \mid \}$. In general, we interpret impartial games as partizan ones in the natural way; $*$ $\cong \{ 0 \mid 0 \mid \dots \mid 0 \}$, etc. Following [1], we define $1_L \cong \{ 0 \mid \dots \mid \}$, and analogously 1_{C_i} and 1_R . Note that we never use \parallel or similar to denote “options of options”, as it could be confused with notation like $\{ 0 \mid \mid 0 \}$, which denotes a particular three-player game with no Center₁ options.

4.1.1. Outcomes, Sums, and Conjugates

In order to speak of the outcome of a partizan game, it is helpful to examine the impartial games that result when we select a player to make the first move.

Definition 4.1. Given a partizan game G , we recursively define N impartial games, the *restrictions* of G , as follows. For $0 \leq i \leq N - 2$, the *Center_i restriction* is $[G]_{C_i} \cong \left\{ \left[G_j^{C_i} \right]_{C_{i+1}} \right\}$, where $G_j^{C_i}$ ranges over the options of G for *Center_i*. The *Right restriction* is $[G]_R \cong \left\{ \left[G_j^R \right]_L \right\}$, where G_j^R ranges over the options of G for *Right*.

Recall that if $i = 0$ then *Center_i* is *Left*. For examples, $[1_L]_L = *$, and $[1_L]_{C_1} = 0$.

The outcome of a two-player game is determined by the pair of impartial outcomes when *Left* or *Right* make the first move. We define the outcome in the N -player case analogously.

Definition 4.2. The *outcome* $o(G)$ of a partizan game G is the ordered N -tuple of the impartial outcomes of the restrictions.

$$o(G) = \left(o([G]_L), o([G]_{C_1}), \dots, o([G]_{C_{N-2}}), o([G]_R) \right).$$

For example, $(\mathcal{N}, \mathcal{P}, \mathcal{O})$ is the three-player outcome of games where *Left* and only *Left* can secure the win, no matter which player moves first. And the outcome $(\{\mathbf{N}, \mathbf{P}\}, \{\mathbf{N}, \mathbf{P}\}, \{\mathbf{N}, \mathbf{P}\}, \{\mathbf{N}, \mathbf{P}\})$ is the four-player outcome of games where the next player and the previous player can each guarantee that they win (and if they play well, they will certainly win together with some other player), no matter which player moves first.

Proposition 4.3. *If $N > 2$, all $(2^N - 1)^N$ potential outcomes are possible.*

For instance, there are 343 possible outcomes of a three-player partizan game, and 50625 possible outcomes of a four-player partizan game.

Proof. Recall that by claim 4 of Proposition 3.2, there are $2^N - 1$ possible impartial outcomes. Given impartial outcomes o_1, \dots, o_N , choose corresponding impartial games with those outcomes G_1, \dots, G_N . Then, using $N - 1$ pairs of braces in each component, define $G \cong \{ \{ \dots \{ G_1 \} \dots \} \mid \dots \mid \{ \dots \{ G_N \} \dots \} \}$. Then $o(G) = (o_1, \dots, o_N)$ by construction. \square

Very analogously to the two-player case, we define the three-player analogue of the disjunctive sum.

Definition 4.4. Let G and H be games. The (*disjunctive*) *sum* $G + H$ is defined recursively by

$$G + H \cong \{ G^L + H, G + H^L \mid G^{C_1} + H, G + H^{C_1} \mid \dots \mid G^R + H, G + H^R \}.$$

The *conjugates* of a game G serve as N -player analogues of the two-player negative $-G$, in the sense that the players' roles are interchanged. As these do not serve the role of an additive inverse, they are reminiscent of the general two-player conjugate in [4], of which the misère adjoint defined in V.6.3 of [8] is a special case.

Definition 4.5. The *first conjugate* of G , denoted by G^\dagger , is defined by

$$G^\dagger \cong \left\{ (G^R)^\dagger \mid (G^L)^\dagger \mid (G^{C_1})^\dagger \mid \dots \mid (G^{C_{N-2}})^\dagger \right\}.$$

In general, the k^{th} conjugate of G is obtained by taking the first conjugate k times, as in $G^{\dagger \dots \dagger}$. We denote the sum of the first $N - 1$ conjugates $G^\dagger + G^{\dagger\dagger} + \dots + G^{\dagger \dots \dagger}$ by G^- .

Note that the N^{th} conjugate of G is G itself. For example, $1_{C_1} \cong 1_L^\dagger$ and 1_R is the $(N - 1)^{\text{st}}$ conjugate of 1_L . If $N = 4$, then $1_{C_2}^- = 1_R + 1_L + 1_{C_1}$.

Proposition 4.6. For all games G , $\mathbf{N} \notin o([G + G^-]_x)$ for $x = C_0, \dots, C_{N-1}$.

Proof. Analogously to Proposition 3.5, the players who do not move next can mirror all moves in the other $N - 1$ terms of the sum. □

Note that there is nothing like an absorbing game in the partizan context.

Proposition 4.7. There is no game G with $o(G + H) = (\emptyset, \dots, \emptyset)$ for all H .

Proof. Let G be a game, and define $H \cong k \cdot 1_L$, where k is greater than the depth of the game tree of G . Then Left can win $G + H$, no matter which player moves first, by simply making all of their moves in H . □

4.1.2. Inequalities

In the two-player setting, we define a preorder by (misère or normal play) favorability to Left. With N players, we similarly define preorders by favorability to an individual player. Our definition is reminiscent of, but incompatible with (see Corollary 4.50), the definitions in [1]. The definition parallels the characterization for two-player inequality in Proposition V.6.1 of [8]. It was independently defined, in essentially this form, in [3].

Definition 4.8. We write $G \leq_L H$ if, for $1 \leq i \leq N$, Left has a winning strategy in $H + X$ moving i^{th} whenever Left has a winning strategy in $G + X$ moving i^{th} . We also write $G \not\leq_L H$ for the negation of $G \leq_L H$. We define \leq_{C_i} and \leq_R analogously.

For example, if $N = 3$, $G \leq_L H$, and $\mathbf{O} \in o([G + X]_R)$, then $\mathbf{O} \in o([H + X]_R)$.

Many results below are phrased only in terms of \leq_L , but they apply equally well, *mutatis mutandis*, to the other relations.

Proposition 4.9. \leq_L is transitive and reflexive.

Proof. Both properties follow immediately from the corresponding properties of implication. \square

Proposition 4.10. If $G \leq_L H$ then $G + J \leq_L H + J$ for any J .

Proof. Suppose that Left can win $(G + J) + X \cong G + (J + X)$ moving i^{th} . Then since $G \leq_L H$, Left can win $H + (J + X) \cong (H + J) + X$ moving i^{th} . \square

Corollary 4.11. If $G \leq_L H$ and $J \leq_L K$ then $G + J \leq_L H + K$.

We define other related notation as well.

Definition 4.12. We write $G <_L H$ if $G \leq_L H$ but $H \not\leq_L G$. We define the strict inequalities for the other players analogously.

Proposition 4.13. If $G <_L H$ and $H \leq_L J$, then $G <_L J$. Similarly, if $G \leq_L H$ and $H <_L J$, then $G <_L J$.

Proof. We prove only the first claim, as the second claim is similar. By Proposition 4.9, $G \leq_L J$, so it suffices to show that $J \not\leq_L G$. Since $G <_L H$, choose X and a play order so that Left can win $H + X$ moving i^{th} but does not have a strategy to win $G + X$ moving i^{th} . Then, since $H \leq_L J$, Left can win $J + X$ moving i^{th} as well. This witnesses that $J \not\leq_L G$. \square

Definition 4.14. We write $G =_L H$ and say that G and H are equal for Left if $G \leq_L H$ and $H \leq_L G$. We define $=_{C_i}$ and $=_R$ analogously.

Proposition 4.15. Note that if this definition is applied to the two-player case ($N = 2$), then $G =_L H$ implies $G =_R H$.

Proof. If Left does not have a winning strategy for $G + X$ under some play order, then Right does under the same play order. Similarly, if Left does have a winning strategy, then Right does not. \square

As is standard, we define $=$ by the fundamental equivalence (see [8]).

Definition 4.16 (Partizan Equality). $G = H$ if $o(G + X) = o(H + X)$ for all X .

Proposition 4.17 (Components of Equality). $G = H$ exactly when $G =_{C_i} H$ for $0 \leq i \leq N - 1$.

Proof. Let X be a game. If $G =_L H$, both $o(G + X)$ and $o(H + X)$ have the same membership (or lack thereof) of \mathbf{N} in the first component, \mathbf{P} in the second component, and an appropriate \mathbf{O}_i in each other component. Each other $=_{C_i}$ covers another potential element in each component, and $=_R$ covers the last potential

element in each component. Thus, $o(G + X) = o(H + X)$. Since X was arbitrary, $G = H$.

The proof of the other direction is immediate. □

Corollary 4.18. *If $G = H$ and $J = K$ then $G + J = H + K$.*

Proof. This follows from Proposition 4.17 (Components of Equality) and $2N$ applications of Corollary 4.11. □

4.2. Inequality Results

As in the impartial case, the lack of provisos in the definition of a partizan outcome allows us to generalize some two-player normal-play results to the N -player setting. Since the player who cannot move is the unique loser, more moves available for Left can never hurt them due to timing issues.

We first characterize comparisons with the game 0 and apply this to a study of G^- , and then consider more general inequalities.

Many results below are stated only for Left, using \leq_L and $=_L$. But, by symmetry, there are corresponding results for each other player.

4.2.1. Comparisons with Zero

We begin by confirming some general properties of preordered monoids.

Lemma 4.19. *Suppose $H + J \leq_L 0$. If $0 \leq_L H$, then $J \leq_L 0$.*

Proof. If $0 \leq_L H$, then by Proposition 4.10 we may add J to both sides to obtain $J \leq_L H + J \leq_L 0$. □

Proposition 4.20. *Suppose $H + J \leq_L 0$. If $0 <_L H$ then $J <_L 0$.*

Proof. If $0 <_L H$ then $0 \leq_L H$ so that $J \leq_L 0$ by Lemma 4.19. But if $J =_L 0$, then $0 \leq_L J$, so that Lemma 4.19 yields $H \leq_L 0$, which would contradict $0 <_L H$. □

Proposition 4.21. *Suppose $H + J <_L 0$. If $0 \leq_L H$ then $J <_L 0$.*

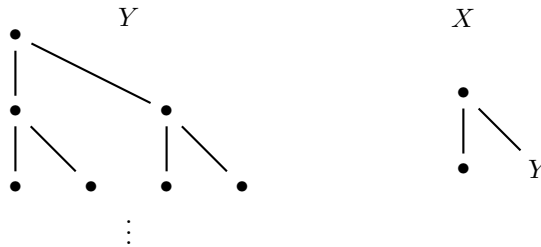
Proof. By Lemma 4.19, $J \leq_L 0$. Since $0 \not\leq_L H + J$ and $0 \leq_L H$, the contrapositive of Corollary 4.11 yields $0 \not\leq_L J$. Thus, $J <_L 0$. □

Next, we examine which games G satisfy $0 \leq_L G$. In the proofs of Lemmas 4.22 and 4.23 to follow, we take inspiration from Theorem 7 of [5].

Lemma 4.22. *If $N > 2$ and a player other than Left or Right has an option in G , then $0 \not\leq_L G$.*

Proof. Suppose that Center_i has an option in G for some i with $1 \leq i \leq N - 2$. Choose m greater than the depth of the game tree of G , and define the games $Y \cong m \cdot (1_L)^-$ (see Definition 4.5) and X to be the game with the two options $X^{C_i} \cong 0$ and $X^{C_{i+1}} \cong Y$.

Example game trees for the case of $N = 3$ are shown below.



Since $N > 2$, Center_{i+1} is not Left. Note that Left can win X with Center_i moving first, since Center_{i+1} loses in the only line of play. But in $G + X$, Left cannot guarantee a win with Center_i moving first.

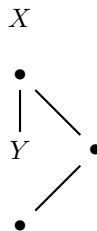
If Center_i moves to some $G^{C_i} + X$, then Center_{i+1} can move to $G^{C_i} + Y$, and then the players other than Left can outlast all of Left's available moves in subpositions of G .

Since Left has a winning strategy in $0 + X$ but not $G + X$ when Center_i moves first, $0 \not\leq_L G$. □

Lemma 4.23. *If $N > 2$ and Right has an option in G , then $0 \not\leq_L G$.*

Proof. Suppose Right has an option in G . Define Y as in the proof of Lemma 4.22 above, and define the game X to have the two options $X^{C_1} \cong Y$ and $X^R \cong 1_L$.

The case of $N = 3$ is illustrated below.



Note that Left can win X moving second, since Center_1 loses in the only line of play. But in $G + X$, Left cannot guarantee a win moving second.

Right can move to some $G^R + X$. If G^R has no Left option, then Left loses immediately. Otherwise, Left moves to some $G^{RL} + X$, in which case Center_1 can move to $G^{RL} + Y$, and Left will run out of moves first.

Since Left has a winning strategy in $0 + X$ but not $G + X$ when moving second, $0 \not\leq_L G$. □

Proposition 4.24. *If $G \cong 0$ or G only has Left options, then $0 \leq_L G$.*

Proof. Suppose Left can win X moving i^{th} . Then Left can win $G + X$ moving i^{th} by making all moves in X , since the other players have no moves available in G . \square

Theorem 4.25 (Nonnegativity Rule). *If $N > 2$, then $0 \leq_L G$ if and only if $G \cong 0$ or G only has Left options.*

Proof. This is a combination of Lemmas 4.22 and 4.23, and Proposition 4.24. \square

Corollary 4.26. *Suppose that $N > 2$, $G \not\cong 0$, $0 \leq_{C_i} G$, and $0 \leq_{C_j} G$. Then $i = j$.*

Proof. By Theorem 4.25 (Nonnegativity Rule), G only has Center_i options and also only has Center_j options. Since $G \not\cong 0$, we must have $i = j$. \square

Corollary 4.27. *If $N > 2$ and $G = 0$, then $G \cong 0$.*

Proof. By Proposition 4.17 (Components of Equality) and Definition 4.14, both $0 \leq_L G$ and $0 \leq_R G$. A contrapositive of Corollary 4.26 yields $G \cong 0$. \square

Note that this dashes any hopes for a good analogue of the two-player result that $1 + (-1) = 0$. For example, $1_L + (1_L)^- \neq 0$. However, we confirm in Theorem 4.45 that non-isomorphic partizan games *can* be equal.

Theorem 4.28 (Nonpositivity Rule). *$G \leq_L 0$ if and only if $\mathbf{N} \notin o([G]_L)$.*

In other words, if Left doesn't have a winning strategy in G moving first, then G is no better for Left than 0, in any context. In the two-player case, this corresponds to the well-known fact that $G \leq 0$ exactly when Right can win G playing second.

Proof. For one direction, assume $G \leq_L 0$. Then since Left does not have a winning strategy in $0+0$ moving first, Left does not have a winning strategy in $G+0$ moving first.

For the other direction, assume $\mathbf{N} \notin o([G]_L)$ and that Left has a winning strategy in $G + X$ moving i^{th} for some i .

If $2 \leq i \leq N$, then Left can win all relevant $G + X'$ moving $(i - 1)^{\text{st}}$. So, by induction on X , Left can win all X' moving $(i - 1)^{\text{st}}$, and so Left can win X moving i^{th} . Now suppose Left can win $G + X$ moving first.

If a winning move is to some $G^L + X$, then since $\mathbf{N} \notin o([G]_L)$, the players other than Left could play in the G^L component to reach some subposition H with $\mathbf{N} \notin o([H]_L)$ and Left having a winning strategy in $H + X$ moving first. $H \leq_L 0$ by induction on G , so that Left can win X moving first.

In the other case, a winning move is instead to some $G + X^L$. Then Left can win $G + X^L$ moving N^{th} . By induction on X , Left can win X^L moving N^{th} as well, so that X^L would be a winning move in X , too. \square

Corollary 4.29. *If $G \cong \{(1_L)^- \mid \dots \}$ then $G =_L 0$.*

Proof. Note that $0 \leq_L G$ by Proposition 4.24 (or Theorem 4.25 (Nonnegativity Rule) if $N > 2$). But since Left does not win G moving first, $G \leq_L 0$ by Theorem 4.2.1 (Nonpositivity Rule). \square

Proposition 4.30. *If G is comparable to 0 under each \leq_{C_i} , then one of the following conditions holds, and all cases are possible.*

1. $G = 0$.
2. $G <_{C_i} 0$ for all i .
3. $G =_{C_j} 0$ for some j and $G <_{C_i} 0$ for $i \neq j$.
4. $G >_{C_j} 0$ for some j and $G <_{C_i} 0$ for $i \neq j$.

Under Cincotti’s convention for \leq_{C_i} , an additional type of case is possible with precisely two equalities (see Section 4 of [1] and [2]).

Proof. The above cases are all possible with the following examples. Condition 1 is satisfied by 0. Condition 2 is satisfied by $1_L + (1_L)^-$. Condition 3 is satisfied by a game with only one option in which Center $_j$ can move to $(1_{C_j})^-$. Condition 4 is satisfied by 1_{C_j} .

If none of the four conditions were satisfied, then $G \not\cong 0$ and $G \geq_{C_i} 0$ for at least two distinct values of i . But this would contradict Corollary 4.26. \square

We also use the two rules for comparisons with 0 to examine the extent to which G^- is similar to a negative for G .

Proposition 4.31. *For any game G , $G + G^- \leq_L 0$.*

Proof. By Proposition 4.6, $\mathbf{N} \notin o([G + G^-]_L)$. By Theorem 4.2.1 (Nonpositivity Rule), $G + G^- \leq_L 0$. \square

To emphasize, $G + G^- \leq_{C_i} 0$ for any i by symmetry, not just $i = 0$.

Corollary 4.32. *If $N > 2$ and $G \not\cong 0$, then $G + G^- <_L 0$.*

Proof. By Proposition 4.31, $G + G^- \leq_L 0$. Since $G \not\cong 0$, $G + G^-$ must have a Right option, so that $0 \not\leq_L G + G^-$ by Theorem 4.25 (Nonnegativity Rule) (or Lemma 4.23). \square

Corollary 4.33. *Suppose that $N > 2$, $G \not\cong 0$, and $0 \leq_L G$. Then $G^- <_L 0$.*

Proof. This follows from Corollary 4.32 and Proposition 4.21. \square

Intuitively, if G never hurts Left, then Left would prefer no change to giving that same benefit to each of the other players.

Note that we cannot turn this around.

Proposition 4.34. *If $N > 2$ and $G \not\cong 0$, then $0 \not\leq_L G^-$.*

Proof. This is immediate from Theorem 4.25 (Nonnegativity Rule). □

4.2.2. General Inequalities

In the two-player case, there is a recursive characterization of \leq (see Thm. II.1.20 in [8]). There is a similar partial characterization of \leq_L in the N -player setting.

Theorem 4.35 (Inequality Test). *Let $G \cong \{G^L \mid G^{C_1} \mid \dots \mid G^{C_{N-1}}\}$ and $H \cong \{H^L \mid H^{C_1} \mid \dots \mid H^{C_{N-1}}\}$. Suppose that for any option G^L there is a corresponding H^L such that $G^L \leq_L H^L$. Further suppose that for $1 \leq i \leq N - 1$ and for all H^{C_i} there is a corresponding G^{C_i} such that $G^{C_i} \leq_L H^{C_i}$. Then $G \leq_L H$.*

Proof. Let X be a game. First, suppose that Left can win $G + X$ moving first. If Left can win by moving to some $G^L + X$, then they can win $G^L + X$ moving last. Using the assumption, choose an H^L so that Left can win $H^L + X$ moving last. Then Left can win $H + X$ moving first by moving to $H^L + X$. If Left cannot win $G + X$ by moving in G , then Left can win some $G + X^L$ moving last. By induction on X , Left can win $H + X^L$ moving last, and Left can still win $H + X$ moving first.

Now suppose that Left can win $G + X$ moving j^{th} for some j with $2 \leq j \leq N$. Set $i = N - j + 1$. So, with Left moving $(j - 1)^{\text{st}}$, Left can win any $G + X^{C_i}$ and any $G^{C_i} + X$. Then, with Left moving $(j - 1)^{\text{st}}$, Left can win any $H + X^{C_i}$ (by induction on X) or $H^{C_i} + X$ (since every H^{C_i} has a corresponding G^{C_i} with $G^{C_i} \leq_L H^{C_i}$). But those are all of the Center_i options of $H + X$, so Left can win $H + X$ moving j^{th} .

For all starting players, we have shown that Left having a winning strategy in $G + X$ implies they have one in $H + X$, so $G \leq_L H$ by Definition 4.8. □

Proposition 4.36. *Suppose that H is obtained from G by replacing various options with options for the same player that are equal to the original ones. More precisely, suppose that each of the following holds:*

- $G \cong \{\mathcal{G}^{C_0} \mid \mathcal{G}^{C_1} \mid \mathcal{G}^{C_2} \mid \dots \mid \mathcal{G}^{C_{N-2}} \mid \mathcal{G}^{C_{N-1}}\}$
- $H \cong \{\mathcal{H}^{C_0} \mid \mathcal{H}^{C_1} \mid \mathcal{H}^{C_2} \mid \dots \mid \mathcal{H}^{C_{N-2}} \mid \mathcal{H}^{C_{N-1}}\}$
- For $0 \leq i \leq N - 1$,
 - for all $G' \in \mathcal{G}^{C_i}$, there exists $H' \in \mathcal{H}^{C_i}$ satisfying $G' = H'$;
 - for all $H' \in \mathcal{H}^{C_i}$, there exists $G' \in \mathcal{G}^{C_i}$ satisfying $G' = H'$.

Then $G = H$.

Proof. By Proposition 4.17 (Components of Equality), all of the options of G obey all inequalities with the corresponding options of H . For example, if $N > 2$, then $H' \leq_{C_1} G'$ for all $G' \in \mathcal{G}^{C_2}$ and $H' \in \mathcal{H}^{C_2}$. Applying Theorem 4.35 (Inequality Test) $2N$ times and using Proposition 4.17 (Components of Equality) again, results in $G = H$. \square

Theorem 4.37 (Deleting Dominated Options). *Let G be a game with at least two Left options G^{L_1} and G^{L_2} , which satisfy $G^{L_1} \leq_L G^{L_2}$. Suppose that H is obtained from G by removing the Left option G^{L_1} . Then $H =_L G$.*

Proof. Since all the options of H are options of G , the Theorem 4.35 (Inequality Test) yields $H \leq_L G$. Since all the options of G are options of H except for G^{L_1} , and $G^{L_1} \leq_L G^{L_2}$, the Theorem 4.35 (Inequality Test) yields $G \leq_L H$. The claim follows from Definition 4.14 for $=_L$. \square

Note that by Proposition 4.15, this reduces to the standard theorem in the two-player case (see Thm. II.2.4 in [8]).

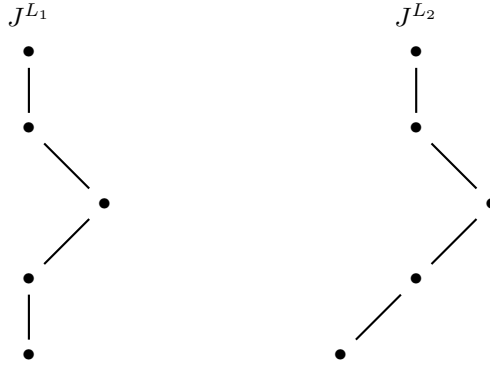
Unfortunately, with more than two players, we only obtain equality for a particular player. In fact, deleting a dominated option need not even preserve the outcome of the game.

Proposition 4.38. *If $N > 2$, there exist games G and H satisfying $G =_L H$ by Theorem 4.37 (Deleting Dominated Options), but $o(G) \neq o(H)$.*

Proof. There exist games J^{L_1} and J^{L_2} such that for the games with no options for players other than Left, $G \cong \{J^{L_1}, J^{L_2} \mid \dots \mid\}$ and $H \cong \{J^{L_2} \mid \dots \mid\}$, we have $G =_L H$, but $o(G) \neq o(H)$.

Take J^{L_1} to be a game which allows each player to move once in turn, starting with Center_1 and ending with Left, and then Center_1 may move once more (so that Center_2 loses). Take J^{L_2} to be the same, except the last option is another move for Left (so that Center_1 loses) instead of a move for Center_1 .

The game trees for the case of $N = 3$ are illustrated below.



By Theorem 4.35 (Inequality Test), $1_{C_1} \leq_L 1_L$. With this as a starting point, we work our way up the game trees to see that multiple applications of Theorem 4.35 (Inequality Test) yield $J^{L_1} \leq_L J^{L_2}$. Thus, by Theorem 4.37 (Deleting Dominated Options), we can delete the dominated option to find $\{J^{L_1}, J^{L_2} \mid \dots \mid\} =_L \{J^{L_2} \mid \dots \mid\}$.

To distinguish the outcomes of the two games, we examine the restrictions $[\{J^{L_1}, J^{L_2} \mid \dots \mid\}]_L$ and $[\{J^{L_2} \mid \dots \mid\}]_L$ to see if Center₂ has a winning strategy.

In $[\{J^{L_2} \mid \dots \mid\}]_L$, the only line of play leads to $[1_L]_{C_1}$. In this case, Center₁ loses, and so Center₂ wins. Thus, $o([\{J^{L_2} \mid \dots \mid\}]_L) = \{\mathbf{N}, \mathbf{O}_2, \dots, \mathbf{O}_{N-1}\}$.

But in $[\{J^{L_1}, J^{L_2} \mid \dots \mid\}]_L$, Left may choose to move to J^{L_1} for a line of play ending with $[1_{C_1}]_{C_1}$ followed by $[0]_{C_2}$. In this case, Center₂ loses. Thus,

$$o([\{J^{L_1}, J^{L_2} \mid \dots \mid\}]_L) = \{\mathbf{N}, \mathbf{O}_3, \dots, \mathbf{O}_{N-1}\} = o([\{J^{L_2} \mid \dots \mid\}]_L) - \{\mathbf{O}_2\}.$$

Since the outcomes of the Left restrictions are not equal, $o(G) \neq o(H)$. □

Theorem 4.39 (Bypassing Reversible Options). *Let G be a game, and suppose that some N^{th} option satisfies $G^{\widehat{L}C_1 \dots \widehat{R}} \leq_L G$. Put*

$$G' \cong \left\{ G^{\widehat{L}C_1 \dots \widehat{R}L}, G^{L'} \mid G^{C_1} \mid \dots \mid G^{C_{N-2}} \mid G^R \right\},$$

where $G^{\widehat{L}C_1 \dots \widehat{R}L}$ ranges over all Left options of $G^{\widehat{L}C_1 \dots \widehat{R}}$, and $G^{L'}$ ranges over all Left options of G except for $G^{\widehat{L}}$. Then $G' =_L G$.

Proof. Suppose Left can win $G + X$ moving first. If they can win by moving to some $G + X^L$, then by induction they could win $G' + X^L$ moving last, so they can win $G' + X$ moving first. And if Left can win by moving to some $G^{L'} + X$ (not $G^{\widehat{L}}$) then they can do the same in $G' + X$.

Define $H \cong G^{\widehat{L}C_1 \dots \widehat{R}} + X$. The interesting case is if Left can win $G + X$ by moving to $G^{\widehat{L}} + X$. Then the other players can respond by moving to H , so Left can win H moving first.

If Left can win H by moving to some $G^{\widehat{L}\widehat{C}_1 \cdots \widehat{R}} + X^L$, then Left can win $G + X^L$ moving last (since $G^{\widehat{L}\widehat{C}_1 \cdots \widehat{R}} \leq_L G$). But then Left could win $G + X$ by moving to $G + X^L$, so we're in the former case again. And if Left can win H by moving to some $G^{\widehat{L}\widehat{C}_1 \cdots \widehat{R}L} + X$, they could do the same in $G' + X$.

Now suppose Left has a winning strategy in $G + X$ moving i^{th} for some i with $2 \leq i \leq N - 1$. Then they can win all $G^{C_{N-i+1}} + X$ and $G + X^{C_{N-i+1}}$ moving $(i - 1)^{\text{st}}$. But $G^{C_{N-i+1}} + X$ are some of the Center_{N-i+1} options of $G' + X$. And, by induction, Left can win the other Center_{N-i+1} options which have the form $G' + X^{C_{N-i+1}}$.

Therefore, $G \leq_L G'$. It remains to show that $G' \leq_L G$.

Suppose Left can win $G' + X$ moving first. Then there are three cases.

If they win by moving to some $G^{L'} + X$, they can do the same in $G + X$. If they win by moving to $G' + X^L$, then by induction they can win by moving to $G + X^L$. If they win by moving to $G^{\widehat{L}\widehat{C}_1 \cdots \widehat{R}L} + X$, this means they could also win $G^{\widehat{L}\widehat{C}_1 \cdots \widehat{R}} + X$ by moving to $G^{\widehat{L}\widehat{C}_1 \cdots \widehat{R}L} + X$. Since $G^{\widehat{L}\widehat{C}_1 \cdots \widehat{R}} \leq_L G$, Left can win $G + X$ moving first.

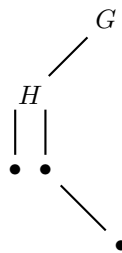
The other cases where Left can win $G' + X$ moving i^{th} for some i with $2 \leq i \leq N - 1$ are routine.

Since we proved both inequalities, $G' =_L G$. □

Unfortunately, as with Theorem 4.37 (Deleting Dominated Options), Theorem 4.39 (Bypassing Reversible Options) need not preserve outcomes (cf. Proposition 4.38).

Proposition 4.40. *If $N > 2$, then there exists a game G such that $G =_L 0$ by Theorem 4.39 (Bypassing Reversible Options), but $o(G) \neq o(0)$.*

Proof. Take $G \cong \{H \mid \cdots \mid\}$ where H only has two options, both for Center_1 ; the options are 0 (to make Center_2 lose) and $1_{C_2} + \cdots + 1_{C_{N-1}}$. The case of $N = 3$ is illustrated below.



Note that $0 \leq_L G$ by Proposition 4.24 (or Theorem 4.25 (Nonnegativity Rule)). Thus, by Theorem 4.39 (Bypassing Reversible Options), we can bypass the Left

option through 0 to find $G =_L 0$. (This could also be verified with Theorem 4.2.1 (Nonpositivity Rule) as in the proof of Corollary 4.29.)

To distinguish the outcomes of the two games, we examine $[G]_L$ and $[0]_L$ to see whether the previous player (Right) has a winning strategy.

In $[0]_L$, Left loses immediately, so $o([0]_L) = \{\mathbf{O}_1, \dots, \mathbf{O}_{N-1}\}$. But in $[G]_L$, Left must move to $[H]_C$. Then Center_1 may choose to move to 0 so that Center_2 loses. Thus,

$$o([G]_L) = \{\mathbf{O}_1, \mathbf{O}_3, \dots, \mathbf{O}_{N-1}\} = o([0]_L) - \{\mathbf{O}_2\}.$$

Since the outcomes of the Left restrictions are not equal, $o(G) \neq o(0)$. □

4.3. Integers

Earlier, we defined games with a single move for various players. For example, $1_{C_1} \cong \{ | 0 | \dots | \}$. Following [1], we can extend these definitions to define various “integers”. We can similarly define $2_L \cong \{ 1_L, 0 | \dots | \}$, $3_L \cong \{ 2_L, 1_L, 0 | \dots | \}$ etc. And analogously for the other players.

In this subsection, as a sort of case study and application of the earlier results, we examine these integers and some related games in detail.

4.3.1. Comparing Integers

Proposition 4.41. *If $0 \leq k < m$, then $k \cdot 1_L <_L m \cdot 1_L$.*

Proof. By Proposition 4.24 (or Theorem 4.25 (Nonnegativity Rule) if $N > 2$), $0 \leq_L 1_L$. Then by Proposition 4.10, we can add $j \cdot 1_L$ to both sides, to obtain $j \cdot 1_L \leq_L (j + 1) \cdot 1_L$ for all j . By Proposition 4.13, a strict inequality propagates, so it remains to show that $(j + 1) \cdot 1_L \not\leq_L j \cdot 1_L$ for all j .

Note that by Proposition 4.6, Left loses $j \cdot 1_L + (j \cdot 1_L)^-$ moving first. But by counting moves, we see that Left wins $(j + 1) \cdot 1_L + (j \cdot 1_L)^-$ moving first. □

Proposition 4.42. *If $1 \leq k$ and $1 \leq i < N$, then $k \cdot 1_{C_i} <_L 0$.*

Proof. Since 1_{C_i} has no Left options, Left loses immediately when moving first, and so $k \cdot 1_{C_i} \leq_L 0$ by Theorem 4.2.1 (Nonpositivity Rule).

If $N = 2$, $k \cdot (-1) \not\leq 0$ is well-known. For $N > 2$, since 1_{C_i} has an option for a player other than Left, Theorem 4.25 (Nonnegativity Rule) yields $k \cdot 1_{C_i} \not\leq_L 0$. Putting both inequalities together, $k \cdot 1_{C_i} <_L 0$. □

Proposition 4.43. *If $N > 2$, $1 \leq i, j \leq N - 1$, $i \neq j$, and $1 \leq k, \ell$, then $k_{C_i} \not\leq_L \ell_{C_j}$.*

By symmetry, this means that integers for distinct non-Left players are incomparable for Left.

Proof. Note that Left wins $k_{C_i} + (1_{C_j})^-$ moving first since Center_{*j*} has no move available. But Left loses $\ell_{C_j} + (1_{C_j})^-$ moving first since Left has no move available after each player has moved once (regardless of which move Center_{*j*} makes). \square

Lemma 4.44. *If $k < m$, then $m \cdot 1_{C_i} <_L k \cdot 1_{C_i}$ for $1 \leq i \leq N - 1$.*

Proof. First, we prove \leq_L . If $k = 0$, then this follows immediately from Theorem 4.35 (Inequality Test) since $m \cdot 1_{C_i}$ has no Left options and $0 \cdot 1_{C_i}$ has no non-Left options. If $k > 0$, then we can add $k \cdot 1_{C_i}$ to both sides of $(m - k) \cdot 1_{C_i} \leq_L 0$ by Proposition 4.10.

To show that the inequality is strict, note that Left wins $k \cdot 1_{C_i} + (m \cdot 1_{C_i})^-$ moving first (since Center_{*i*} runs out of moves), but not $m \cdot 1_{C_i} + (m \cdot 1_{C_i})^-$. \square

Theorem 4.45. *For $k \geq 0$, $k_L = k \cdot 1_L$.*

Proof. $k = 0$ and $k = 1$ are trivial, so we assume $k \geq 2$. By definition, $k_L \equiv \{0, 1_L, 2_L, \dots, (k - 1)_L \mid \dots \mid \}$. By induction and Proposition 4.36 allowing the replacement of options with equals, we may assume $k_L = \{0, 1_L, 2 \cdot 1_L, \dots, (k - 1) \cdot 1_L \mid \dots \mid \}$.

By Proposition 4.41, we see that $(k - 1) \cdot 1_L$ is the best of these options for Left. Thus, we can repeatedly Theorem 4.37 (Delete Dominated Options) in k_L to find $k_L =_L k \cdot 1_L$. By Proposition 4.17 (Components of Equality), it remains to show that $k_L =_{C_i} k \cdot 1_L$ for $1 \leq i \leq N - 1$.

First, we show that $k_L \leq_{C_i} k \cdot 1_L$. Theorem 4.35 (Inequality Test) adapted for \leq_{C_i} says that for any G^{C_i} we need a corresponding H^{C_i} , but neither game has a Center_{*i*} option. And neither game has options for players other than Left and Center_{*i*}, either. So we just need to verify that for all $(k \cdot 1_L)^L$, there is a corresponding k_L^L with $k_L^L \leq_{C_i} (k \cdot 1_L)^L$. But the only option $(k \cdot 1_L)^L$ is $(k - 1) \cdot 1_L$, which is also a Left option of k_L (up to equality), so that $(k - 1) \cdot 1_L \leq_{C_i} (k - 1) \cdot 1_L$ suffices.

Next, we show that $k \cdot 1_L \leq_{C_i} k_L$. Similarly to the other direction, we need only handle each Left option of k_L . It remains to check that $(k - 1) \cdot 1_L \leq_{C_i} j \cdot 1_L$ for $j \leq k - 1$. This follows immediately from Lemma 4.44 with players switched.

Since $k \cdot 1_L \leq_{C_i} k_L$ and $k_L \leq_{C_i} k \cdot 1_L$, we have $k_L =_{C_i} k \cdot 1_L$ for $1 \leq i \leq N - 1$. Since we also verified that $k_L =_L k \cdot 1_L$, Proposition 4.17 (Components of Equality) yields $k_L = k \cdot 1_L$, as desired. \square

4.3.2. Sums of Integers

Proposition 4.46. *Evaluating the outcome of a sum of integers (such as 3_{C_2}) reduces to the case of evaluating the outcome of a game of the form $k_0 \cdot 1_{C_0} + \dots + k_{N-1} \cdot 1_{C_{N-1}}$ for nonnegative integers k_0, \dots, k_{N-1} .*

Proof. Note that by Theorem 4.45, a sum of multiple integers for the same player can be reduced to a sum of ones; for example, $k_L + j_L = (k + j) \cdot 1_L$. Corollary 4.18 tells us that we can make such replacements throughout a sum of integers for various players. By Definition 4.16 (Partizan Equality), the outcome doesn't change when a game is replaced by an equal one. \square

Proposition 4.47. *Let k_0, \dots, k_{N-1} be nonnegative integers with minimum k_{\min} , and i be an integer with $0 \leq i \leq N - 1$. If j is the least nonnegative integer with $k_{i+j} = k_{\min}$ or $k_{i+j-N} = k_{\min}$, then*

$$o\left([k_0 \cdot 1_{C_0} + \dots + k_{N-1} \cdot 1_{C_{N-1}}]_{C_i}\right) = \{\mathbf{O}_0, \dots, \mathbf{O}_{N-1}\} - \{\mathbf{O}_j\}.$$

Proof. Suppose that Center_i moves first in $k_0 \cdot 1_{C_0} + \dots + k_{N-1} \cdot 1_{C_{N-1}}$. There is only one line of play. After Nk_{\min} moves, all coefficients that were equal to k_{\min} have been reduced to 0, so that the next player to move (starting with Center_i) with a corresponding 0 coefficient will lose, and all other players will win. \square

Corollary 4.48. *In a sum of the form $kC_i + K \cdot (1_{C_i})^-$ with $K > k$, Center_i loses regardless of how they play or which player moves first.*

Proof. By Propositions 4.46 and 4.47, all players other than Center_i have winning strategies, regardless of which player moves first. But the players other than Center_i have no choice in their moves, so there is no other way they could play to allow Center_i to win. \square

4.3.3. Games Less than One

In the two-player context, there are results such as $\{-1 \mid \} = 0$, since a move that benefits Right can't help Left. But with three or more players, things are more subtle.

Theorem 4.49. *If $N > 2$, then for $1 \leq i \leq N - 1$,*

$$0 <_L \dots <_L \{3_{C_i} \mid \dots \mid \} <_L \{2_{C_i} \mid \dots \mid \} <_L \{1_{C_i} \mid \dots \mid \} <_L 1_L.$$

Proof. There are four claims that must be verified.

1. $0 \leq_L \{k_{C_i} \mid \dots \mid \}$ for all k .
2. $\{k_{C_i} \mid \dots \mid \} \not\leq_L 0$ for all k .
3. $\{m_{C_i} \mid \dots \mid \} \leq_L \{k_{C_i} \mid \dots \mid \}$ for $k < m$.
4. $\{k_{C_i} \mid \dots \mid \} \not\leq_L \{m_{C_i} \mid \dots \mid \}$ for $k < m$.

Claim 1 follows from Theorem 4.25 (Nonnegativity Rule) (Theorem 4.25).

Claim 2 follows from the fact that Left wins each $\{k_{C_i} \mid \dots \mid\}$ moving first (here we use $N > 2$), but does not win 0 moving first.

Claim 3 follows from Theorem 4.45 to convert the integers into equal sums of 1_{C_i} , Proposition 4.36 to replace them in the options, Lemma 4.44 to compare those sums of 1_{C_i} , and one application of Theorem 4.35 (Inequality Test) to obtain the desired inequality.

Claim 4 requires more care. To witness the inequality, we define a game G differently in the cases $i = 1$ and $i > 1$.

If $i = 1$, then define G to be the game with only one option in which Center_2 can move to $(m - 1) \cdot 1_L + (m - 1) \cdot 1_{C_2} + \sum_{3 \leq j \leq N-1} m \cdot 1_{C_j}$. If $2 \leq i \leq N - 1$,

then define G to be the game with only one option in which Center_1 can move to $(m - 1) \cdot 1_L + (m - 1) \cdot 1_{C_1} + \sum_{2 \leq j \leq N-1, j \neq i} m \cdot 1_{C_j}$.

Either way, Left has a winning strategy in $\{k_{C_i} \mid \dots \mid\} + G$ moving first. The only lines of play lead to a position of the form $\widehat{k}_{C_i} + (m - 1) \cdot (1_{C_i})^-$ for some $\widehat{k} \leq k - 1$, with Left to move. From here, by Corollary 4.48, Center_i will lose even if they play well. However, Left does not have a winning strategy in $\{m_{C_i} \mid \dots \mid\} + G$ moving first. Assuming Center_i plays well, play leads to a position equal to $((m - 1) \cdot 1_L) + ((m - 1) \cdot 1_L)^-$ with Left to move, so that Left does not have a winning strategy by Proposition 4.6. \square

Corollary 4.50. *If $N = 3$, our \leq_L is incompatible with the one defined by Cincotti in Subsection 2.3 of [1].*

Proof. Note that $0 <_L \{1_{C_1} \mid \mid\}$ by Theorem 4.49. But under Cincotti’s recursive definition, we would have $0 \geq_L \{1_{C_i} \mid \mid\}$ since $1_{C_i} \not\leq_L 0$. \square

5. Conclusions

Throughout this paper, we have seen a variety of generalizations of two-player theorems, and regularity in results that apply for $N > 2$. It appears that the normal play convention considered in this paper may be the easiest to investigate for N players, without discarding parts of the game tree (as in [1]) or making any assumptions on how players play (as in [3] and other papers mentioned in I.4 of [8]).

That said, when considering combinatorial games with more than two players, there is a world of gaps and fundamental questions that remain unsettled.

For just the case of impartial three-player games, there are other similar play conventions worth considering, each depending on the winner(s) and loser(s) of 0, analogous to misère play. In [6], Propp analyzed the convention in which Previous is

the unique winner of 0. But that still leaves four other non-trivial play conventions that do not seem to have been investigated in the literature. For example, consider the similar convention in which Other is the unique winner of 0. The two conventions other than normal play with a unique loser for 0 may be particularly difficult to analyze.

Question 1. For each of the remaining four non-trivial play conventions, investigate the sum table analogous to Table 1.

Even when restricting ourselves to normal play, much is still unknown. As noted at the end of Subsection 2.2.2, we can build many equal impartial games by replacing one absorbing subposition with another. Are there other equal games? For instance, perhaps $3 \cdot * = 6 \cdot *$ in the three-player case.

We might hope to find a recursive test for equality similar to the two-player misère Theorem V.3.6 from [8]. However, no simple translation of that result can handle the three-player case under normal play. A key part of that argument is Lemma V.3.3, which uses $T \cong \{G_1^-, G_2^-, \dots, G_k^-, U\}$ and then considers $G + T$ and $H + T$. In that case, options such as $G_i + T$ and $G + G_i^-$ have an option of the form $G_i + G_i^-$ with $\mathbf{N} \notin o(G_i + G_i^-)$. We can therefore conclude that $\mathbf{O} \notin o(G_i + T)$. But without a property such as “ $\mathbf{O} \in o(G_i + G_i^-)$ for all G_i ” or “ $\mathbf{P} \in o(G_i + G_i^-)$ for some G_i ”, we can’t conclude anything further about options of $G + T$ such as $G_i + T$, no matter what we know about $G + U$.

Question 2. For $N > 2$, is there a non-isomorphic pair of equal impartial games that do not have an absorbing subposition?

Even without more ability to test for equality, it would be nice to complete Proposition 2.16 about the outcomes of $3 \cdot G$. While there were no surprises under normal play for doubling, Claim 9 from Section 4 of [6] suggests that there could be another obstruction to the outcome of $3 \cdot G$ aside from the obstructions that apply to all sums and Proposition 2.15 about move mirroring.

Question 3. Under normal play, is there a game G with three-player outcome \mathcal{N} such that $\mathbf{P} \in o(3 \cdot G)$?

Restricting ourselves to the particular game of N -player NIM for $N > 2$, it is not certain how similar things must be to the three-player case.

Question 4. Are all NIM positions N -periodic?

This need not hold *outside* of normal play, even for $N = 3$. For example, in the three-player convention in which Other is the unique winner of 0, $o(2 \cdot *3) = \mathcal{O}$, but $o(3 \cdot * + 2 \cdot *3) = \emptyset$.

In the proof of Corollary 3.28, we show that certain very large NIM positions are absorbing in the context of all impartial games. The author suspects this can be improved considerably.

Question 5. If $N > 2$, do $N \cdot *2$ and $2 \cdot *N$ have the absorbing property in the Nim Quotient?

If so, then $2(N - 1) \cdot *2$ and $N \cdot *N$ are absorbing (for all impartial games) by Theorem 3.14 (Absorbing Game Construction) (Theorem 3.14).

The above is far from an exhaustive list of avenues for future research. Loopy games, nondisjunctive compounds, other case studies such as Rhombination from [3], etc. are all wide open.

Acknowledgments. The author would like to thank the anonymous referee for their helpful suggestions and attention to detail, especially in the case of Subsection 3.3 which underwent significant revision. The author received early suggestions from Richard Nowakowski and Richard Biggs. He also wishes to thank Stephen Goodloe and Gregory Puleo for their feedback and support. Finally, Yuki Irie and Nathan Fox were sources of inspiring questions.

References

- [1] A. Cincotti, Three-player partizan games, *Theoret. Comput. Sci.* **332** (2005), 367-389.
- [2] A. Cincotti, N -player partizan games, *Theoret. Comput. Sci.* **411** (2010), 3224-3234.
- [3] K. A. Doles (née Greene), <http://hdl.handle.net/10339/82171> *Exploration of the Three-Player Partizan Game of Rhombination*, Masters thesis, Wake Forest University, Winston-Salem, 2017.
- [4] U. Larsson, R. J. Nowakowski, and C. P. Santos, Absolute combinatorial game theory, *ArXiv* (2016), <https://arxiv.org/abs/1606.01975> arXiv:1606.01975v1.
- [5] G. A. Mesdal and P. Ottaway, Simplification of partizan games in misère play, *Integers* **7(1)** (2007), G6.
- [6] J. Propp, Three-player impartial games, *Theoret. Comput. Sci.* **233** (2000), 263-278.
- [7] <https://math.stackexchange.com/users/474832/saltSalt>, Extending Conway Games to n players, *Mathematics Stack Exchange* <https://math.stackexchange.com/q/2406302> **2406302** (2017), <https://math.stackexchange.com/visions/2406302/39> revision 39.
- [8] A. N. Siegel, *Combinatorial Game Theory*, Grad. Stud. Math., Amer. Math. Soc. **146**, Providence, 2013.
- [9] J. Wästlund, The strange algebra of combinatorial games, *ArXiv* (2009), <https://arxiv.org/abs/0912.0448> arXiv:0912.0448v1.

Appendix

In this appendix, we follow the compact notation for impartial games used in V.2 of [8] for analyzing two-player misère play.

Specifically:

- $*GHJ$ denotes the game $\{*G, *H, *J\}$; for example, note that $*21$ denotes $\{*2, *\}$, rather than a nim-heap of size twenty-one;
- $*G_{\#}$ denotes the game $\{*G\}$;
- $*G_H$ denotes $*G + *H$.

Some examples of this notation from [8] are as follows.

- $*2_{\#}320 \cong \{\{ *2 \}, *3, *2, 0\}$
- $*2_{\#\#}2 \cong \{\{ *2 \} \} + *2$
- $*2_{2\#\#} \cong \{\{ *2 + *2 \} \}$

The following tables are all for impartial games with three players.

	\emptyset	\mathcal{N}	\mathcal{O}	\mathcal{P}	\mathcal{NO}	\mathcal{OP}	\mathcal{PN}
\emptyset	$*2_{\#}2$	\times	\times	\times	\times	\times	\times
\mathcal{N}	$*3$	$*(1_{\#}1)20$	$*2$	\times	$*(1_{\#}1)(20)0$	\times	\times
\mathcal{O}	$*2_2$	\times	$*2_{\#}$	\times	\times	\times	\times
\mathcal{P}	$*2_21_{\#}$	\times	\times	\times	\times	\times	\times
\mathcal{NO}	$*2_121$	$*21$	$*2_11$	$*(2_10)_{\#}$	$*2_1$	$*(1_{\#}1)1$	$*1_{\#}$
\mathcal{OP}	$*(2_1)_{\#}$	\times	$*(2_11)_{\#}$	$*2_11_{\#}$	\times	0	\times
\mathcal{PN}	$*(2_1)_{\#\#}$	$*2_10$	$*(1_{\#}2_1)_{\#}$	\times	$*$	\times	\times

Table 7: Doubling examples — The row is $o(G)$ and the column is $o(G + G)$.

	\emptyset	\mathcal{O}	\mathcal{P}	\mathcal{OP}
\emptyset	$*2_23$	\times	\times	\times
\mathcal{N}	$*2$	$*((1_{\#}1)1)20$	$?$	$?$
\mathcal{O}	$*2_{\#}$	$*1_{\#}2$	\times	\times
\mathcal{P}	$*2_{\#}1_{\#}$	\times	\times	\times
\mathcal{NO}	$*21$	$*(2_10)_{\#}$	$*(21)_{\#\#\#}$	$*1_{\#}$
\mathcal{OP}	$*2_11_{\#}$	$*(2_10)_{\#\#}$	$*(21)_{\#}$	0
\mathcal{PN}	$*2_{\#}1_{\#}0$	$*2_10$	$*1_{\#}0$	$*$

Table 8: Trebling Examples — The row is $o(G)$ and the column is $o(3 \cdot G)$.

$\emptyset + \emptyset = \emptyset$	$*2_{\#}2$	$*2_{\#}2$	$\mathcal{O} + \mathcal{PN} = \mathcal{N}$	$*2_2$	*
$\emptyset + \mathcal{N} = \emptyset$	$*2_{\#}2$	$*2$	$\mathcal{O} + \mathcal{PN} = \mathcal{P}$	$*2_{\#}$	*
$\emptyset + \mathcal{N} = \mathcal{N}$	$*2_{\#}\#21_{\#}$	$*2$	$\mathcal{O} + \mathcal{PN} = \mathcal{PN}$	$*1_{\#}2$	*
$\emptyset + \mathcal{O} = \emptyset$	$*2_{\#}2$	$*2_{\#}$	$\mathcal{P} + \mathcal{P} = \emptyset$	$*2_{\#}\#$	$*2_{\#}\#$
$\emptyset + \mathcal{P} = \emptyset$	$*2_{\#}2$	$*2_{\#}\#$	$\mathcal{P} + \mathcal{NO} = \emptyset$	$*2_{\#}\#$	$*1_{\#}1$
$\emptyset + \mathcal{NO} = \emptyset$	$*2_22$	$*1_{\#}1$	$\mathcal{P} + \mathcal{NO} = \mathcal{N}$	$*2_21_{\#}$	$*(1_{\#}0)_{\#}$
$\emptyset + \mathcal{NO} = \mathcal{N}$	$*2_22$	$*1_{\#}$	$\mathcal{P} + \mathcal{NO} = \mathcal{O}$	$*2_{\#}\#$	$*1_{\#}$
$\emptyset + \mathcal{NO} = \mathcal{O}$	$*2_{\#}2$	$*1_{\#}1$	$\mathcal{P} + \mathcal{NO} = \mathcal{NO}$	$*2_21_{\#}$	$*1_{\#}$
$\emptyset + \mathcal{NO} = \mathcal{NO}$	$*2_{\#}2$	$*1_{\#}$	$\mathcal{P} + \mathcal{OP} = \emptyset$	$*2_{\#}\#$	$*2_1_{\#}$
$\emptyset + \mathcal{OP} = \emptyset$	$*2_{\#}2$	0	$\mathcal{P} + \mathcal{OP} = \mathcal{P}$	$*2_{\#}\#$	0
$\emptyset + \mathcal{PN} = \emptyset$	$*2_22$	*	$\mathcal{P} + \mathcal{PN} = \emptyset$	$*2_2_{\#}$	$*2_1_{\#}\#$
$\emptyset + \mathcal{PN} = \mathcal{N}$	$*2_{\#}2$	*	$\mathcal{P} + \mathcal{PN} = \mathcal{N}$	$*2_{\#}\#$	*
$\mathcal{N} + \mathcal{N} = \emptyset$	$*2$	$*20$	$\mathcal{NO} + \mathcal{NO} = \emptyset$	$*31$	$*(1_{\#}0)_{\#}$
$\mathcal{N} + \mathcal{N} = \mathcal{N}$	$*2$	$*1_{\#}20$	$\mathcal{NO} + \mathcal{NO} = \mathcal{N}$	$*1_{\#}$	$*1_{\#}1$
$\mathcal{N} + \mathcal{N} = \mathcal{O}$	$*2$	$*2$	$\mathcal{NO} + \mathcal{NO} = \mathcal{O}$	$*2_11$	$*2_11$
$\mathcal{N} + \mathcal{N} = \mathcal{NO}$	$*2$	$*1_{\#}10$	$\mathcal{NO} + \mathcal{NO} = \mathcal{P}$	$*1_{\#}$	$*(1_{\#}0)_{\#}$
$\mathcal{N} + \mathcal{O} = \emptyset$	$*2$	$*2_2$	$\mathcal{NO} + \mathcal{NO} = \mathcal{NO}$	$*1_{\#}1$	$*1_{\#}1$
$\mathcal{N} + \mathcal{O} = \mathcal{N}$	$*2$	$*2_{\#}$	$\mathcal{NO} + \mathcal{NO} = \mathcal{OP}$	$*(1_{\#}1)1$	$*(1_{\#}1)1$
$\mathcal{N} + \mathcal{P} = \emptyset$	$*2_{\#}\#$	$*2_{\#}\#$	$\mathcal{NO} + \mathcal{NO} = \mathcal{PN}$	$*1_{\#}$	$*1_{\#}$
$\mathcal{N} + \mathcal{P} = \mathcal{N}$	$*2$	$*2_{\#}\#$	$\mathcal{NO} + \mathcal{OP} = \emptyset$	$*1_{\#}1$	$*2_1_{\#}$
$\mathcal{N} + \mathcal{NO} = \emptyset$	$*2_{\#}2$	$*1_{\#}$	$\mathcal{NO} + \mathcal{OP} = \mathcal{N}$	$*1_{\#}$	$*(1_{\#}0)_{\#}\#$
$\mathcal{N} + \mathcal{NO} = \mathcal{N}$	$*1_{\#}10$	$*1_{\#}$	$\mathcal{NO} + \mathcal{OP} = \mathcal{O}$	$*1_{\#}$	$*2_1_{\#}$
$\mathcal{N} + \mathcal{NO} = \mathcal{O}$	$*2$	$*2_11$	$\mathcal{NO} + \mathcal{OP} = \mathcal{NO}$	$*1_{\#}$	0
$\mathcal{N} + \mathcal{NO} = \mathcal{P}$	$*2_{\#}\#$	$*1_{\#}$	$\mathcal{NO} + \mathcal{PN} = \emptyset$	$*2_11$	$*1_{\#}0$
$\mathcal{N} + \mathcal{NO} = \mathcal{NO}$	$*20$	$*1_{\#}$	$\mathcal{NO} + \mathcal{PN} = \mathcal{N}$	$*(1_{\#}0)_{\#}$	$*1_{\#}0$
$\mathcal{N} + \mathcal{NO} = \mathcal{OP}$	$*2$	$*(1_{\#}1)1$	$\mathcal{NO} + \mathcal{PN} = \mathcal{O}$	$*(1_{\#}0)_{\#}$	*
$\mathcal{N} + \mathcal{NO} = \mathcal{PN}$	$*2$	$*1_{\#}$	$\mathcal{NO} + \mathcal{PN} = \mathcal{P}$	$*1_{\#}$	$*2_1_{\#}\#$
$\mathcal{N} + \mathcal{OP} = \emptyset$	$*2_{\#}\#$	$*2_1_{\#}$	$\mathcal{NO} + \mathcal{PN} = \mathcal{NO}$	$*2_11$	*
$\mathcal{N} + \mathcal{OP} = \mathcal{N}$	$*2$	0	$\mathcal{NO} + \mathcal{PN} = \mathcal{OP}$	$*1_{\#}$	*
$\mathcal{N} + \mathcal{PN} = \emptyset$	$*2_{\#}\#2$	*	$\mathcal{NO} + \mathcal{PN} = \mathcal{PN}$	$*1_{\#}1$	*
$\mathcal{N} + \mathcal{PN} = \mathcal{N}$	$*1_{\#}10$	*	$\mathcal{OP} + \mathcal{OP} = \emptyset$	$*2_1_{\#}$	$*2_1_{\#}$
$\mathcal{N} + \mathcal{PN} = \mathcal{O}$	$*2_{\#}\#$	*	$\mathcal{OP} + \mathcal{OP} = \mathcal{O}$	$*1_{\#}\#$	$*(1_{\#}0)_{\#}\#$
$\mathcal{N} + \mathcal{PN} = \mathcal{NO}$	$*2$	*	$\mathcal{OP} + \mathcal{OP} = \mathcal{P}$	$*1_{\#}\#$	$*2_1_{\#}$
$\mathcal{O} + \mathcal{O} = \emptyset$	$*2_{\#}$	$*2_2$	$\mathcal{OP} + \mathcal{OP} = \mathcal{OP}$	0	0
$\mathcal{O} + \mathcal{O} = \mathcal{O}$	$*2_{\#}$	$*2_{\#}$	$\mathcal{OP} + \mathcal{PN} = \emptyset$	$*2_1_{\#}$	$*2_1_{\#}\#$
$\mathcal{O} + \mathcal{P} = \emptyset$	$*2_{\#}$	$*2_{\#}\#$	$\mathcal{OP} + \mathcal{PN} = \mathcal{N}$	$*2_1_{\#}$	*
$\mathcal{O} + \mathcal{NO} = \emptyset$	$*2_2$	$*2_1$	$\mathcal{OP} + \mathcal{PN} = \mathcal{P}$	$*(1_{\#}0)_{\#}\#$	$*1_{\#}\#$
$\mathcal{O} + \mathcal{NO} = \mathcal{N}$	$*2_{\#}$	$*1_{\#}$	$\mathcal{OP} + \mathcal{PN} = \mathcal{PN}$	0	*
$\mathcal{O} + \mathcal{NO} = \mathcal{O}$	$*2_2$	$*1_{\#}1$	$\mathcal{PN} + \mathcal{PN} = \emptyset$	$*2_1_{\#}\#$	$*2_1_{\#}\#$
$\mathcal{O} + \mathcal{NO} = \mathcal{NO}$	$*1_{\#}2$	$*1_{\#}$	$\mathcal{PN} + \mathcal{PN} = \mathcal{N}$	*	$*1_{\#}0$
$\mathcal{O} + \mathcal{OP} = \emptyset$	$*1_{\#}2$	$*2_1_{\#}$	$\mathcal{PN} + \mathcal{PN} = \mathcal{O}$	$*1_{\#}\#$	$*2_1_{\#}\#$
$\mathcal{O} + \mathcal{OP} = \mathcal{O}$	$*2_{\#}$	0	$\mathcal{PN} + \mathcal{PN} = \mathcal{NO}$	*	*
$\mathcal{O} + \mathcal{PN} = \emptyset$	$*2_{\#}$	$*1_{\#}0$			

Table 9: Examples of pairs of impartial games witnessing all outcomes of sums