NEW CONGRUENCES FOR 3-REGULAR PARTITIONS WITH DESIGNATED SUMMANDS

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Abstract

In 2002, Andrews, Lewis and Lovejoy introduced and studied the partition function $PD(n)$, the number of partitions of n with designated summands. Recently, congruences involving the number of ℓ-regular partitions with designated summands, denoted by $PD_\ell(n)$, have been explored for $\ell = 2$ and $\ell = 3$. In this paper, we present new arithmetic properties of $PD_3(n)$, including a complete characterization of this function modulo 3.

1. Introduction

A partition of an integer $n \geq 0$ is a non-increasing sequence of positive integers, $\lambda_1 \geq \cdots \geq \lambda_s$, such that $n = \lambda_1 + \cdots + \lambda_s$. The λ_is are called the parts of the partition.

Andrews, Lewis, and Lovejoy [1] introduced and studied many properties of a new class of objects called partitions with designated summands. These partitions are constructed by taking ordinary partitions and tagging exactly one of each part size. For instance, the ten partitions of 4 with designated summands are:

$$4', \;\; 3'+1', \;\; 2'+2, \;\; 2+2', \;\; 2'+1'+1, \;\; 2'+1+1',$$

$$1'+1+1+1, \;\; 1+1'+1+1, \;\; 1+1+1'+1, \;\; 1+1+1+1'.$$

The arithmetic aspects of the number of partitions with designated summands, denoted by $PD(n)$, have been studied in [1, 3, 6, 8, 14]. For instance, in [1] the

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authors proved that for all $n \geq 0$

$$PD(3n+2) \equiv 0 \pmod{3}.$$

In [1], the authors also presented some arithmetic properties of $PD_2(n)$, the number of 2-regular partitions of n with designated summands (a partition of n is called ℓ-regular if there is no part divisible by the positive integer ℓ). Subsequently, in [3], additional congruences satisfied by $PD_2(n)$ were found. In [12], the authors considered $PD_3(n)$, the number of 3-regular partitions of n with designated summands. They proved that the generating function for $PD_3(n)$ is given by

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{(q^6;q^6)_\infty^2 (q^9;q^9)_\infty}{(q;q)_\infty (q^2;q^2)_\infty (q^{18};q^{18})_\infty}, \tag{1}$$

where

$$\begin{aligned}
&(a;q)_0 = 1,\\
&(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \text{for all } n \geq 1,\\
&(a;q)_\infty = \lim_{n\to\infty}(a;q)_n, |q| < 1.
\end{aligned}$$

The authors then proved the following:

Theorem 1 ([12], Theorem 2). *For each $n \geq 0$,*

$$\begin{aligned}
PD_3(6n+3) &\equiv 0 \pmod{4},\\
PD_3(6n+5) &\equiv 0 \pmod{12},\\
PD_3(12n+8) &\equiv 0 \pmod{48},\\
PD_3(24n+4) &\equiv 0 \pmod{9},\\
PD_3(24n+20) &\equiv 0 \pmod{144},\\
PD_3(24n+22) &\equiv 0 \pmod{36},\\
PD_3(48n+38) &\equiv 0 \pmod{12}.
\end{aligned}$$

In this paper, using elementary generating function manipulations, we provide new congruences satisfied by $PD_3(n)$, including a complete characterization of this function modulo 3. Namely, we prove the following:

Theorem 2. *For all $n \geq 0$,*

$$PD_3(n) \equiv \begin{cases} r(n) \pmod{3}, & \text{if } n = k(k+1) + 3m(m+1) + 1,\\ 0 \pmod{3}, & \text{otherwise}, \end{cases} \tag{2}$$

where $r(n)$ is the number of representations of n as $k(k+1) + 3m(m+1) + 1$,

which provides a complete characterization of $PD_3(n)$ modulo 3, and

Theorem 3. *For all $n \geq 0$,*

$$PD_3(24n+16) \equiv 0 \pmod{2}, \tag{3}$$
$$PD_3(9n+6) \equiv 0 \pmod{3}, \tag{4}$$
$$PD_3(32n+8) \equiv 0 \pmod{4}, \tag{5}$$
$$PD_3(36n) \equiv 0 \pmod{4}, \tag{6}$$
$$PD_3(12n+9) \equiv 0 \pmod{8}, \tag{7}$$
$$PD_3(18n+9) \equiv 0 \pmod{8}, \tag{8}$$
$$PD_3(18n+15) \equiv 0 \pmod{8}, \tag{9}$$
$$PD_3(24n+17) \equiv 0 \pmod{8}, \tag{10}$$
$$PD_3(24n+23) \equiv 0 \pmod{8}, \tag{11}$$
$$PD_3(6n+4) \equiv 0 \pmod{9}. \tag{12}$$

This paper is organized as follows. In Section 2, we recall some basic properties of Ramanujan's theta functions $\phi(q)$ and $\psi(q)$ and we also present some useful identities. Section 3 is devoted to proving Theorem 2 and the new congruences (3)–(12) satisfied by $PD_3(n)$. All of the proofs presented involve elementary generating function dissections and manipulations.

2. Preliminaries

We recall Ramanujan's theta functions

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{13}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{14}$$

By Entry 25 (i), (ii), (v), and (vi) in [4, p. 40], we have

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{15}$$

and

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2. \tag{16}$$

Throughout this paper, we define

$$f_k := (q^k;q^k)_\infty$$

in order to shorten the notation. Thus, we can rewrite (1) in the form

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_6^2 f_9}{f_1 f_2 f_{18}}. \tag{17}$$

In the next lemmas, we recall some identities that will be useful in the proof of Theorem 3. The first lemma presents the 2-dissections of f_1^2, certain quotients of powers of f_1 and f_3, and $\frac{1}{f_1^t}$ for $t \in \{2, 4, 8\}$.

Lemma 1. *The following identities hold:*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{18}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{19}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{20}$$

$$\frac{1}{f_1^8} = \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q\frac{f_4^{16}}{f_2^{24}} + 16q^2\frac{f_4^4 f_8^8}{f_2^{20}}, \tag{21}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{22}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{23}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{24}$$

Proof. Identity (18) follows from (16) after replacing q by $-q$. By (13) and (14) we can rewrite (15) in the form

$$\frac{f_2^5}{f_1^2 f_4^2} = \frac{f_8^5}{f_4^2 f_{16}^2} + 2q\frac{f_{16}^2}{f_8},$$

from which we obtain (19) after multiplying both sides by $\frac{f_4^2}{f_2^5}$.

By (13) and (14) we can rewrite (16) in the form

$$\frac{f_2^{10}}{f_1^4 f_4^4} = \frac{f_4^{10}}{f_2^4 f_8^4} + 4q\frac{f_8^4}{f_4^2},$$

from which (20) follows. The identity in (21) is just (20) squared.

Equation (22) appears in [2] as Theorem 4.17. Replacing q by $-q$ in (22) and using the fact that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we obtain (23). Identity (24) is Eq. (30.10.3) in [9]. □

In the next lemma, we recall the 3-dissections of f_1f_2, f_2/f_1^2, $\frac{1}{f_1^3}$, and $\psi(q)$.

Lemma 2. *The following identities hold:*

$$f_1f_2 = \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2}, \tag{25}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4f_9^6}{f_3^8f_{18}^3} + 2q\frac{f_6^3f_9^3}{f_3^7} + 4q^2\frac{f_6^2f_{18}^3}{f_3^6}, \tag{26}$$

$$\begin{aligned}\frac{1}{f_1^3} = \frac{f_9^3}{f_3^{12}}\Bigg\{ & f_3^2\frac{\phi(-q^9)^6}{\phi(-q^3)^2} + 8q^3f_3^2\frac{\phi(-q^9)^3\psi(q^9)^3}{\phi(-q^3)\psi(q^3)} + 16q^6f_3^2\frac{\psi(q^9)^6}{\psi(q^3)^2} \\ & +3qf_3f_9^3\frac{\phi(-q^9)^3}{\phi(-q^3)} + 12q^4f_3f_9^3\frac{\psi(q^9)^3}{\psi(q^3)} + 9q^2f_9^6\Bigg\},\end{aligned} \tag{27}$$

$$\psi(q) = \frac{f_6f_9^2}{f_3f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{28}$$

Proof. Identity (25) was proven in [11]. A proof of (26) can be seen in [10]. Identity (27) is Equation (14) in [7]. Identity (28) follows from the first equality in Corollary (ii) [4, p. 49]. □

The next lemma appears in [13, Eq. 5.1].

Lemma 3. *The following identity holds:*

$$\frac{f_2^4f_3^8}{f_1^8f_6^4} = 1 + 8q\frac{f_2f_6^5}{f_1^5f_3}. \tag{29}$$

The next result is a consequence of Jacobi's triple product identity.

Lemma 4. *The following identity holds:*

$$f_1^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{30}$$

Proof. See Section 1.3 in [5]. □

In order to prove some of the congruences, we make use of the following well-known results in the next lemma, which can be proven using the binomial theorem.

Lemma 5. *Given a prime p and positive integers k and m, we have*

$$\begin{aligned} f_k^{4m} &\equiv f_{2k}^{2m} \pmod 4, \\ f_k^{9m} &\equiv f_{3k}^{3m} \pmod 9, \\ f_k^{pm} &\equiv f_{pk}^{m} \pmod p. \end{aligned}$$

3. Proofs of the New Congruences

This section is devoted to proving Theorem 2 and Theorem 3. In order to do so, we prove some additional identities involving $\sum_{n=0}^{\infty} PD_3(n)q^n$. We begin by recalling Equations (4), (5), (6), (9), and (12) in [12]:

$$\sum_{n=0}^{\infty} PD_3(2n)q^n = \frac{f_3 f_6^3}{f_1^3 f_{18}}, \tag{31}$$

$$\sum_{n=0}^{\infty} PD_3(2n+1)q^n = \frac{f_2^2 f_3^3 f_{18}}{f_1^4 f_6 f_9}, \tag{32}$$

$$\sum_{n=0}^{\infty} PD_3(3n)q^n = \frac{f_3^4 f_6^2}{f_1^4 f_2^2}\left(\frac{f_2^2 f_3^6}{f_1^2 f_6^6} - 2q\frac{f_1 f_6^3}{f_2 f_3^3}\right), \tag{33}$$

$$\sum_{n=0}^{\infty} PD_3(4n)q^n = \frac{f_2^6 f_3^6}{f_1^9 f_6^2 f_9}, \tag{34}$$

$$\sum_{n=0}^{\infty} PD_3(6n+5)q^n = 12\frac{f_2^5 f_3^5 f_6}{f_1^{11}}. \tag{35}$$

3.1. Proof of Theorem 2

Firstly we consider the case when n is even. We shall show that $PD_3(2n) \equiv 0 \pmod 3$. This is a straightforward consequence of using Lemma 5 in (31). Indeed,

$$\sum_{n=0}^{\infty} PD_3(2n)q^n \equiv \frac{f_3 f_{18}}{f_3 f_{18}} \equiv 1 \pmod 3.$$

Thus, for all $n \geq 1$, $PD_3(2n) \equiv 0 \pmod 3$.

Now we consider the case when n is odd. The generating function for $PD_3(2n+1)$ is (32) above. From (14), using Lemma 5, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} PD_3(2n+1)q^n &\equiv \frac{f_2^2 f_6^2}{f_1^4} \equiv \psi(q)\psi(q^3)\\ &\equiv \sum_{k,m=0}^{\infty} q^{\frac{k(k+1)}{2}+3\frac{m(m+1)}{2}} \pmod 3,\end{aligned}$$

from which it follows that

$$PD_3(2n+1) \equiv \begin{cases} r(2n+1) \pmod 3, & \text{if } 2n+1 = k(k+1)+3m(m+1)+1,\\ 0 \pmod 3, & \text{otherwise,}\end{cases}$$

where $r(n)$ is the number of representations of n as $k(k+1)+3m(m+1)+1$. This completes the proof of (2) in the case when n is odd.

3.2. Proof of Theorem 3

By (27) and (31), we have

$$\sum_{n=0}^{\infty} PD_3(2n)q^n = \frac{f_6^3 f_9^3}{f_3^{11} f_{18}} \left\{ f_3^2 \frac{\phi(-q^9)^6}{\phi(-q^3)^2} + 8q^3 f_3^2 \frac{\phi(-q^9)^3 \psi(q^9)^3}{\phi(-q^3)\psi(q^3)} + 9q^2 f_9^6 \right.$$
$$\left. +16q^6 f_3^2 \frac{\psi(q^9)^6}{\psi(q^3)^2} + 3q f_3 f_9^3 \frac{\phi(-q^9)^3}{\phi(-q^3)} + 12q^4 f_3 f_9^3 \frac{\psi(q^9)^3}{\psi(q^3)} \right\}.$$

By taking the terms involving q^{3n+2}, we obtain

$$\sum_{n=0}^{\infty} PD_3(6n+4)q^n = 9\frac{f_2^3 f_3^9}{f_1^{11} f_6}, \tag{36}$$

from which (12) follows directly.

From (36) and using (29), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} PD_3(6n+4)q^n &= 9\frac{f_3 f_6^3}{f_1^3 f_2} \left(1 + 8q\frac{f_2 f_6^5}{f_1^5 f_3}\right) \\ &= 9\frac{f_3 f_6^3}{f_1^3 f_2} + 72q\frac{f_6^8}{f_1^8}.\end{aligned}$$

Now, using (21) and (22), it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} PD_3(6n+4)q^n = {} & 9\frac{f_4^6 f_6^6}{f_2^{10} f_{12}^2} + 27q\frac{f_4^2 f_6^4 f_{12}^2}{f_2^8} + 72q\frac{f_4^{28} f_6^8}{f_2^{28} f_8^8} \\ & + 576q^2\frac{f_4^{16} f_6^8}{f_2^{24}} + 1152q^3\frac{f_4^4 f_6^8 f_8^8}{f_2^{20}}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n=0}^{\infty} PD_3(12n+4)q^n &= 9\frac{f_2^6 f_3^6}{f_1^{10} f_6^2} + 576q\frac{f_2^{16} f_3^8}{f_1^{24}} \\ &\equiv f_2 f_6 \pmod 2,\end{aligned}$$

from which we obtain (3).

The congruence in (4) follows directly from (33). Indeed,

$$\begin{aligned}\sum_{n=0}^{\infty} PD_3(3n)q^n &= \frac{f_3^{10}}{f_1^6 f_6^4} - 2q\frac{f_3 f_6^5}{f_1^3 f_2^3} \\ &\equiv \frac{f_3^8}{f_6^4} - 2q f_6^4 \pmod 3,\end{aligned}$$

from which we see that the coefficients of q^{3n+2} in $\sum_{n=0}^{\infty} PD_3(3n)q^n$ are congruent to 0 modulo 3, completing the proof of (4).

From (34), using (23), we have

$$\begin{aligned}\sum_{n=0}^{\infty} PD_3(4n)q^n &\equiv \frac{f_1^3 f_3^2}{f_9} \equiv \frac{f_1^3}{f_3}\frac{f_3^3}{f_9} \pmod 4\\ &\equiv \left(\frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}\right)\left(\frac{f_{12}^3}{f_{36}} - 3q^3\frac{f_6^2 f_{36}^3}{f_{12} f_{18}^2}\right) \pmod 4\\ &\equiv \frac{f_4^3 f_{12}^2}{f_{36}} - 3q\frac{f_2^2 f_{12}^6}{f_4 f_6^2 f_{36}} - 3q^3\frac{f_4^3 f_6^2 f_{36}^3}{f_{12}^2 f_{18}^2} + 9q^4\frac{f_2^2 f_{12}^2 f_{36}^3}{f_4 f_{18}^2} \pmod 4.\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} PD_3(8n)q^n \equiv \frac{f_2^3 f_6^2}{f_{18}} + q^2\frac{f_1^2 f_6^2 f_{18}^3}{f_2 f_9^2} \pmod 4,$$

from which, using (18) and (19), we deduce

$$\begin{aligned}\sum_{n=0}^{\infty} PD_3(8n)q^n \equiv \frac{f_2^3 f_6^2}{f_{18}} &+ q^2\frac{f_6^2 f_{18}^3}{f_2}\left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}\right)\\ &\times\left(\frac{f_{72}^5}{f_{18}^5 f_{144}^2} + 2q^9\frac{f_{36}^2 f_{144}^2}{f_{18}^5 f_{72}}\right) \pmod 4.\end{aligned}$$

By expanding this product and taking the odd part we are left with

$$\sum_{n=0}^{\infty} PD_3(16n+8)q^n \equiv \left(2q^5\frac{f_4^5 f_{18}^2 f_{72}^2}{f_2^2 f_8^2 f_{36}} - 2q\frac{f_8^2 f_{36}^5}{f_4 f_{72}^2}\right)\frac{f_3^2}{f_9^2} \pmod 4.$$

Using (18) and (19) once again, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} PD_3(16n+8)q^n \equiv &\left(2q^5\frac{f_4^5 f_{18}^2 f_{72}^2}{f_2^2 f_8^2 f_{36}} - 2q\frac{f_8^2 f_{36}^5}{f_4 f_{72}^2}\right)\left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_6 f_{48}^2}{f_{24}}\right)\\ &\times\left(\frac{f_{72}^5}{f_{18}^5 f_{144}^2} + 2q^9\frac{f_{36}^2 f_{144}^2}{f_{18}^5 f_{72}}\right) \pmod 4,\end{aligned}$$

from which we see that the even part of $\sum_{n=0}^{\infty} PD_3(16n+8)q^n$ is congruent to 0 modulo 4, completing the proof of (5).

In order to prove (6), we use Lemma 5 to rewrite (34) modulo 4:

$$\sum_{n=0}^{\infty} PD_3(4n)q^n \equiv \frac{f_2^6 f_3^6}{f_1 f_2^4 f_6^2 f_9} \equiv \frac{f_2^2}{f_1}\frac{f_3^6}{f_6^2 f_9} \equiv \psi(q)\frac{f_3^6}{f_6^2 f_9} \pmod 4.$$

Now, by (28), it follows that

$$\sum_{n=0}^{\infty} PD_3(4n)q^n \equiv \frac{f_3^6}{f_6^2 f_9}\left(\frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}\right) \pmod 4,$$

from which we obtain

$$\sum_{n=0}^{\infty} PD_3(12n)q^n \equiv \frac{f_1^5 f_3}{f_2 f_6} \equiv f_1 f_2 \frac{f_3}{f_6} \pmod 4.$$

Finally, using (25), we extract the terms involving q^{3n}. This yields

$$\sum_{n=0}^{\infty} PD_3(36n)q^n \equiv \frac{f_3^4}{f_6^2} \equiv 1 \pmod 4,$$

which implies (6).

From (32) and using (26) we obtain

$$\sum_{n=0}^{\infty} PD_3(2n+1)q^n = \frac{f_3^3 f_{18}}{f_6 f_9}\left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)^2.$$

Extracting the terms of the form q^{3n+1} on both sides of this identity, we are left with

$$\sum_{n=0}^{\infty} PD_3(6n+3)q^{3n+1} = \frac{f_3^3 f_{18}}{f_6 f_9}\left(2q\frac{f_6^7 f_9^9}{f_3^{15} f_{18}^3} + 16q^4\frac{f_6^4 f_{18}^6}{f_3^{12}}\right),$$

from which it follows that

$$\sum_{n=0}^{\infty} PD_3(6n+3)q^n \equiv 2\frac{f_2^6 f_3^8}{f_1^{12} f_6^2} \equiv 2\frac{f_2^2 f_6^2}{f_1^4} \pmod 8. \tag{37}$$

Now, using (20) we extract the even parts on both sides of this congruence to obtain

$$\sum_{n=0}^{\infty} PD_3(12n+9)q^n \equiv 8\frac{f_2^2 f_3^2 f_4^4}{f_1^8} \pmod 8,$$

which implies (7).

From Lemma 5 we see that

$$\frac{f_2^2 f_6^2}{f_1^4} \equiv f_3^4 \pmod 4.$$

Then, by (37), we deduce

$$\sum_{n=0}^{\infty} PD_3(6n+3)q^n \equiv 2f_3^4 \pmod 8,$$

from which (8) and (9) follow.

We close this section by proving (10) and (11). By Lemma 5,

$$\frac{f_2^5 f_3^5 f_6}{f_1^{11}} \equiv \frac{f_3 f_6^3}{f_1} \pmod 2.$$

From (35), it follows that

$$\sum_{n=0}^{\infty} PD_3(6n+5)q^n \equiv 4f_6^3 \frac{f_3}{f_1} \pmod 8.$$

Now, using (24), we 2-dissect $\sum_{n=0}^{\infty} PD_3(6n+5)q^n$ modulo 8 to obtain

$$\sum_{n=0}^{\infty} PD_3(12n+5)q^n \equiv 4\frac{f_2 f_3^4 f_8 f_{12}^2}{f_1^2 f_4 f_6 f_{24}} \equiv 4\frac{f_6 f_8}{f_4} \pmod 8,$$

and

$$\sum_{n=0}^{\infty} PD_3(12n+11)q^n \equiv 4\frac{f_3^4 f_4^2 f_{24}}{f_1^2 f_8 f_{12}} \equiv 4\frac{f_2^3 f_6^2 f_{24}}{f_8 f_{12}} \pmod 8,$$

from which we obtain (8) and (9), respectively.

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