#A9 INTEGERS 20A (2020)

PELL EQUATION: A REVISIT THROUGH PERIODIC $\mathcal{F}_{2^l}$-CONTINUED FRACTIONS

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Received: 3/9/19, Revised: 1/6/20, Accepted: 3/20/20, Published: 4/27/20

Abstract

In this note, we discuss the periodicity of $\mathcal{F}_{2^l}$-continued fractions, where $l \in \mathbb{N}$. It is used to determine the solvability of the Pell equation $X^2 - DY^2 = 1$, under the condition that $(X, Y) \in \mathbb{Z} \times 2^l\mathbb{Z}$. In particular, we establish a correspondence between the set of solutions of the Pell equation (under the given condition) and the set of $\mathcal{F}_{2^l}$-convergents of $\sqrt{D}$, where D is a non-square positive integer.

1. Introduction

A Diophantine equation of the form $X^2 - DY^2 = 1$ is known as a Pell equation and $X^2 - DY^2 = -1$ as a negative Pell equation, where D is a non-square positive integer. Finding solution(s) to the Pell equation has been an interesting problem for a long time. A solution (X, Y) to this equation gives that $X + \sqrt{D}Y$ has norm 1, and hence it is a unit in $\mathbb{Z}[\sqrt{D}]$. Thus, this equation is directly related to algebraic number theory and describes units in quadratic integer rings $\mathbb{Z}[\sqrt{D}]$. Furthermore, these equations are also related to Archimedes' cattle problem and Chebyshev polynomials. For details, we refer the reader to the surveys by Lenstra [1, 2].

A solution to $X^2 - DY^2 = \pm 1$ serves as a best rational approximation of $\sqrt{D}$. The best approximations of a real number are characterized by convergents of the regular continued fraction of the number. Euler introduced the continued fraction approach to solve the Pell equation. Later, Lagrange proved that a solution to this equation depends on the regular continued fraction expansion of $\sqrt{D}$ which is purely periodic. He showed that every Pell equation has infinitely many solutions (see [5, 11]).

In this note, we look for solutions to the Pell equation over $\mathbb{Z} \times 2^l\mathbb{Z}$ for $l \geq 1$. We know that the negative Pell equation is not always solvable in $\mathbb{Z} \times \mathbb{Z}$, and its solvability depends on the parity of the period length of the regular continued fraction of $\sqrt{D}$. For example, suppose $D = 29$; then the regular continued fraction of $\sqrt{D}$ is periodic with an odd period length. We know that $(70, 13)$ is one of the solutions to $X^2 - DY^2 = -1$, but $(70, 13) \notin \mathbb{Z} \times 2^l\mathbb{Z}$ for $l \geq 1$. Suppose $(X, Y) \in \mathbb{Z} \times 2^l\mathbb{Z}$ is a solution of a negative Pell equation, then the integer X is odd and $X^2 - DY^2 \equiv 1 \mod 4$, which is a contradiction. Thus, we see that a negative Pell equation is not solvable over $\mathbb{Z} \times 2^l\mathbb{Z}$ for any $l \in \mathbb{N}$. Further, it is well known that $X^2 - DY^2 = 1$ is always solvable over $\mathbb{Z} \times \mathbb{Z}$. Suppose (X_0, Y_0) is a solution of $X^2 - DY^2 = 1$ in $\mathbb{Z} \times \mathbb{Z}$. Then (X_1, Y_1), obtained by comparing $(X_1 + \sqrt{D}Y_1) = (X_0 + \sqrt{D}Y_0)^{2^l}$, is a solution of the Pell equation in $\mathbb{Z} \times 2^l\mathbb{Z}$. Given a solution $(X_1, Y_1) \in \mathbb{Z} \times 2^l\mathbb{Z}$, one can find infinitely many solutions, $(X_{n+1}, Y_{n+1}) \in \mathbb{Z} \times 2^l\mathbb{Z}$ for $n \geq 0$, by the following equation:

$$(X_{n+1} + \sqrt{D}Y_{n+1}) = (X_1 + \sqrt{D}Y_1)^{2^n}.$$

Suppose $l \in \mathbb{N}$. Consider a subset $\mathcal{X}_{2^l}$ of $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ given by

$$\mathcal{X}_{2^l} = \left\{ \frac{p}{2^l q} : \ p, q \in \mathbb{Z} \ , q > 0, \gcd(p, 2q) = 1 \right\} \cup \{\infty\}.$$

If $(P, Q) \in \mathbb{Z} \times 2^l\mathbb{Z}$ is a solution to $X^2 - DY^2 = 1$, then $P/Q \in \mathcal{X}_{2^l}$. The set $\mathcal{X}_{2^l}$ is related to $\mathcal{F}_{2^l}$-continued fractions introduced by Sarma et al. in [4, 8]. A finite continued fraction of the form

$$\frac{1}{0+} \ \frac{2^l}{b+} \ \frac{\epsilon_1}{a_1+} \ \frac{\epsilon_2}{a_2+} \ \cdots \frac{\epsilon_n}{a_n} \quad (n \geq 0)$$

or an infinite continued fraction of the form

$$\frac{1}{0+} \ \frac{2^l}{b+} \ \frac{\epsilon_1}{a_1+} \ \frac{\epsilon_2}{a_2+} \ \cdots \frac{\epsilon_n}{a_n+} \cdots ,$$

where b is an odd integer, $a_1, a_2, \ldots$ are even positive integers, and $\epsilon_1, \epsilon_2, \cdots \in \{\pm 1\}$, is called an *$\mathcal{F}_{2^l}$-continued fraction.* Every irrational number has a unique infinite $\mathcal{F}_{2^l}$-continued fraction expansion. The expression

$$\frac{P_i}{Q_i} = \frac{1}{0+} \ \frac{2^l}{b+} \ \frac{\epsilon_1}{a_1+} \ \frac{\epsilon_2}{a_2+} \ \cdots \frac{\epsilon_i}{a_i}$$

for $i \geq 0$ is called the *i-th $\mathcal{F}_{2^l}$-convergent* which belongs to $\mathcal{X}_{2^l}$. The $\mathcal{F}_{2^l}$-continued fractions also characterize best approximations of a real number by elements of $\mathcal{X}_{2^l}$. A rational number $p/q \in \mathcal{X}_{2^l}$ is called a *best approximation of a real number α by an element of $\mathcal{X}_{2^l}$*, if for every $p'/q' \in \mathcal{X}_{2^l}$ different from p/q with $0 < q' \leq q$, we have $|q\alpha - p| < |q'\alpha - p'|$.

Thus, we have a class of continued fractions satisfying a kind of best approximation property. Note that these continued fractions are closely related to the continued fractions with even partial quotients (ECF) [9], but the study shows that an ECF has no approximation property [9, 10]. Thus, we raise a question to solve the Pell equation over $\mathbb{Z} \times 2^l\mathbb{Z}$ using $\mathcal{F}_{2^l}$-continued fractions.

The organization of this article is as follows. Section 2 recalls the known properties of the $\mathcal{F}_{2^l}$-continued fractions. We derive certain results which we will use to prove our main results. Section 3 deals with the question of periodicity of an $\mathcal{F}_{2^l}$-continued fraction. In particular, we show that an irrational number has a periodic $\mathcal{F}_{2^l}$-continued fraction if and only if it is a quadratic surd. The notion of pure periodicity of $\mathcal{F}_{2^l}$-continued fractions is introduced and related results are proved. We derive results on the periodicity of the continued fractions with even partial quotients proved in [3]. In Section 4, we achieve our main results describing the solution set of the Pell equation over $\mathbb{Z} \times 2^l\mathbb{Z}$. We conclude this section by adding a remark on the contribution of our results in algebraic number theory.

2. Preliminaries

Here, we summarize a few results of $\mathcal{F}_{2^l}$-continued fractions; for more details, we refer to [4, 8]. For the basic properties of regular continued fractions and semi-regular continued fractions, we refer to [6, 7]. Further, we derive certain results related to $\mathcal{F}_{2^l}$-continued fractions which will be used later in this paper.

Given an $\mathcal{F}_{2^l}$-continued fraction

$$\frac{1}{0+}\ \frac{2^l}{b+}\ \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+}\ \frac{\epsilon_3}{a_3+}\ \cdots\ \frac{\epsilon_n}{a_n+}\ \cdots,$$

the continued fraction

$$\frac{\epsilon_i}{a_i+}\ \frac{\epsilon_{i+1}}{a_{i+1}+}\ \cdots\ \frac{\epsilon_n}{a_n+}\ \cdots$$

is called the *fin* at the i-th stage of the $\mathcal{F}_{2^l}$-continued fraction for $i \geq 1$. Here, we record certain propositions describing properties of $\mathcal{F}_{2^l}$-continued fractions discussed in [8].

Proposition 2.1. *Let* $\frac{1}{0+}\ \frac{2^l}{b+}\ \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+}\ \frac{\epsilon_3}{a_3+}\ \cdots\ \frac{\epsilon_n}{a_n+}\ \cdots$ *be the* $\mathcal{F}_{2^l}$*-continued fraction of a real number* α*. Let* y_i *denote the* i*-th fin of the continued fraction. Then*

1. $b = 2\lfloor 2^{l-1}\alpha \rfloor + 1$, *the nearest odd integer to* $2^{l-1}\alpha$;
2. $a_i = 2\left\lfloor \frac{1}{2}\left(1 + \frac{1}{|y_i|}\right)\right\rfloor$, *the nearest even integer to* $1/|y_i|$, *for* $i \geq 1$;
3. $\epsilon_i = \operatorname{sign}(y_i)$, *for* $i \geq 1$;
4. $y_{i+1} = \frac{1}{|y_i|} - a_i$.

Proposition 2.2. *Suppose* $\alpha = \frac{1}{0+} \frac{2^l}{b+} \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \frac{\epsilon_3}{a_3+} \ldots$ *is an* $\mathcal{F}_{2^l}$*-continued fraction and* $\{\frac{p_i}{q_i}\}_{i=0}^{\infty}$ *is the sequence of* $\mathcal{F}_{2^l}$*-convergents of* α*. Suppose* $(p_{-1}, q_{-1}) = (1, 0)$ *and* $(p_0, q_0) = (b, 2^l)$*. Let* y_i *be the fin at the* i*-th stage of the* $\mathcal{F}_{2^l}$*-continued fraction of* α*. Then for* $i \geq 0$*, we have*

(1) $p_i q_{i-1} - q_i p_{i-1} = \pm 2^l$;

(2) $p_{i+1} = a_{i+1} p_i + \epsilon_{i+1} p_{i-1}$ *and* $q_{i+1} = a_{i+1} q_i + \epsilon_{i+1} q_{i-1}$;

(3) the sequence $\{q_i\}_{i \geq 0}$ *is strictly increasing;*

(4) $\dfrac{p_i}{q_i} \neq \dfrac{p_j}{q_j}$ *for* $i \neq j$;

(5) for $i \geq 1$, $|y_i| \leq 1$;

(6) $\alpha = \dfrac{x_{i+1} p_i + \epsilon_{i+1} p_{i-1}}{x_{i+1} q_i + \epsilon_{i+1} q_{i-1}}$, *where* $x_i = \dfrac{1}{|y_i|}$.

Proposition 2.3. *Suppose* $x \in \mathbb{R}$ *has an eventually constant* $\mathcal{F}_{2^l}$*-continued fraction. Then* $x \in \mathbb{Q}$ *if and only if all but finitely many partial numerators are* -1 *and all but finitely many partial denominators are* 2.

Corollary 2.4. *If* α *is an irrational number, then there are infinitely many* $i \in \mathbb{N}$ *such that* $\epsilon_i / a_i \neq -1/2$.

In Section 1, we introduced the definition of a best approximation by an element in $\mathcal{X}_{2^l}$. Here, we record a result on best approximation properties of $\mathcal{F}_{2^l}$-continued fractions.

Theorem 2.5. *Let* α *be an irrational number and* $r/s \in \mathcal{X}_{2^l}$*. Then* r/s *is a best approximation of a real number* α *by an element of* $\mathcal{X}_{2^l}$ *if and only if* r/s *is an* $\mathcal{F}_{2^l}$*-convergent of* α.

Lemma 2.6. *Let* α *be a real number and* p_i/q_i *be the sequence of* $\mathcal{F}_{2^l}$*-convergents of* α*. If* p_n/q_n *is an* $\mathcal{F}_{2^l}$*-convergent of* α *with* $\epsilon_{n+1}/a_{n+1} \neq -1/2$*, then*

$$|\alpha - p_n/q_n| < 2^l/q_n^2.$$

Proof. We know that

$$\begin{aligned} |\alpha - \frac{p_n}{q_n}| &= |\frac{x_{n+1} p_n + \epsilon_{n+1} p_{n-1}}{x_{n+1} q_n + \epsilon_{n+1} q_{n-1}} - \frac{p_n}{q_n}| \\ &= \frac{2^l}{|x_{n+1} q_n + \epsilon_{n+1} q_{n-1}| q_n}. \end{aligned}$$

If $\epsilon_{n+1} = 1$, then $x_{n+1} q_n + \epsilon_{n+1} q_{n-1} > q_n$. If $\epsilon_{n+1} = -1$, then $a_{n+1} \geq 4$ so that $x_{n+1} \geq 3$, and hence $x_{n+1} q_n + \epsilon_{n+1} q_{n-1} > q_n$. Thus, we get $|\alpha - p_n/q_n| < 2^l/q_n^2$. $\square$

Using Corollary 2.4, we have the following corollary of Lemma 2.6.

Corollary 2.7. *If α is an irrational number, then there are infinitely many $p/q \in \mathcal{X}_{2^l}$ such that $|\alpha - p/q| < 2^l/q^2$.*

3. Periodic $\mathcal{F}_{2^l}$-continued Fractions

An $\mathcal{F}_{2^l}$-continued fraction is called *periodic* of *period length* $m \geq 1$ with an *initial block of length* $n \geq 1$ if $y_n \neq y_{n+r}$, for $r \geq 1$, but $y_{n+i} = y_{(n+km)+i}$, that is,

$$\epsilon_{n+i} = \epsilon_{(n+km)+i} \text{ and } a_{n+i} = a_{(n+km)+i},$$

for $1 \leq i \leq m$ and $k \geq 0$. A periodic continued fraction having no initial term is called *purely periodic*. In this section, we discuss that a periodic $\mathcal{F}_{2^l}$-continued fraction reaches to a quadratic surd and vice versa. Recall that a *quadratic surd* is a solution of a quadratic equation $Ax^2 + Bx + C = 0$ with integer coefficients $A \neq 0, B$, and C such that the discriminant $D = B^2 - 4AC$ is not a perfect square. Here, we record an observation which we will use further.

Lemma 3.1. *A real number α is a quadratic surd if and only if $s\alpha + t$ is a quadratic surd, where $0 \neq s \in \mathbb{Q}$ and $t \in \mathbb{Q}$.*

Lemma 3.2. *Suppose α is an irrational number and y_i is the i-th fin of the $\mathcal{F}_{2^l}$-continued fraction expansion of α. If $y_k = y_r$ for some k, r with $r > k$, then $y_{k+j} = y_{r+j}$ for each $j \geq 1$. In particular, the continued fraction is periodic.*

Proof. By the fourth statement of Proposition 2.1, $y_{k+1} = \frac{1}{|y_k|} - a_k$, where a_k is the nearest even integer to $1/|y_k|$. Thus, we get that the statement is true for $j = 1$. Now suppose $y_{k+j-1} = y_{r+j-1}$. Using the fact that y_i is an irrational for each $i \geq 1$, we get $a_{k+j-1} = a_{r+j-1}$ and by the induction hypothesis, we have $y_{k+j} = y_{r+j}$ for each $j \geq 1$. We can find the smallest n such that $y_{n+1} = y_{s+1}$ for some $s > n$ (then $1 \leq n < k$), and choose the smallest $m > n$ such that $y_{n+1} = y_{m+1}$. Thus, the continued fraction is periodic of the length m with an initial block of length n. $\square$

Theorem 3.3. *Suppose α is an irrational number. The $\mathcal{F}_{2^l}$-continued fraction of α is periodic if and only if α is a quadratic surd.*

Proof. Suppose the $\mathcal{F}_{2^l}$-continued fraction of α is periodic and given by

$$x = \frac{1}{0+} \; \frac{2^l}{b+} \; \frac{\epsilon_1}{a_1+} \; \cdots \; \frac{\epsilon_n}{a_n+} \; \frac{\epsilon_{n+1}}{a_{n+1}+} \; \cdots \; \frac{\epsilon_{n+m}}{a_{n+m}+} \; \frac{\epsilon_{n+1}}{a_{n+1}+} \; \cdots \; \frac{\epsilon_{n+m}}{a_{n+m}+} \; \frac{\epsilon_{n+1}}{a_{n+1}+} \; \cdots,$$

where $n \geq 0$ and $m \geq 1$. Let P_i/Q_i denote the i-th convergent and y_i be the i-th fin of the continued fraction. Then $y_{n+1} = y_{(n+mk)+1}$, for $k \geq 0$. Using Proposition

2.2, we have $y_{i+1} = \frac{P_i - \alpha Q_i}{\alpha Q_{i-1} - P_{i-1}}$ so that

$$\frac{P_n - \alpha Q_n}{\alpha Q_{n-1} - P_{n-1}} = \frac{P_{n+m} - \alpha Q_{n+m}}{\alpha Q_{(n+m)-1} - P_{(n+m)-1}},$$

which gives that α is a root of the quadratic polynomial $Rx^2 + Sx + T$, where $R = Q_{n-1}Q_{n+m} - Q_{n+m-1}Q_n$, $S = (Q_nP_{n+m-1} - P_{n+m}Q_{n-1} + P_nQ_{n+m-1} - P_{n-1}Q_{n+m})$ and $T = P_nP_{n+m-1} + P_{n-1}P_{n+m}$. By assumption, α is an irrational, and hence it is a quadratic surd. For the converse part, we assume that α is a quadratic surd. By Lemma 3.1, $y_1 = 2^l\alpha - b$ is also a quadratic surd. Thus, there exist $0 \neq R_0 \in \mathbb{Z}$ and $S_0, T_0 \in \mathbb{Z}$ such that

$$R_0y_1^2 + S_0y_1 + T_0 = 0.$$

The $\mathcal{F}_{2^l}$-continued fraction of α is given by

$$\alpha = \frac{1}{0+}\ \frac{2^l}{b+}\ \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+}\ \cdots\ \frac{\epsilon_n}{a_n+} \cdots .$$

Let y_k be the fin at the k-th stage for $k \geq 1$, and A_k/B_k denotes the k-th convergent of the semi-regular continued fraction of y_1. For $k \geq 1$,

$$y_1 = \frac{A_k + y_{k+1}A_{k-1}}{B_k + y_{k+1}B_{k-1}}.$$

Then, we have

$$R_ky_{k+1}^2 + S_ky_{k+1} + T_k = 0,$$

where

$$\begin{aligned}
R_{k+1} &= R_0A_{k-1}^2 + S_0A_{k-1}B_{k-1} + T_0B_{k-1}^2,\\
S_{k+1} &= 2A_kA_{k-1}R_0 + (A_kB_{k-1} + B_kA_{k-1})S_0 + 2B_kB_{k-1}T_0,\\
T_{k+1} &= R_0A_k^2 + S_0A_kB_k + T_0B_k^2,
\end{aligned}$$

and the discriminant remains unchanged for each k. Note that $R_{k+1} = T_k$. If T_k and T_{k+1} are bounded, then R_k and S_k are also bounded as the discriminant is bounded. Again, note that $P_k = bB_k + A_k$ and $Q_k = 2^lB_k$. By Corollary 2.4, the cardinality of the set

$$K_\alpha = \{k \in \mathbb{N} \mid \epsilon_{k+1}/a_{k+1} \neq -1/2 \text{ in the } \mathcal{F}_{2^l}\text{-continued fraction of } \alpha\}$$

is infinite. Let $k^* \in K_\alpha$, then by Lemma 2.6

$$|B_{k^*}y_1 - A_k| < \frac{1}{B_{k^*}}.$$

We can write $A_{k^*} = B_{k^*}y_1 + \frac{\delta}{B_{k^*}}$, for some δ with $|\delta| < 1$. Using this value, we get

$$|T_{k^*+1}| < |2R_0y_1| + |S_0| + |R_0|,$$

and hence T_{k^*+1} is bounded. Now we claim that T_{k^*} is also bounded for $k^* \in K_\alpha$. If $\epsilon_{k^*}/a_{k^*} \neq -1/2$, then $k^*-1 \in K_\alpha$ and we are done. So let $\epsilon_{k^*}/a_{k^*} = -1/2$. If $y_{k^*+1} > 0$, then $x_{k^*} > 2$, and

$$|B_{k^*-1}y - A_{k^*-1}| = \frac{1}{|B_{k^*-1} + y_{k^*}B_{k^*-2}|} = \frac{1}{|x_{k^*}B_{k^*-1} - B_{k^*-2}|} < \frac{1}{B_{k^*-1}}.$$

Now suppose $y_{k^*+1} < 0$. Then $a_{k^*+1} \geq 4$ so that $x_{k^*+1} > 3$ and equivalently $|y_{k^*+1}| < 1/3$. We know that $1/|y_{k^*}| - 2 = y_{k^*+1}$ and $|y_{k^*+1}| < 1$, and therefore $5/3 < 1/|y_{k^*}| < 7/3$. Using this inequality, we get

$$|B_{k^*-1}y - A_{k^*-1}| = \frac{1}{|B_{k^*-1}x_{k^*} - B_{k^*-2}|} < \frac{3}{2B_{k^*-1}}.$$

We apply the same method to get the boundedness of T_{k^*} as in the case of T_{k^*+1}, for each $k^* \in K_\alpha$. Thus, we get $R_{k+1}, S_{k+1}, T_{k+1}$ are bounded for infinitely many k, that is, for all $k \in K_\alpha$ and the discriminant remains unchanged. But there are only finitely many polynomials with a given discriminant and bounded coefficients. Thus, the sequence y_{k+1} with $k \in K_\alpha$ has entries from a finite set. Thus, there exist integers $r, s \in \mathbb{N}$ with $r < s$ such that $y_{r+1} = y_{s+1}$. The result is achieved by Lemma 3.2. □

Here, we state a result on the periodicity of the continued fractions with even partial quotients (see [3]) as a corollary of Theorem 3.3.

Corollary 3.4. *The continued fraction of an irrational number β with even partial quotient is periodic if and only if β is a quadratic surd.*

Theorem 3.5. *Suppose $l \geq 1$ and α is a quadratic surd with $0 < \alpha < 1/2^{l-1}$. The $\mathcal{F}_{2^l}$-continued fraction of α is purely periodic if and only if $\bar{\alpha} < 0$.*

Proof. Suppose α is a quadratic surd with $0 < \alpha < 1/2^{l-1}$, and then $b = 1$. Let $\bar{\alpha} < 0$. Suppose the $\mathcal{F}_{2^l}$-continued fraction of α is not purely periodic and it is given by

$$\alpha = \frac{1}{0+}\ \frac{2^l}{1+}\ \frac{\epsilon_1}{a_1+}\ \cdots\ \frac{\epsilon_m}{a_n+}\ \frac{\epsilon_{n+1}}{a_{n+1}+}\ \cdots\ \frac{\epsilon_{n+m}}{a_{n+m}+}\ \frac{\epsilon_{n+1}}{a_{n+1}+}\ \frac{\epsilon_{n+2}}{a_{n+2}+}\ \cdots\ \frac{\epsilon_{n+m}}{a_{n+m}+}\ \cdots,$$

where $n \geq 1, m \geq 1$ with $y_n \neq y_{n+m}$ and $y_{n+i} = y_{n+m+i}$ for $i \geq 1$. Let P_i/Q_i be the i-th convergent. Then, for $i \geq 0$,

$$\bar{\alpha} \quad = \quad \frac{1}{0+}\ \frac{2^l}{1+}\ \frac{\epsilon_1}{a_1+}\ \cdots\ \frac{\epsilon_i}{a_i + \overline{y_{i+1}}} = \frac{P_i + \overline{y_{i+1}}P_{i-1}}{Q_i + \overline{y_{i+1}}Q_{i-1}},$$

and

$$\overline{y_{i+1}} = \frac{P_i - Q_i\bar{\alpha}}{Q_{i-1}\bar{\alpha} - P_{i-1}}.$$

We know that $P_i > 0$, (since, $\alpha > 0$) which gives that $\overline{y_{i+1}} < 0$ for $i \geq 0$. Further, we claim that $\overline{y_{i+1}} < -1$. Note that $P_i \geq P_{i-1}$. Suppose $-1 < \overline{y_{i+1}} < 0$, then $-1 < \frac{P_i - Q_i\bar{\alpha}}{Q_{i-1}\bar{\alpha} - P_{i-1}} < 0$, but $P_{i-1} > 0$, $Q_i > Q_{i-1}$ and $\bar{\alpha} < 0$ give that $P_{i-1} > P_i$, which is not possible. Thus, $\overline{y_{i+1}} < -1$ for $i \geq 0$. Since, $\overline{y_{n+1}} = \overline{y_{n+m+1}}$, we get

$$\frac{\epsilon_n}{\overline{y_n}} - \frac{\epsilon_{n+m}}{\overline{y_{n+m}}} = a_{n+m} - a_n. \tag{3.1}$$

We split the discussion into two cases. First, suppose $a_{n+m} \neq a_n$. Note that $\overline{y_n} < -1$ and $\overline{y_{n+m}} < -1$, then

$$-2 < \frac{\epsilon_n}{\overline{y_n}} - \frac{\epsilon_{n+m}}{\overline{y_{n+m}}} < 2.$$

The R.H.S. of (3.1) is an even integer, we get $\frac{\epsilon_n}{\overline{y_n}} = \frac{\epsilon_{n+m}}{\overline{y_{n+m}}}$, equivalently, $a_n = a_{n+m}$ which is a contradiction. Now suppose $a_n = a_{n+m}$, then $\epsilon_n \neq \epsilon_{n+m}$. Again, by Equation (3.1)

$$\frac{\epsilon_n}{\overline{y_n}} = \frac{\epsilon_{n+m}}{\overline{y_{n+m}}},$$

which implies that $\overline{y_n}$ and $\overline{y_{n+m}}$ have different signs which is a contradiction.

Now for the converse part, we assume that α with $0 < \alpha < 1/2^{l-1}$ has a purely periodic continued fraction. By Theorem 3.3, we know that α is a quadratic surd with $b = 1$. Then there exists a positive integer m such that $2^l\alpha - 1 = y_{m+1}$ with

$$\alpha = \frac{P_m + y_{m+1}P_{m-1}}{Q_m + y_{m+1}Q_{m-1}},$$

and so $2^lQ_{m-1}\alpha^2 + (Q_m - Q_{m-1} - 2^lP_{m-1})\alpha + (P_{m-1} - P_m) = 0$. We know that $(P_{m-1} - P_m) < 0$, and hence $\bar{\alpha}$ is negative. □

Let D be a positive integer which is not a perfect square, then the irrational conjugate of $\sqrt{D}$ is negative. Hence, we have the following corollary.

Corollary 3.6. *If D is a non-square positive integer, then the $\mathcal{F}_{2^l}$-continued fraction of $\sqrt{D}$ is purely periodic.*

Again we go back to continued fractions with even partial quotients. The following lemma states a relation between periodic $\mathcal{F}_{2^l}$-continued fractions and continued fractions with even partial quotients.

Lemma 3.7. *Let $\beta > 1$ be a quadratic surd. Then β has the purely periodic continued fraction with even partial quotient if and only if the $\mathcal{F}_{2^l}$-continued fraction of $\frac{1+\beta}{2^l\beta}$ is purely periodic.*

Using Lemma 3.7, we state a result from [3] as a corollary of Theorem 3.5.

Corollary 3.8. *A quadratic surd $\beta > 1$ has a purely periodic continued fraction with even partial quotient if and only if $-1 < \bar{\beta} < 0$.*

The following proposition record a pattern of partial numerator ϵ_i and denominator a_i in the $\mathcal{F}_{2^l}$-continued fraction expansion of $\sqrt{D}$.

Proposition 3.9. *Suppose D is a non-square positive integer and m is the period length of the $\mathcal{F}_{2^l}$-continued fraction of $\sqrt{D}$. If $m = 1$, then $a_1 = 2b$ with $\epsilon_1 = -1 = 4^l D - b^2$ and if $m > 1$, then $\epsilon_{1+i} = \epsilon_{m-i}$* and *$a_i = a_{m-i}$ for an integer i, $1 \leq i \leq m/2$.*

Proof. Suppose $m = 1$. Then $y_1 = 2^l\sqrt{D} - b$ so that

$$\sqrt{D} = \frac{1}{0+}\ \frac{2^l}{b+}\ \frac{\epsilon_1}{a_1 + (2^l\sqrt{D} - b)}.$$

Thus, $\sqrt{D}$ is a root of the polynomial

$$4^l x^2 + 2^l(a_1 - 2b)x + (b^2 - a_1 b - \epsilon_1).$$

Hence, $a_1 - 2b = 0$, equivalently, $a_1 = 2b$ and $(b^2 - a_1 b - \epsilon_1) \equiv 0 \mod 4$. Using these values, we get $\epsilon_1 = -1 = 4^l D - b^2$. Now, suppose $m > 1$. Then

$$2^l\sqrt{D} - b \quad = \quad \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_m}{a_m + (2^l\sqrt{D} - b)}. \tag{3.2}$$

If y_i denote the fin at the i-th stage, then

$$2^l\sqrt{D} - b = y_1 = \frac{\epsilon_1}{a_1 + y_2},\ y_2 = \frac{\epsilon_2}{a_2 + y_3}, \ldots,\ y_m = \frac{\epsilon_m}{a_m + y_1}.$$

For $i \geq 1$, the number x_i is given by

$$x_i = \frac{\epsilon_i}{y_i} = a_i + \frac{\epsilon_{i+1}}{a_{i+1}+}\ \frac{\epsilon_{i+2}}{a_{i+2}}\ \cdots.$$

Then

$$x_1 = a_1 + \frac{\epsilon_2}{x_2},\ x_2 = a_2 + \frac{\epsilon_3}{x_3}, \ldots,\ x_m = a_m + \frac{\epsilon_1}{x_1},$$

equivalently,

$$\frac{-\epsilon_2}{\overline{x_2}} = a_1 - \overline{x_1},\ \frac{-\epsilon_3}{\overline{x_3}} = a_2 - \overline{x_2},\ \ldots, \frac{-\epsilon_1}{\overline{x_1}} = a_m - \overline{x_m}.$$

Thus,

$$\frac{-\epsilon_1}{\overline{x_1}} = a_m + \frac{\epsilon_m}{a_{m-1}+}\ \frac{\epsilon_{m-1}}{a_{m-2}+}\ \cdots\ \frac{\epsilon_2}{a_1 - \overline{x_1}}. \tag{3.3}$$

Note that $\frac{-\epsilon_1}{\overline{x_1}} = 2^l\sqrt{D} + b$, or say, $\frac{-\epsilon_1}{\overline{x_1}} - 2b = 2\sqrt{D} - b$. Using Equations (3.2) and (3.3), we get $a_m = 2b, \epsilon_m = \epsilon_1$. Further, using the fact that every irrational has a unique $\mathcal{F}_{2^l}$-continued fraction, we get

$$\epsilon_{1+i} = \epsilon_{m-i} \text{ and } a_i = a_{m-i}$$

for an integer i with $1 \leq i \leq m/2$. $\square$

4. Pell Equation

In this section, D denotes a positive integer which is not a perfect square. By Corollary 3.6, the $\mathcal{F}_{2^l}$-continued fraction is purely periodic. For $i \geq 0$, P_i/Q_i denotes the i-th convergent of the $\mathcal{F}_{2^l}$-continued fraction of $\sqrt{D}$. The following theorem states that certain $\mathcal{F}_{2^l}$-convergents of $\sqrt{D}$ serve as a solution to $X^2 - DY^2 = 1$.

Theorem 4.1. *Suppose the $\mathcal{F}_{2^l}$-continued fraction of $\sqrt{D}$ is periodic of period length m. If $m = 1$, then each P_i/Q_i is a solution to the Pell equation $X^2 - DY^2 = 1$ for $i \geq 0$. If $m > 1$, then P_{mk-1}/Q_{mk-1} is a solution to the Pell equation $X^2 - DY^2 = 1$ for every $k \geq 1$.*

Proof. Suppose the $\mathcal{F}_{2^l}$-continued fraction expansion of $\sqrt{D}$ is given by

$$\sqrt{D} = \frac{1}{0+}\ \frac{2^l}{b+}\ \frac{\epsilon_1}{a_1+}\ \cdots\ \frac{\epsilon_m}{a_m+}\ \frac{\epsilon_1}{a_1+}\ \cdots\ \frac{\epsilon_m}{a_m+}\ \frac{\epsilon_1}{a_1+}\ \cdots.$$

If $m = 1$, then by Proposition 3.9, $P_0^2 - DQ_0^2 = -\epsilon_1 = 1$. Further, we can write

$$\sqrt{D} = \frac{1}{0+}\ \frac{2^l}{b+}\ \frac{\epsilon_1}{a_1 + (2^l\sqrt{D} - b)} \text{ or } \sqrt{D} = \frac{P_1 + (2^l\sqrt{D} - b)P_0}{Q_1 + (2^l\sqrt{D} - b)Q_0}.$$

On comparing rational and irrational parts, we get

$$P_1 = b^2 + 4^l D, \text{ and } Q_1 = 2^{l+1}b$$

so that $P_1^2 - DQ_1^2 = (b^2 - 4^l D)^2 = (P_0^2 - DQ_0^2)^2 = \epsilon_1^2$. Now suppose the result is true up to some $i > 1$, that is, $P_i^2 - DQ_i^2 = 1$. Again,

$$\sqrt{D} = \frac{P_{i+1} + (2^l\sqrt{D} - b)P_i}{Q_{i+1} + (2^l\sqrt{D} - b)Q_i}.$$

On comparing rational and irrational part, we get $P_{i+1} = bP_i + 2^l DQ_i$ and $Q_{i+1} = bQ_i + 2^l P_i$ so that

$$P_{i+1}^2 - DQ_{i+1}^2 = (P_i^2 - DQ_i^2)(b^2 - 4^l D) = 1.$$

Now suppose $m > 1$. Then for $k \geq 1$,

$$\sqrt{D} = \frac{P_{mk} + (2^l\sqrt{D} - b)P_{mk-1}}{Q_{mk} + (2^l\sqrt{D} - b)Q_{mk-1}}.$$

We get $Q_{mk} = bQ_{mk-1} + 2^l P_{mk-1}$ and $P_{mk} = 2^l DQ_{mk-1} + bP_{mk-1}$ so that

$$\pm 2^l = Q_{mk}P_{mk-1} - P_{mk}Q_{mk-1} = 2^l(P_{mk-1}^2 - DQ_{mk-1}^2). \tag{4.1}$$

Applying this modulo 4, one can see that $P_{mk-1}^2 - DQ_{mk-1}^2 = 1$ for each $k \geq 1$. $\square$

Lemma 4.2. *Suppose $l \geq 1$, and $0 < K \leq 2^{l-1}$. If $r/2^l s \in \mathcal{X}_{2^l}$ is such that*

$$|2^l s\alpha - r| < \frac{K}{2^l s},$$

then $r/2^l s$ is an $\mathcal{F}_{2^l}$-convergent of α.

Proof. Suppose $p/2^l q \in \mathcal{X}_{2^l}$ with $0 < q \leq s$ and $|2^l q\alpha - p| < |2^l s\alpha - r|$. Then

$$|2^l q\alpha - p| < \frac{K}{2^l s}.$$

We have

$$\frac{1}{2^l qs} \leq |\frac{p}{2^l q} - \frac{r}{2^l s}| \leq |\alpha - \frac{p}{2^l q}| + |\alpha - \frac{r}{2^l s}| < \frac{K}{4^l sq} + \frac{K}{4^l s^2}.$$

Thus, $q > s\left(\frac{2^l}{K} - 1\right)$. By assumption $0 < K < 2^{l-1}$, and so $q > s\left(\frac{2^l}{K} - 1\right) \geq s$, which yields a contradiction. Thus, for $p/2^l q \in \mathcal{X}_{2^l}$ with $0 < q \leq s$ and $|2^l q\alpha - p| \geq |2^l s\alpha - r|$ so that $r/2^l s$ is a best approximation of α by an element of $\mathcal{X}_{2^l}$, and hence an $\mathcal{F}_{2^l}$-convergent of α. $\square$

Theorem 4.3. *Let D be a positive integer which is not a perfect square. If $(X, Y) \in \mathbb{Z} \times \mathbb{Z}$ is a solution of the Pell equation $X^2 - DY^2 = 1$ with $Y \in 2^l\mathbb{Z}$. Then X/Y is an $\mathcal{F}_{2^l}$-convergent of $\sqrt{D}$.*

Proof. Suppose $(P, 2^l Q)$ is a solution to $X^2 - DY^2 = 1$, then

$$\begin{aligned}
P^2 - D2^{2l}Q^2 &= 1 \\
(P - 2^l Q\sqrt{D})(P + 2^l Q\sqrt{D}) &= 1 \\
(P - 2^l Q\sqrt{D})^2 + (P - 2^l Q\sqrt{D})2^{l+1}Q\sqrt{D} &= 1 \\
(P - 2^l Q\sqrt{D})2^l Q &< \frac{1}{2\sqrt{D}}.
\end{aligned}$$

Note that $P - 2^l Q\sqrt{D} > 0$. By Lemma 4.2, we get that $P/2^l Q$ is an $\mathcal{F}_{2^l}$-convergent of $\sqrt{D}$ (since, $1/2\sqrt{D} < 1$). $\square$

Lemma 4.4. *If P_i/Q_i denotes the i-th convergent of the $\mathcal{F}_{2^l}$-continued fraction of $\sqrt{D}$, then*

1. $P_i^2 - DQ_i^2 = P_{km+i}^2 - DQ_{mk+i}^2$, *for* $0 \leq i \leq (m-1)$;
2. $P_i^2 - DQ_i^2 = 1$ *if and only if* $i = mk - 1$, *for some* $k \in \mathbb{N}$;
3. $|P_i^2 - DQ_i^2| = |P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2|$, *for* $0 \leq i \leq \lfloor \frac{m}{2} \rfloor - 1$.

Proof. Suppose $i \geq 0$. The $i+1$-th fin is given by

$$y_{i+1} = \frac{\sqrt{D}Q_i - P_i}{P_{i-1} - \sqrt{D}Q_{i-1}}.$$

We can write y_{i+1} in the following way:

$$y_{i+1} = \frac{M_{i+1} + 2^l\sqrt{D}}{N_{i+1}},$$

where $M_{i+1} = \pm(P_iP_{i-1} - DQ_iQ_{i-1})$ and $N_{i+1} = \pm(P_{i-1}^2 - DQ_{i-1}^2)$. Since, the continued fraction of $\sqrt{D}$ is purely periodic of length m, $y_i = y_{km+i}$, for $1 \leq i \leq m$ and $k \geq 0$. On comparing the rational and irrational parts, we get that

$$M_i = M_{mk+i} \text{ and } N_i = N_{mk+i}.$$

Thus, $P_{i-1}^2 - DQ_{i-1}^2 = P_{mk+(i-1)}^2 - DQ_{mk+(i-1)}^2$, for $1 \leq i \leq m$ and $k \geq 0$, and we get the first statement. Now suppose $P_i^2 - DQ_i^2 = 1$ so that $N_{i+2} = 1$. Then

$$|y_{i+2}| = |M_{i+2} + 2^l\sqrt{D}| < 1,$$

and hence $-M_{i+2} - 1 < 2^l\sqrt{D} < -M_{i+2} + 1$. Observe that M_i is an odd integer for each i. Thus, the above inequality gives that $M_{i+2} = -b$ so that

$$y_{i+2} = 2^l\sqrt{D} - b = y_{mk+1},$$

for $k \geq 0$. Thus, we get $i+2 = mk+1$, equivalently, $i = mk-1$. The converse statement is clear from the proof of Theorem 4.1. For the third statement, recall that

$$y_{m-(i+1)} = \frac{\epsilon_{i+2}(P_i + \sqrt{D}Q_i)}{P_{i+1} + \sqrt{D}Q_{i+1}}.$$

Now we can write

$$P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2 \quad = \quad (P_{m-(i+2)} + \frac{\epsilon_{i+2}(P_i + \sqrt{D}Q_i)}{P_{i+1} + \sqrt{D}Q_{i+1}}Q_{m-(i+2)})A,$$

where $A = (P_{m-(i+2)} + \sqrt{D}Q_{m-(i+2)})$ and $0 \leq i \leq \lfloor \frac{m}{2} \rfloor - 1$. Using the value of $y_{m-(i+1)}$ and comparing the rational and irrational terms, we get

$$B(Q_{m-(i+2)}P_{i+1} + Q_{m-(i+3)}P_i) \quad = \quad \pm\epsilon_{i+2}(P_iP_{m-(i+2)} + DQ_iQ_{m-(i+2)}) \tag{4.2}$$
$$B(Q_{m-(i+2)}Q_{i+1} + Q_{m-(i+3)}Q_i) \quad = \quad \pm\epsilon_{i+2}(P_iQ_{m-(i+2)} + DQ_iP_{m-(i+2)}) \tag{4.3}$$

where $B = (P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2)$. By Equation (4.2) and (4.3),

$$P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2 = \epsilon_{i+2}(P_i^2 - DQ_i^2),$$

and hence

$$|P_{m-(i+2)}^2 - DQ_{m-(i+2)}^2| = |(P_i^2 - DQ_i^2)|.$$

□

Combining the results of Theorems 4.1, 4.3 and Lemma 4.4, we obtain our main result which can be stated as follows.

Theorem 4.5. *If D is a positive integer which is not a perfect square, then*

1. *the Pell equation $X^2 - DY^2 = 1$ is always solvable in $\mathbb{Z} \times 2^l\mathbb{Z}$;*
2. *the solution set of $X^2 - DY^2 = 1$ is given by*

$$\{(P_{mk-1}, Q_{mk-1}) \mid k \in \mathbb{N}\},$$

where $\frac{P_{mk-1}}{Q_{mk-1}}$ is the $(mk-1)$-th convergent of the $\mathcal{F}_{2^l}$-continued fraction of $\sqrt{D}$ with period m.

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