INFINITE SERIES CONTAINING GENERALIZED q-HARMONIC NUMBERS

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Abstract

In this paper, by using Abel's Lemma on summation by parts, we mainly show that several types of infinite sums involving the generalized q-harmonic numbers can be expressed in terms of linear combinations of the generalized Hurwitz q-zeta values, which are natural q-generalizations of Sofo's results.

1. Introduction

With an arbitrary complex sequence $\{\tau_k\}$, a forward difference and a backward difference are expressions of the forms

$$\Delta\tau_k = \tau_{k+1} - \tau_k \quad \text{and} \quad \nabla\tau_k = \tau_k - \tau_{k-1}\,,$$

respectively. Given two sequences $\{f_k\}$ and $\{g_k\}$, Abel's Lemma on summation by parts [1, 3] can be stated as

$$\sum_{k=m}^{n} f_k \Delta g_k = f_n g_{n+1} - f_m g_m - \sum_{k=m}^{n-1} g_{k+1}\Delta f_k\,.$$

When $m = 1$, $n \to \infty$ and g_k is replaced by g_{k-1}, it is reformed as

$$\sum_{k=1}^{\infty} f_k \nabla g_k = \lim_{n\to\infty} f_n g_n - f_1 g_0 - \sum_{k=1}^{\infty} g_k \Delta f_k\,. \tag{1}$$

If we use the summation

$$\sum_{k=m}^{n} f_k \Delta g_k = f_{n+1} g_{n+1} - f_m g_m - \sum_{k=m}^{n} g_{k+1}\Delta f_k\,,$$

then we have

$$\sum_{k=1}^{\infty} f_k \nabla g_k = \lim_{n\to\infty} f_{n+1} g_n - f_1 g_0 - \sum_{k=1}^{\infty} g_k \Delta f_k \,. \tag{2}$$

There are also some other forms of Abel's Lemma on summation by parts (see, e.g., Chu [3]). Since then, Chu and more authors have applied Abel's Lemma on summation by parts to varieties of numbers or functions to obtain relations and identities.

The generalized harmonic numbers in power r are those rational numbers given by

$$H_n^{(r)} = \sum_{j=1}^{n} \frac{1}{j^r} \quad (n, r \in \mathbb{N} := \{1, 2, 3, \cdots\}),$$

where the empty sum $H_0^{(r)}$ is conventionally understood to be zero. When $r = 1$, they reduce to the classical harmonic numbers $H_n = \sum_{j=1}^{n} 1/j$. These numbers play an important role in combinatorics, number theory, analysis of algorithms and many other areas.

There are various curious identities in the literature involving the generalized harmonic numbers $H_n^{(r)}$. A classical result that goes back to the time of Euler [7, 9] is

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^r} = \frac{1}{2}(\zeta(r)^2 + \zeta(2r)),$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $\Re e(s) > 1$, denotes the well-known Riemann zeta function. Sofo [6] expresses the general sums of types

$$\sum_{n=1}^{\infty} \frac{(a-1)H_n^{(s)}}{(n+1)(n+a)}, \quad \sum_{n=1}^{\infty} \frac{aH_n^{(s)}}{n(n+a)}, \quad \sum_{n=1}^{\infty} \frac{H_n^{(s)}}{\binom{n+k}{k}}, \quad \sum_{n=1}^{\infty} \frac{H_n^{(s)}}{n\binom{n+k}{k}}$$

in terms of linear combinations of zeta values.

Wang [8] defines the (a, b)-*harmonic numbers of order* m by

$$h_n^{<m>}(a, b) = \sum_{k=1}^{n} \frac{1}{(ak - a + b)^m} \quad (n, m \in \mathbb{N})$$

with $h_0^{<m>}(a, b) = 0$, where a and b are positive real numbers. For $m = 2, 3$, Wang gives some interesting infinite series including $h_n^{<2>}(a, b)$ and $h_n^{<3>}(a, b)$. In particular, for special values of a and b, Wang expresses some infinite series in terms of linear combinations of π^2, π^4, $\ln 2$ and some zeta values.

For $q \in \mathbb{C}$ with $|q| < 1$ and any real x, denote the q-number $[x]_q$ by

$$[x]_q = \frac{1 - q^x}{1 - q} \,.$$

We have $[x]_q \to x$ $(q \to 1)$.

There are some different definitions for q-zeta functions. For $0 < x \leq 1$, $s \in \mathbb{C}$, and $\Re e(s) > 1$, define a *Hurwitz q-zeta function* $\zeta_q(s,x)$ [5] as

$$\zeta_q(s,x) = \sum_{n=0}^{\infty} \frac{q^{(n+x)r}}{[n+x]_{q^r}^s}\,. \tag{3}$$

When $x = 1$, $\zeta_q(s) = \zeta_q(s,1)$ is the q-zeta function. In this paper, following [2, 4], define a *generalized q-harmonic number* $h_n^{(s)}(x)$ as

$$h_n^{(s)}(x) = \sum_{k=1}^{n} \frac{1}{[k-1+x]_{q^r}^s}\,. \tag{4}$$

The well-known q-binomial coefficients are defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

with q-factorials $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. For convenience, we introduce another two q-binomial coefficients. We define the *shifted q-binomial coefficients* $\binom{n+x}{k}_q$ by

$$\binom{n+x}{k}_q = \frac{[n+x]_q[n+x-1]_q \cdots [n+x-k+1]_q}{[k]_q!}.$$

We define the *(q_1,q_2)-binomial coefficients* $\binom{n}{k}_{q_1,q_2}$ by

$$\binom{n}{k}_{(q_1,q_2)} = \frac{[n]_{q_2}!}{[k]_{q_1}![n-k]_{q_2}!}.$$

The motivation of the present paper arises from Sofo's work [6] on some infinite series involving the generalized harmonic numbers $H_n^{(r)}$. Our main aim is to establish a q-version of those infinite series. In order to be able to reach our goal, we introduce some more notation. We define the *generalized Hurwitz q-zeta functions* $\zeta_{t,q}(s,x)$ by

$$\zeta_{t,q}(s,x) = \sum_{n=0}^{\infty} \frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\,, \tag{5}$$

where $r, t \in \mathbb{N}$, $0 < x \leq 1$, and $s \in \mathbb{C}$ with $\Re e(s) > 1$. We define the *generalized q-harmonic numbers* $h_{t,n}^{(s)}(x)$ by

$$h_{t,n}^{(s)}(x) = h_{t,n,q}^{(s)}(x) := \sum_{k=1}^{n} \frac{q^{rt(k-1+x)}}{[k-1+x]_{q^r}^s}\,.$$

It is clear that the right-hand side of (5) is absolutely convergent as $0 < q \le 1$. When $s = 1$ and $0 < q < 1$, the right-hand side of (5) is also absolutely convergent. We have $h_{t,n}^{(s)}(x) \to \zeta_{t,q}(s,x)$ $(n \to \infty)$.

In the present paper, with the help of Abel's Lemma on summation by parts, we mainly show that infinite sums of types

$$\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)} [a-1]_{q^r}}{[n+a-1+x]_{q^r} [n+x]_{q^r}}, \quad \sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n-1+x)} [a]_{q^r}}{[n+a-1+x]_{q^r} [n-1+x]_{q^r}},$$
$$\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}}{\binom{n-1+x+k}{k}_{q^r}}, \qquad \sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}}{[n+x-1]_{q^r} \binom{n-1+x+k}{k}_{q^r}}$$

can be expressed in terms of linear combinations of the generalized Hurwitz q-zeta values $\zeta_{t,q}(s,x)$.

2. Main Theorems

We shall show our main results.

Theorem 1. *For $r \in \mathbb{N}$, $0 < x \le 1$, $0 < q < 1$, and $s \in \mathbb{C}$ with $\Re e(s) > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{q^{(n-1+x)r} h_n^{(s)}(x)}{[n+x]_{q^r} [n-1+x]_{q^r}} = \zeta_q(s+1, x)\,.$$

Proof. In (2), put $f_n = h_n^{(s)}(x)$. Then by $\Delta f_n = \frac{1}{[n+x]_{q^r}^s}$, we have

$$\sum_{k=1}^{\infty} h_k^{(s)}(x) \nabla g_k = \lim_{n\to\infty} f_{n+1} g_n - f_1 g_0 - \sum_{k=1}^{\infty} \frac{g_k}{[k+x]_{q^r}^s}\,.$$

Letting

$$g_n = -\frac{q^{r(n+x)}}{[n+x]_{q^r}}\,,$$

we get

$$\nabla g_n = -\frac{q^{r(n+x)}}{[n+x]_{q^r}} + \frac{q^{r(n-1+x)}}{[n-1+x]_{q^r}} = \frac{q^{r(n-1+x)}}{[n+x]_{q^r} [n-1+x]_{q^r}}\,.$$

Since

$$\lim_{n\to\infty} f_{n+1} g_n = -\lim_{n\to\infty} \frac{q^{r(n+x)} h_{n+1}^{(s)}(x)}{[n+x]_{q^r}} = 0\,,$$

we have the desired result. □

When $x = 1$ in Theorem 1, we have the following corollary.

Corollary 1. *For $r \in \mathbb{N}$, $0 < q < 1$, and $s \in \mathbb{C}$ with $\Re e(s) > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{q^{nr} h_n^{(s)}(1)}{[n]_{q^r}[n+1]_{q^r}} = \zeta_q(s+1)\,.$$

Remark 1. When $q \to 1$, Corollary 1 is reduced to

$$\sum_{n=1}^{\infty} \frac{H_n^{(s)}}{n(n+1)} = \zeta(s+1) \qquad ([6, p.21]).$$

We now begin to show some q-versions of Sofo's results.

Theorem 2. *For $0 < q < 1$, $0 < x \le 1$ and $r, s, t, a \in \mathbb{N}$ with $t \ge s$, $a > 1$, we have*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}[a-1]_{q^r}}{[n+a-1+x]_{q^r}[n+x]_{q^r}} \\
&\qquad = \sum_{b=1}^{a-1} \frac{(-1)^{s-1} q^{rb}}{[b]_{q^r}^s} \left(\zeta_{t-s+1,q}(1,x) - q^{-rb(t-s+1)}\zeta_{t-s+1,q}(1,x+b)\right) \\
&\qquad\quad + \sum_{m=2}^{s} (-1)^{s-m} \zeta_{t-s+m,q}(m,x) h_{1,a-1}^{(s-m+1)}(1)\,.
\end{aligned}$$

When $q \to 1$, $\zeta_{t-s+1,1}(1,x) - \zeta_{t-s+1,1}(1,x+b)$ is interpreted as

$$h_b^{<1>}(1,x) = \sum_{k=1}^{b} \frac{1}{k-1+x}\,. \tag{6}$$

Proof. Consider the following expansion:

$$\begin{aligned}
&-\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}[a-1]_{q^r}}{[n+a-1+x]_{q^r}[n+x]_{q^r}} \\
&\qquad = \sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}\left(q^{r(a-1)}[n+x]_{q^r} - [n+a-1+x]_{q^r}\right)}{[n+a-1+x]_{q^r}[n+x]_{q^r}} \\
&\qquad = \sum_{n=1}^{\infty} h_{t,n}^{(s)}(x) \left(\frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}} - \frac{q^{r(n+x)}}{[n+x]_{q^r}} \right) \\
&\qquad = \sum_{n=1}^{\infty} h_{t,n}^{(s)}(x) \left(\left(\frac{q^{r(n+1+x)}}{[n+1+x]_{q^r}} + \cdots + \frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}} \right) \right. \\
&\qquad\qquad \left. - \left(\frac{q^{r(n+x)}}{[n+x]_{q^r}} + \cdots + \frac{q^{r(n+a-2+x)}}{[n+a-2+x]_{q^r}} \right) \right)\,.
\end{aligned}$$

If in (2) we put $f_n = h_{t,n}^{(s)}(x)$ and

$$g_n = \frac{q^{r(n+1+x)}}{[n+1+x]_{q^r}} + \cdots + \frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}}\,,$$

then we get

$$\Delta f_n = \frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\,.$$

Since

$$\lim_{n\to\infty} f_{n+1}g_n = \lim_{n\to\infty} h_{t,n+1}^{(s)}(x)\left(\frac{q^{r(n+1+x)}}{[n+1+x]_{q^r}} + \cdots + \frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}}\right) = 0\,,$$

we have

$$\begin{aligned}
&-\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)} [a-1]_{q^r}}{[n+a-1+x]_{q^r}[n+x]_{q^r}} \\
&\qquad = -\frac{q^{rtx}}{[x]_{q^r}^s}\left(\frac{q^{r(1+x)}}{[1+x]_{q^r}} + \cdots + \frac{q^{r(a-1+x)}}{[a-1+x]_{q^r}}\right) \\
&\qquad\qquad - \sum_{n=1}^{\infty} \frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\left(\frac{q^{r(n+1+x)}}{[n+1+x]_{q^r}} + \cdots + \frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}}\right) \\
&\qquad = -\sum_{n=0}^{\infty} \frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\left(\frac{q^{r(n+1+x)}}{[n+1+x]_{q^r}} + \cdots + \frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}}\right) \\
&\qquad = -\sum_{b=1}^{a-1}\sum_{n=0}^{\infty} \frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\frac{q^{r(n+b+x)}}{[n+b+x]_{q^r}}\,.
\end{aligned}$$

By partial fraction decomposition, we have

$$\frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\frac{q^{r(n+b+x)}}{[n+b+x]_{q^r}} = \frac{A_1}{[n+x]_{q^r}} + \frac{A_2}{[n+x]_{q^r}^2} + \cdots + \frac{A_s}{[n+x]_{q^r}^s} + \frac{B_1}{[n+b+x]_{q^r}}\,,$$

where

$$A_j = \frac{(-1)^{s-j} q^{rb} q^{r(t-s+j)(n+x)}}{[b]_{q^r}^{s-j+1}} \quad (1 \le j \le s)$$

and

$$B_1 = \frac{(-1)^s q^{rb} q^{r(t-s+1)(n+x)}}{[b]_{q^r}^s}\,.$$

Thus we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}[a-1]_{q^r}}{[n+a-1+x]_{q^r}[n+x]_{q^r}}\\
&\quad=\sum_{b=1}^{a-1}\sum_{n=0}^{\infty}\left(\frac{A_1}{[n+x]_{q^r}}+\frac{A_2}{[n+x]_{q^r}^2}+\cdots+\frac{A_s}{[n+x]_{q^r}^s}+\frac{B_1}{[n+b+x]_{q^r}}\right)\\
&\quad=\sum_{b=1}^{a-1}\sum_{n=0}^{\infty}\left(\frac{(-1)^{s-1}q^{rb}q^{r(t-s+1)(n+x)}}{[b]_{q^r}^s}\left(\frac{1}{[n+x]_{q^r}}-\frac{1}{[n+b+x]_{q^r}}\right)\right)\\
&\qquad+\sum_{b=1}^{a-1}\sum_{n=0}^{\infty}\sum_{m=2}^{s}\frac{(-1)^{s-m}q^{rb}q^{r(t-s+m)(n+x)}}{[b]_{q^r}^{s-m+1}[n+x]_{q^r}^m}\\
&\quad=\sum_{b=1}^{a-1}\frac{(-1)^{s-1}q^{rb}}{[b]_{q^r}^s}\sum_{n=0}^{\infty}q^{r(t-s+1)(n+x)}\left(\frac{1}{[n+x]_{q^r}}-\frac{1}{[n+b+x]_{q^r}}\right)\\
&\qquad+\sum_{m=2}^{s}(-1)^{s-m}\sum_{b=1}^{a-1}\frac{q^{rb}}{[b]_{q^r}^{s-m+1}}\sum_{n=0}^{\infty}\frac{q^{r(t-s+m)(n+x)}}{[n+x]_{q^r}^m}\\
&\quad=\sum_{b=1}^{a-1}\frac{(-1)^{s-1}q^{rb}}{[b]_{q^r}^s}(\zeta_{t-s+1,q}(1,x)-q^{-rb(t-s+1)}\zeta_{t-s+1,q}(1,x+b))\\
&\qquad+\sum_{m=2}^{s}(-1)^{s-m}\zeta_{t-s+m,q}(m,x)h_{1,a-1}^{(s-m+1)}(1)\,,
\end{aligned}$$

which completes the proof. □

Remark 2. In order to guarantee the absolute convergence of the infinite series used in the proof, it is necessary that to require $t \geq s$.

When $q \to 1$ and $x = 1$ in Theorem 2, we get the following formula, which is given by Sofo [6, Lemma 1.1].

Corollary 2. *For $s, a \in \mathbb{N}$ with $a > 1$, we have*

$$\sum_{n=1}^{\infty}\frac{(a-1)H_n^{(s)}}{(n+1)(n+a)}=\sum_{j=1}^{a-1}\frac{(-1)^{s+1}H_j}{j^s}+\sum_{i=2}^{s}(-1)^{s-i}H_{a-1}^{(s-i+1)}\zeta(i)\,.$$

Theorem 3. *For $0 < q < 1$, $0 < x \le 1$ and $r, s, t, a \in \mathbb{N}$ with $t \ge s$, $a > 1$, we have*

$$\begin{aligned}\sum_{n=1}^{\infty} & \frac{h_{t,n}^{(s)}(x) q^{r(n-1+x)}[a]_{q^r}}{[n+a-1+x]_{q^r}[n-1+x]_{q^r}} \\ &= \zeta_{t+1,q}(s+1,x) + \sum_{m=2}^{s} (-1)^{s-m}\zeta_{t-s+m,q}(m,x) h_{1,a-1}^{(s-m+1)}(1) \\ &\quad + \sum_{b=1}^{a-1} \frac{(-1)^{s-1}q^{rb}}{[b]_{q^r}^s}\left(\zeta_{t-s+1,q}(1,x) - q^{-rb(t-s+1)}\zeta_{t-s+1,q}(1,x+b)\right).\end{aligned}$$

When $q \to 1$, $\zeta_{t-s+1,1}(1,x) - \zeta_{t-s+1,1}(1,x+b)$ is interpreted as $h_b^{<1>}(1,x)$ in (6).

Proof. Consider the following expansion:

$$\begin{aligned}-\sum_{n=1}^{\infty} & \frac{h_{t,n}^{(s)}(x) q^{r(n-1+x)}[a]_{q^r}}{[n+a-1+x]_{q^r}[n-1+x]_{q^r}} \\ &= \sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n-1+x)}(q^{ra}[n-1+x]_{q^r} - [n+a-1+x]_{q^r})}{[n+a-1+x]_{q^r}[n-1+x]_{q^r}} \\ &= \sum_{n=1}^{\infty} h_{t,n}^{(s)}(x)\left(\frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}} - \frac{q^{r(n-1+x)}}{[n-1+x]_{q^r}}\right) \\ &= \sum_{n=1}^{\infty} h_{t,n}^{(s)}(x)\left(\left(\frac{q^{r(n+x)}}{[n+x]_{q^r}} + \cdots + \frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}}\right)\right. \\ &\qquad \left. - \left(\frac{q^{r(n-1+x)}}{[n-1+x]_{q^r}} + \cdots + \frac{q^{r(n+a-2+x)}}{[n+a-2+x]_{q^r}}\right)\right).\end{aligned}$$

If in (2) we put $f_n = h_{t,n}^{(s)}(x)$ and

$$g_n = \frac{q^{r(n+x)}}{[n+x]_{q^r}} + \cdots + \frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}},$$

then we get

$$\Delta f_n = \frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}.$$

Since

$$\lim_{n\to\infty} f_{n+1}g_n = \lim_{n\to\infty} h_{t,n+1}^{(s)}(x)\left(\frac{q^{r(n+x)}}{[n+x]_{q^r}} + \cdots + \frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}}\right) = 0,$$

we have

$$
\begin{aligned}
-\sum_{n=1}^{\infty}&\frac{h_{t,n}^{(s)}(x)q^{r(n-1+x)}[a]_{q^r}}{[n+a-1+x]_{q^r}[n-1+x]_{q^r}}\\
&=-\frac{q^{rtx}}{[x]_{q^r}^s}\left(\frac{q^{rx}}{[x]_{q^r}}+\cdots+\frac{q^{r(a-1+x)}}{[a-1+x]_{q^r}}\right)\\
&\quad-\sum_{n=1}^{\infty}\frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\left(\frac{q^{r(n+x)}}{[n+x]_{q^r}}+\cdots+\frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}}\right)\\
&=-\sum_{n=0}^{\infty}\frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\left(\frac{q^{r(n+x)}}{[n+x]_{q^r}}+\cdots+\frac{q^{r(n+a-1+x)}}{[n+a-1+x]_{q^r}}\right)\\
&=-\sum_{b=0}^{a-1}\sum_{n=0}^{\infty}\frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\frac{q^{r(n+b+x)}}{[n+b+x]_{q^r}}\\
&=-\sum_{n=0}^{\infty}\frac{q^{r(t+1)(n+x)}}{[n+x]_{q^r}^{s+1}}-\sum_{b=1}^{a-1}\sum_{n=0}^{\infty}\frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\frac{q^{r(n+b+x)}}{[n+b+x]_{q^r}}\\
&=-\zeta_{t+1,q}(s+1,x)-\sum_{b=1}^{a-1}\sum_{n=0}^{\infty}\frac{q^{rt(n+x)}}{[n+x]_{q^r}^s}\frac{q^{r(n+b+x)}}{[n+b+x]_{q^r}}\,.
\end{aligned}
$$

With the help of Theorem 2, we get the desired result. □

When $q \to 1$ and $x = 1$ in Theorem 3, we get the following formula, which is given by Sofo [6, Lemma 1.2].

Corollary 3. *For $s, a \in \mathbb{N}$ with $a > 1$, we have*

$$
\sum_{n=1}^{\infty}\frac{aH_n^{(s)}}{n(n+a)}=\zeta(s+1)+\sum_{j=1}^{a-1}\frac{(-1)^{s+1}H_j}{j^s}+\sum_{i=2}^{s}(-1)^{s-i}H_{a-1}^{(s-i+1)}\zeta(i)\,.
$$

Now we consider infinite series involving the shifted q-binomial coefficients.

Theorem 4. *For $0 < q < 1$, $0 < x \leq 1$ and $r, s, t, k \in \mathbb{N}$ with $t \geq s$, $k > 1$, we have*

$$
\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}}{\binom{n-1+x+k}{k}_{q^r}}=\sum_{a=2}^{k}\frac{[-1]_{q^{-r}}^{a-2}[a]_{q^{-r}}[a-1]_{q^{-r}}}{[a-1]_{q^r}}\binom{k}{a}_{(q^{-r},q^r)}A_a\,,
$$

where

$$
\begin{aligned}
A_a&=\sum_{m=2}^{s}(-1)^{s-m}\zeta_{t-s+m,q}(m,x)h_{1,a-1}^{(s-m+1)}(1)\\
&\quad+\sum_{b=1}^{a-1}\frac{(-1)^{s-1}q^{rb}}{[b]_{q^r}^s}\left(\zeta_{t-s+1,q}(1,x)-q^{-rb(t-s+1)}\zeta_{t-s+1,q}(1,x+b)\right).
\end{aligned}
$$

When $q \to 1$, $\zeta_{t-s+1,1}(1,x)-\zeta_{t-s+1,1}(1,x+b)$ is interpreted as $h_b^{<1>}(1,x)$ in (6).

Proof. Consider the following expansion:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}}{\binom{n-1+x+k}{k}_{q^r}}&=\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}[k]_{q^r}!}{[n+x+k-1]_{q^r}\cdots[n+x]_{q^r}}\\&=\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}[k]_{q^r}!}{[n+x]_{q^r}}\prod_{a=2}^{k}\frac{1}{[n+x+a-1]_{q^r}}.\end{aligned}$$

By partial fraction decomposition, we have

$$\begin{aligned}&\prod_{a=2}^{k}\frac{1}{[n+x+a-1]_{q^r}}\\&\qquad=\frac{C_2}{[n+x+1]_{q^r}}+\frac{C_3}{[n+x+2]_{q^r}}+\cdots+\frac{C_k}{[n+x+k-1]_{q^r}},\end{aligned}$$

where

$$\begin{aligned}C_a&=\lim_{n\to -x-a+1}\frac{[n+x+a-1]_{q^r}}{[n+x+1]_{q^r}\cdots[n+x+k-1]_{q^r}}\\&=\lim_{n\to -x-a+1}\frac{1}{[n+x+1]_{q^r}\cdots[n+x+a-2]_{q^r}[n+x+a]_{q^r}\cdots[n+x+k-1]_{q^r}}\\&=\frac{1}{[-a+2]_{q^r}\cdots[-1]_{q^r}[1]_{q^r}\cdots[-a+k]_{q^r}}\quad(2\le a\le k).\end{aligned}$$

Hence, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}}{\binom{n-1+x+k}{k}_{q^r}}&=\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}[k]_{q^r}!}{[n+x]_{q^r}}\sum_{a=2}^{k}\frac{C_a}{[n+x+a-1]_{q^r}}\\&=\sum_{a=2}^{k}\frac{[k]_{q^r}!C_a}{[a-1]_{q^r}}\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}[a-1]_{q^r}}{[n+x]_{q^r}[n+x+a-1]_{q^r}}.\end{aligned}$$

Note that

$$\begin{aligned}\frac{[k]_{q^r}!C_a}{[a-1]_{q^r}}&=\frac{[k]_{q^r}!}{[-a+2]_{q^r}\cdots[-1]_{q^r}[1]_{q^r}\cdots[-a+k]_{q^r}[a-1]_{q^r}}\\&=\frac{[k]_{q^r}![-1]_{q^{-r}}^{a-2}}{[a-2]_{q^{-r}}\cdots[1]_{q^{-r}}[1]_{q^r}\cdots[-a+k]_{q^r}[a-1]_{q^r}}\\&=\frac{[k]_{q^r}![-1]_{q^{-r}}^{a-2}[a]_{q^{-r}}[a-1]_{q^{-r}}}{[a]_{q^{-r}}[a-1]_{q^{-r}}[a-2]_{q^{-r}}\cdots[1]_{q^{-r}}[1]_{q^r}\cdots[-a+k]_{q^r}[a-1]_{q^r}}\\&=\frac{[-1]_{q^{-r}}^{a-2}[a]_{q^{-r}}[a-1]_{q^{-r}}}{[a-1]_{q^r}}\binom{k}{a}_{(q^{-r},q^r)}.\end{aligned}$$

With the help of Theorem 2, we get the desired result. □

When $q \to 1$ and $x = 1$ in Theorem 4, we get the following formula, which is given by Sofo [6, Theorem 2.1].

Corollary 4. *For $s, k \in \mathbb{N}$ with $k > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(s)}}{\binom{n+k}{k}} = \sum_{r=2}^{k} (-1)^r r \binom{k}{r} \left(\sum_{j=1}^{r-1} \frac{(-1)^{s+1} H_j}{j^s} + \sum_{i=2}^{s} (-1)^{s-i} H_{r-1}^{(s-i+1)} \zeta(i) \right).$$

Theorem 5. *For $0 < q < 1$, $0 < x \le 1$ and $r, s, t, k \in \mathbb{N}$ with $t \ge s$, $k > 1$, we have*

$$\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}}{[n+x-1]_{q^r} \binom{n-1+x+k}{k}_{q^r}} = \sum_{a=2}^{k} \frac{[-1]_{q^{-r}}^{a-1} [a]_{q^{-r}}}{[a]_{q^r}} \binom{k}{a}_{(q^{-r}, q^r)} \left(\zeta_{t+1,q}(s+1, x) + A_a \right),$$

where

$$A_a = \sum_{m=2}^{s} (-1)^{s-m} \zeta_{t-s+m,q}(m, x) h_{1,a-1}^{(s-m+1)}(1) + \sum_{b=1}^{a-1} \frac{(-1)^{s-1} q^{rb}}{[b]_{q^r}^s} \left(\zeta_{t-s+1,q}(1, x) - q^{-rb(t-s+1)} \zeta_{t-s+1,q}(1, x+b) \right).$$

When $q \to 1$, $\zeta_{t-s+1,1}(1, x) - \zeta_{t-s+1,1}(1, x+b)$ is interpreted as $h_b^{<1>}(1, x)$ in (6).

Proof. Consider the following expansion:

$$\sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)}}{[n+x-1]_{q^r} \binom{n-1+x+k}{k}_{q^r}} = \sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)} [k]_{q^r}!}{[n+x+k-1]_{q^r} \cdots [n+x-1]_{q^r}}$$
$$= \sum_{n=1}^{\infty} \frac{h_{t,n}^{(s)}(x) q^{r(n+x)} [k]_{q^r}!}{[n+x-1]_{q^r}} \prod_{a=1}^{k} \frac{1}{[n+x+a-1]_{q^r}}.$$

By partial fraction decomposition, we have

$$\prod_{a=1}^{k} \frac{1}{[n+x+a-1]_{q^r}} = \frac{D_1}{[n+x]_{q^r}} + \frac{D_2}{[n+x+1]_{q^r}} + \cdots + \frac{D_k}{[n+x+k-1]_{q^r}},$$

where

$$\begin{aligned} D_a &= \lim_{n \to -x-a+1} \frac{[n+x+a-1]_{q^r}}{[n+x]_{q^r} \cdots [n+x+k-1]_{q^r}} \\ &= \lim_{n \to -x-a+1} \frac{1}{[n+x]_{q^r} \cdots [n+x+a-2]_{q^r} [n+x+a]_{q^r} \cdots [n+x+k-1]_{q^r}} \\ &= \frac{1}{[-a+1]_{q^r} \cdots [-1]_{q^r} [1]_{q^r} \cdots [-a+k]_{q^r}} \quad (1 \le a \le k). \end{aligned}$$

Hence, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}}{[n+x-1]_{q^r}\binom{n-1+x+k}{k}_{q^r}} &= \sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}[k]_{q^r}!}{[n+x-1]_{q^r}}\sum_{a=1}^{k}\frac{D_a}{[n+x+a-1]_{q^r}}\\ &= \sum_{a=1}^{k}\frac{[k]_{q^r}!D_a}{[a]_{q^r}}\sum_{n=1}^{\infty}\frac{h_{t,n}^{(s)}(x)q^{r(n+x)}[a]_{q^r}}{[n+x-1]_{q^r}[n+x+a-1]_{q^r}}.\end{aligned}$$

Note that

$$\begin{aligned}\frac{[k]_{q^r}!D_a}{[a]_{q^r}} &= \frac{[k]_{q^r}!}{[-a+1]_{q^r}\cdots[-1]_{q^r}[1]_{q^r}\cdots[-a+k]_{q^r}[a]_{q^r}}\\ &= \frac{[k]_{q^r}![-1]_{q^{-r}}^{a-1}}{[a-1]_{q^{-r}}\cdots[1]_{q^{-r}}[1]_{q^r}\cdots[-a+k]_{q^r}[a]_{q^r}}\\ &= \frac{[k]_{q^r}![-1]_{q^{-r}}^{a-1}[a]_{q^{-r}}}{[a]_{q^{-r}}[a-1]_{q^{-r}}[a-2]_{q^{-r}}\cdots[1]_{q^{-r}}[1]_{q^r}\cdots[-a+k]_{q^r}[a]_{q^r}}\\ &= \frac{[-1]_{q^{-r}}^{a-1}[a]_{q^{-r}}}{[a]_{q^r}}\binom{k}{a}_{(q^{-r},q^r)}.\end{aligned}$$

With the help of Theorem 3, we get the desired result. □

When $q \to 1$ and $x = 1$ in Theorem 5, we get the following formula, which is given by Sofo [6, Theorem 2.2].

Corollary 5. *For $s, k \in \mathbb{N}$ with $k > 1$, we have*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{H_n^{(s)}}{n\binom{n+k}{k}}\\ &= \zeta(s+1)+\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left(\sum_{j=1}^{r-1}\frac{(-1)^{s+1}H_j}{j^s}+\sum_{i=2}^{s}(-1)^{s-i}H_{r-1}^{(s-i+1)}\zeta(i)\right).\end{aligned}$$

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