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ON NUMERICAL SEMIGROUPS GENERATED BY COMPOUND SEQUENCES

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Abstract

Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be two sequences of positive integers such that $a_i < b_i$ for each i and $\gcd(a_i, b_j) = 1$ for each pair i, j, $i \geq j$. The compound sequence formed from these two sequences is

$$c_0 = a_1 a_2 a_3 \cdots a_k, c_1 = b_1 a_2 a_3 \cdots a_k, c_2 = b_1 b_2 a_3 \cdots a_k, \ldots, c_k = b_1 b_2 b_3 \cdots b_k.$$

Two important special cases are the geometric sequence and the supersymmetric sequence. We determine $\mathrm{F}(S)$, $g(S)$, and $\mathrm{PF}(S)$, for semigroups generated by the compound sequence of any two sequences.

1. Introduction

A numerical semigroup S is a submonoid of $\mathbb{Z}_{\geq 0}$ whose complement $\mathbb{Z}_{\geq 0} \setminus S$ is finite. For the complement to be finite, it is necessary and sufficient that $\gcd(S) = 1$. For a given subset A of positive integers, we write

$$\langle A \rangle = \big\{ a_1 x_1 + \cdots + a_k x_k : a_i \in A, x_i \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N} \big\}.$$

Note that $\langle A \rangle$ is a submonoid of $\mathbb{Z}_{\geq 0}$, and that $S = \langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

We say that A is a set of generators of the semigroup S, or that the semigroup S is generated by the set A, when $S = \langle A \rangle$. Further, A is a minimal set of generators for S if A is a set of generators of S and no proper subset of A generates S. Every semigroup has a unique minimal set of generators. The embedding dimension $e(S)$ of S is the size of the minimal set of generators.

If S is a numerical semigroup, the finite set $\mathbb{Z}_{\geq 0} \setminus S$ is called the gap set of S, and is denoted by $\mathrm{G}(S)$. The cardinality of the gap set is the genus of S and denoted by $g(S) = |\mathrm{G}(S)|$. The largest element in $\mathrm{G}(S)$ is the Frobenius number of S, and is denoted by $\mathrm{F}(S)$. A positive integer $n \notin S$ is a pseudo-Frobenius number if $n+s \in S$ for each $s \in S$, $s > 0$, and the set of pseudo-Frobenius numbers is denoted by $\mathrm{PF}(S)$. The cardinality of the set $\mathrm{PF}(S)$ is called the type of S, and denoted by $t(S)$.

The Apéry set of $S = \langle A \rangle$ corresponding to any fixed $a \in S$, denoted by $\mathrm{Ap}(S,a)$, consists of those $n \in S$ for which $n - a \notin S$. Thus, $\mathrm{Ap}(S,a)$ is the set of minimum integers in $S \cap \mathbf{C}$ as $\mathbf{C}$ runs through the complete set of residue classes modulo a.

The integers $\mathrm{F}(S)$ and $g(S)$, and the set $\mathrm{PF}(S)$, can be computed from the Apéry set $\mathrm{Ap}(S,a)$ of S corresponding to any $a \in S$ via the following proposition.

Proposition 1 (BS62, Sel77, Tri03)**.** *Let S be a numerical semigroup, let $a \in S$, and let $Ap(S,a)$ be the Apéry set of S corresponding to a. Then*

(i)
$$F(S) = \max\Big(Ap(S,a)\Big) - a;$$

(ii)
$$g(S) = \frac{1}{a}\left(\sum_{n \in Ap(S,a)} n\right) - \frac{a-1}{2};$$

(iii)
$$PF(S) = \Big\{n - a : n \in Ap(S,a), n + \boldsymbol{m}_i > \boldsymbol{m}_{i+n}, i = 1, \ldots, a-1\Big\},$$

where $\boldsymbol{m}_i \in Ap(S,a)$ and $\boldsymbol{m}_i \equiv i \pmod{a}$.

Symmetric numerical semigroups are probably the numerical semigroups that have been most studied in the literature. The motivation and introduction of these semigroups is due mainly to Kunz [3], who proved that a one-dimensional analytically irreducible Noetherian local ring is Gorenstein if and only if its value semigroup is symmetric. Symmetric numerical semigroups always have odd Frobenius number. A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it, and *symmetric* if it is irreducible and $\mathrm{F}(S)$ is odd. The following Proposition lists some equivalent conditions for symmetric numerical semigroups.

Proposition 2 (Corollary 4.5, Proposition 4.10, Corollary 4.11, RG-S09)**.** *For a numerical semigroup S, the following statements are equivalent:*

(i) *S is symmetric.*

(ii) *$PF(S) = \{F(S)\}$.*

(iii) *$t(S) = 1$.*

(iv) *If $Ap(S,n) = \{a_0 < a_1 < \ldots < a_{n-1}\}$, then $a_i + a_{n-1-i} = a_{n-1}$ for each $i \in \{0, \ldots, n-1\}$.*

(v) *$n \notin S$ if and only if $F(S) - n \in S$ and $F(S)$ is odd.*

(vi) *$F(S)$ is odd and $g(S) = \frac{1+F(S)}{2}$.*

Numerical semigroups generated by compound sequences were introduced and studied by Kiers, O'Neill & Ponomarenko [2]. These are examples of symmetric numerical semigroups. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be two sequences of positive integers such that $a_i < b_i$ for each i and $\gcd(a_i, b_j) = 1$ for each pair i, j, $i \geq j$. The *compound sequence* formed from these two sequences is

$$c_0 = a_1 a_2 a_3 \cdots a_k, c_1 = b_1 a_2 a_3 \cdots a_k, c_2 = b_1 b_2 a_3 \cdots a_k, \ldots, c_k = b_1 b_2 b_3 \cdots b_k. \quad (1)$$

Note that $\gcd(c_0, c_1, c_2, \ldots, c_k) = 1$. Two important special cases are

- The compound sequence for $a_1 = \ldots = a_k = a$ and $b_1 = \ldots = b_k = b$, $\gcd(a, b) = 1$ is the geometric sequence
$$a^k, a^{k-1}b, a^{k-2}b^2, \ldots, b^k.$$
- For pairwise coprime positive integers $a_1, \ldots, a_k$, the compound sequence for $a_2, a_3, \ldots, a_k$ and $a_1, a_2, \ldots, a_{k-1}$ is the supersymmetric sequence
$$\frac{P}{a_1}, \frac{P}{a_2}, \ldots, \frac{P}{a_k},$$
where $P = a_1 a_2 \cdots a_k$.

Kiers et al. determine an Apéry set and the Frobenius number, among other functions, for semigroups generated by compound sequences of any two given sequences. In this short note, we give an alternate and direct proof of their result on Apéry sets, and use this to compute $F(S)$, $g(S)$, and $PF(S)$.

2. Main Results

Theorem 1 (Theorem 15, KNP16)**.** *Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be two sequences of positive integers such that $a_i < b_i$ for each i and $\gcd(a_i, b_j) = 1$ for each pair i, j, $i \geq j$. Then the Apéry set for the numerical semigroup S generated by the compound sequence $\{c_0, c_1, c_2, \ldots, c_k\}$ of these two sequences is given by*

$$Ap(S, c_0) = \left\{ \sum_{i=1}^{k} c_i x_i : 0 \leq x_i \leq a_i - 1, i = 1, \ldots, k \right\}.$$

Proof. For convenience, let $\mathbf{v}(x_1, \ldots, x_k) = \sum_{i=1}^{k} c_i x_i$, $0 \leq x_i \leq a_1 - 1$, $i = 1, \ldots, k$, and note that each $\mathbf{v}(x_1, \ldots, x_k) \in S$. Let $\mathbf{v}(x_1, \ldots, x_k) = \mathbf{v}(y_1, \ldots, y_k)$ with $x_i, y_i \in \{0, \ldots, a_i - 1\}$ for each i, so that

$$c_1 x_1 + \cdots + c_{k-1} x_{k-1} + c_k x_k = c_1 y_1 + \cdots + c_{k-1} y_{k-1} + c_k y_k. \quad (2)$$

Since $a_k \mid c_i$ for $i \in \{1, \ldots, k-1\}$ and $\gcd(a_k, c_k) = 1$, reducing Equation (2) modulo a_k gives $x_k \equiv y_k \pmod{a_k}$. Therefore $x_k = y_k$, and Equation (2) reduces to

$$c_1x_1 + \cdots + c_{k-2}x_{k-2} + c_{k-1}x_{k-1} = c_1y_1 + \cdots + c_{k-2}y_{k-2} + c_{k-1}y_{k-1}. \quad (3)$$

Reducing Equation (3) modulo a_{k-1} leads to $x_{k-1} = y_{k-1}$, and continuing this argument $x_i = y_i$, $i \in \{1, \ldots, k\}$. Hence $\{\mathbf{v}(x_1, \ldots, x_k) : 0 \le x_i \le a_i - 1, i = 1, \ldots, k\}$ is a complete residue system modulo c_0.

To show that $N \in \mathrm{Ap}\big(S, c_0\big)$ if and only if $N = \mathbf{v}(x_1, \ldots, x_k)$ with x_i satisfying $0 \le x_i \le a_i - 1$ for each i, we must show that each such $\mathbf{v}(x_1, \ldots, x_k) - c_0 \notin S$.

Suppose, by way of contradiction, that $\mathbf{v}(x_1, \ldots, x_k) - c_0 \in S$, so that

$$\mathbf{v}(x_1, \ldots, x_k) - c_0 = \mathbf{v}(y_1, \ldots, y_k) \quad (4)$$

with each $y_i \ge 0$. The transformation $(y_{k-1}, y_k) \mapsto (y_{k-1} + b_k, y_k - a_k)$ maintains the value of $\mathbf{v}(y_1, \ldots, y_k)$, and we repeatedly apply this until $0 \le y_k \le a_k - 1$. Note that the corresponding y_{k-1} is greater than 0. Next we repeatedly apply the transformation $(y_{k-2}, y_{k-1}) \mapsto (y_{k-2} + b_{k-1}, y_{k-1} - a_{k-1})$ until $0 \le y_{k-1} \le a_{k-1} - 1$. The corresponding $y_{k-2} > 0$, and the value of $\mathbf{v}(y_1, \ldots, y_k)$ is maintained. Continuing with this process with successive transformations $(y_i, y_{i+1}) \mapsto (y_i + b_{i+1}, y_{i+1} - a_{i+1})$, $i > 1$ leads to the same value of $\mathbf{v}(y_1, \ldots, y_k)$, but with $0 \le y_i \le a_i - 1$ for each $i > 1$ and $y_{i-1} > 0$. Therefore we may additionally assume $0 \le y_i \le a_i - 1$ for each $i > 1$ and $y_1 > 0$ in Equation (4).

Reducing both sides of Equation (4) modulo a_k and arguing as we did following Equation (2) leads to $x_k = y_k$. Cancelling equal terms c_kx_k and c_ky_k from Equation (4) and reducing modulo a_{k-1} leads to $x_{k-1} = y_{k-1}$ following the argument after Equation (3). Repeating this line of argument we are led to the equation

$$c_1x_1 - c_0 = c_1y_1, \quad (5)$$

with $0 \le x_1 \le a_1 - 1$ and $y_1 > 0$. Equation (5) reduces to $b_1(x_1 - y_1) = a_1$, and hence to $x_1 \equiv y_1 \pmod{a_1}$ since $\gcd(a_1, b_1) = 1$. Together with $y_1 < x_1$ from Equation (5) and $0 \le x_1 \le a_1 - 1$, we have $y_1 < 0$. This contradiction proves $\mathbf{v}(x_1, \ldots, x_k) - c_0 \notin S$. $\square$

Theorem 2. *Using the notation of Theorem 1,*

$$F(S) = \left(\sum_{i=1}^{k}(a_i - 1)c_i\right) - c_0.$$

In particular, if S_1 is the semigroup generated by geometric sequence $a^k, a^{k-1}b, \ldots, b^k$ and S_2 the semigroup generated by the supersymmetric sequence $\frac{P}{a_1}, \frac{P}{a_2}, \ldots, \frac{P}{a_k}$, where $P = a_1a_2\cdots a_k$, then

(i)
$$F(S_1) = \sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1} + b^{k+1}),$$
where $\sigma_k = a^k + a^{k-1}b + \cdots + b^k$ *and* $\sigma_{k+1} = a^{k+1} + a^k b + \cdots + b^{k+1}$*;*

(ii)
$$F(S_2) = (k-1)\sigma_k - \sigma_{k-1},$$
where $\sigma_k = P$ *and* $\sigma_{k-1} = \frac{P}{a_1} + \cdots + \frac{P}{a_k}$.

Proof. The general result for $\mathrm{F}(S)$ is a direct consequence of Proposition 1 and Theorem 1.

To obtain the result for the semigroup S_1 generated by the geometric sequence, we set $a_1 = \cdots = a_k = a$ and $b_1 = \cdots = b_k = b$ in the general case. Then $c_i = a^{k-i}b^i$, $i = 0, \ldots, k$, and so

$$\begin{aligned}
\mathrm{F}(S_1) &= (a-1)\sum_{i=1}^{k} a^{k-i}b^i - a^k \\
&= (a-1)\big(\sigma_k(a,b) - a^k\big) - a^k \\
&= a \cdot \sigma_k(a,b) - \sigma_k(a,b) - a^{k+1} \\
&= \big(\sigma_{k+1}(a,b) - b^{k+1}\big) - \sigma_k(a,b) - a^{k+1} \\
&= \sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1} + b^{k+1}).
\end{aligned}$$

To obtain the result for the semigroup S_2 generated by supersymmetric sequences, we replace a_i by a_{i+1}, b_i by a_i, and k by $k-1$. Then $c_i = P/a_{i+1}$, $i = 0, \ldots, k-1$, and so

$$\begin{aligned}
\mathrm{F}(S_2) &= P\sum_{i=1}^{k-1} \frac{a_{i+1}-1}{a_{i+1}} - \frac{P}{a_1} \\
&= (k-1)P - \sum_{i=1}^{k} \frac{P}{a_i} \\
&= (k-1)\sigma_k - \sigma_{k-1}.
\end{aligned}$$

□

Remark 1. The results in parts (i) and (ii) appear in [7, Theorem 1] and [8, Theorem 1], respectively.

Theorem 3. *Using the notation of Theorem 1,*

$$g(S) = \frac{1}{2}\left(\sum_{i=1}^{k}(a_i - 1)c_i - c_0 + 1\right).$$

In particular, if S_1 is the semigroup generated by geometric sequence $a^k, a^{k-1}b, \dots, b^k$ and S_2 the semigroup generated by the supersymmetric sequence $\frac{P}{a_1}, \frac{P}{a_2}, \dots, \frac{P}{a_k}$, where $P = a_1 a_2 \cdots a_k$, then

(i)
$$g(S_1) = \frac{1}{2}\left(\sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1} + b^{k+1}) + 1\right),$$

where $\sigma_k = a^k + a^{k-1}b + \cdots + b^k$ and $\sigma_{k+1} = a^{k+1} + a^k b + \cdots + b^{k+1}$;

(ii)
$$g(S_2) = \frac{1}{2}\left((k-1)\sigma_k - \sigma_{k-1} + 1\right),$$

where $\sigma_k = P$ and $\sigma_{k-1} = \frac{P}{a_1} + \cdots + \frac{P}{a_k}$.

Proof. The general result for $g(S)$ is a direct consequence of Proposition 1 and Theorem 1. More specifically,

$$\begin{aligned}
g(S) &= \frac{1}{c_0} \sum_{\substack{0 \le x_i \le a_i - 1 \\ 1 \le i \le k}} (c_1 x_1 + \cdots + c_k x_k) - \frac{c_0 - 1}{2} \\
&= \frac{1}{c_0} \sum_{i=1}^{k} \left(\prod_{\substack{1 \le j \le k \\ j \ne i}} a_j \right) c_i \sum_{x_i = 0}^{a_i - 1} x_i - \frac{c_0 - 1}{2} \\
&= \frac{1}{c_0} \sum_{i=1}^{k} \frac{c_0}{a_i} c_i \cdot \frac{1}{2} a_i (a_i - 1) - \frac{c_0 - 1}{2} \\
&= \frac{1}{2} \left(\sum_{i=1}^{k} (a_i - 1) c_i - c_0 + 1 \right).
\end{aligned}$$

To obtain the result for the semigroup S_1 generated by the geometric sequence, we set $a_1 = \cdots = a_k = a$ and $b_1 = \cdots = b_k = b$ in the general case. Then $c_i = a^{k-i} b^i$, $i = 0, \dots, k$, and so

$$\begin{aligned}
g(S_1) &= \frac{1}{2} \left((a-1) \sum_{i=1}^{k} a^{k-i} b^i - a^k + 1 \right) \\
&= \frac{1}{2} \left((a-1)\big(\sigma_k(a,b) - a^k\big) - a^k + 1 \right) \\
&= \frac{1}{2} \left(a \cdot \sigma_k(a,b) - \sigma_k(a,b) - a^{k+1} + 1 \right) \\
&= \frac{1}{2} \left(\big(\sigma_{k+1}(a,b) - b^{k+1}\big) - \sigma_k(a,b) - a^{k+1} + 1 \right) \\
&= \frac{1}{2} \left(\sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1} + b^{k+1}) + 1 \right).
\end{aligned}$$

To obtain the result for the semigroup S_2 generated by supersymmetric sequences, we replace a_i by a_{i+1}, b_i by a_i, and k by $k-1$. Then $c_i = P/a_{i+1}$, $i = 0, \ldots, k-1$, and so

$$\begin{aligned} g(S_2) &= \frac{1}{2}\left(P\sum_{i=1}^{k-1}\frac{a_{i+1}-1}{a_{i+1}} - \frac{P}{a_1} + 1\right) \\ &= \frac{1}{2}\left((k-1)P - \sum_{i=1}^{k-1}\frac{P}{a_{i+1}} - \frac{P}{a_1} + 1\right) \\ &= \frac{1}{2}\left((k-1)P - \sum_{i=1}^{k}\frac{P}{a_i} + 1\right) \\ &= \frac{1}{2}\Big((k-1)\sigma_k - \sigma_{k-1} + 1\Big). \end{aligned}$$

□

Remark 2. The results in parts (i) and (ii) appear in [7, Theorem 1] and [8, Theorem 1], respectively.

Theorem 4. *Using the notation of Theorem 1,*

$$PF(S) = \left\{\sum_{i=1}^{k}(a_i - 1)c_i - c_0\right\}.$$

In particular, if S_1 is the semigroup generated by geometric sequence $a^k, a^{k-1}b, \ldots, b^k$ and S_2 the semigroup generated by the supersymmetric sequence $\frac{P}{a_1}, \frac{P}{a_2}, \ldots, \frac{P}{a_k}$, where $P = a_1a_2\cdots a_k$, then

(i)
$$PF(S_1) = \left\{\sigma_{k+1}(a,b) - \sigma_k(a,b) - (a^{k+1} + b^{k+1})\right\},$$
where $\sigma_k = a^k + a^{k-1}b + \cdots + b^k$ and $\sigma_{k+1} = a^{k+1} + a^k b + \cdots + b^{k+1}$;

(ii)
$$PF(S_2) = \{(k-1)\sigma_k - \sigma_{k-1}\},$$
where $\sigma_k = P$ and $\sigma_{k-1} = \frac{P}{a_1} + \cdots + \frac{P}{a_k}$.

Proof. We use Proposition 1 and Theorem 1 to prove the general result for PF(S). As in the proof of Theorem 1, let $\mathbf{v}(x_1, \ldots, x_k) = \sum_{i=1}^{k} c_i x_i$, $0 \le x_i \le a_1 - 1$, $i = 1, \ldots, k$. Each $n \in$ PF(S) is of the form $\mathbf{v}(x_1, \ldots, x_k) - c_0$, $0 \le x_i \le a_1 - 1$, $i = 1, \ldots, k$, and

$$\mathbf{v}(x_1, \ldots, x_k) + \mathbf{v}(a_1 - x_1, \ldots, a_k - x_k) = \mathbf{v}(a_1 - 1, \ldots, a_k - 1).$$

implies $\mathbf{v}(x_1, \ldots, x_k) - c_0 \notin \mathrm{PF}(S)$ if $x_i < a_i - 1$ for at least one $i \in \{1, \ldots, k\}$. The only remaining element is $\mathbf{v}(a_1 - 1, \ldots, a_k - 1)$, and so this must belong to $\mathrm{PF}(S)$.

The special cases given in parts (i) and (ii) follow from their derivations in Theorem 2. □

Remark 3. The results in parts (i) and (ii) appear in [7, Theorem 2] and [8, Theorem 3], respectively. The general result as well as the results in parts (i) and (ii) are also direct consequences of Proposition 2 and Theorems 2 and 3.

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