



HIGH DEGREE b -NIVEN NUMBERS

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Received: 7/21/18, Revised: 7/29/20, Accepted: 9/30/21, Published: 10/25/21

Abstract

Let b be a numeration base. A b -Niven number is one that is divisible by the sum of its base b digits. We introduce high degree b -Niven numbers. These are b -Niven numbers that have a power greater than 1 that is a b -Niven number. Our main result shows that for each degree there exists an infinite number of bases b for which an infinite number of b -Niven numbers of that degree exists. The high degree b -Niven numbers are given by explicit formulas and have all digits different from zero. Moreover, given a degree d , we show that there exists an infinite number of bases for which an infinite number of b -Niven numbers, simultaneously of all degrees between 1 and d , exist.

1. Introduction

Niven (or Harshad) numbers are numbers divisible by the sum of their decimal digits. Niven numbers have been extensively studied. See, for instance [3] [4] [5], and [7]. Of interest, and mentioned in the literature, are also b -Niven numbers. These are numbers divisible by the sum of their base b digits ;See, for example, [6]. Other variants of Niven numbers can be found in [1] and [2].

In [8] the author introduced degree 2 b -Niven numbers. These are b -Niven numbers for which the squares also are b -Niven numbers. For example, it follows from the divisibility test by $b - 1$ and from $(b - 1)^2 = [(b - 2)1]_b$ that $[b - 1]_b$ is a degree 2 b -Niven number if $b \geq 3$. The goal of this paper is to introduce high degree b -Niven numbers. These are b -Niven numbers for which a high power is also a b -Niven number. For example, it follows from the divisibility test by $b - 1$ and from $(b - 1)^3 = [(b - 3)2(b - 1)]_b$ that $[b - 1]_b$ is a degree 3 b -Niven number if $b \geq 4$. Our main results, Theorems 1 and 2, show that for each degree there exists an infinite number of bases in which an infinite number of b -Niven numbers of that degree appears. We observe that the high degree b -Niven numbers shown in this paper are

given by explicit formulas and have all digits different from zero. In Example 1 we show that the number $[(6510)(6510)]_{6511}$ is 6511-Niven of degree 15.

This distinction is important if one wants to avoid the trivial examples of b -Niven numbers given by the powers of $[10]_b$.

Theorem 3 shows that, given a degree d , there exists an infinite number of bases for which an infinite number of b -Niven numbers, simultaneously of all degrees between 1 and d , exists.

All numeration bases in our results are odd. In particular, we do not know how to show the existence of an infinite number of high degree (or even quadratic) 10-Niven numbers with all digits different from zero. Nevertheless, particular examples of high degree 10-Niven numbers are easy to find. For example, 3 is 10-Niven of degrees 1 through 5, but not of degree 6, and 6 is 10-Niven of degrees 1 through 10, but not of degree 11.

The examples naturally lead to the following questions, which are open.

Question 1. Does there exist an integer that is not a power of 10 that is 10-Niven of an infinite number of degrees?

Question 2. Does there exist an integer that is not 10-Niven of an infinite number of degrees?

2. Statements of the Main Results

In what follows let $b \geq 2$ be a numeration base. We let $s_b(N)$ denote the sum of the base b digits of the integer N . If x is a string of digits, we denote by $(x)^k$ the base 10 integer obtained by repeating x k times. We denote by $[x]_b$ the value of the string in base b .

Definition 1. Let $m \geq 1$ be an integer. An integer N is called a *degree m b -Niven number* if N and N^m are b -Niven numbers.

It is shown in [8] that if $b \equiv 3 \pmod{4}$ then there exists an infinite number of degree 2 b -Niven numbers. It is obvious that any power of $[10]_b$ is a degree m b -Niven number for any m and for any b . Moreover, if N is a degree m b -Niven number then $10_b \times N$ is also a degree m b -Niven number.

The following theorem gives a criterion for the existence of even degree b -Niven numbers with all digits different from zero.

Theorem 1. *Let $b \geq 2, k \geq 1, n \geq 1$ be integers. Consider the number:*

$$N_k = [b^{2^k} - 1]_b.$$

Assume that $n = 2^q p, p \geq 0$ odd, $q \geq 0, b \geq \binom{2n}{n}$, and $b = (4\ell + 2)p + 1, \ell \geq 0$. Then N_k is a degree $2n$ b -Niven number. In particular, for any even degree, there exists

an infinite number of bases in which an infinite number of b -Niven numbers of that degree, with all digits different from zero, exists.

The proof of Theorem 1 is done in Section 3.

Estimates for the size of b in Theorem 1 can be obtained from standard estimates for binomial coefficients, for example $\frac{4^n}{\sqrt{n}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt[3]{n}}$.

The following theorem gives a criterion for the existence of odd degree b -Niven numbers with all digits different from zero.

Theorem 2. *Let $b \geq 2, k \geq 1, n \geq 1$ be integers. Consider the numbers*

$$N_k = [b^{2^k} - 1]_b.$$

Assume that $\frac{n+1}{2} = 2^q p, p \geq 0$ odd, $q \geq 0, b \geq \binom{n}{\frac{n+1}{2}}$, and $b = (4\ell + 2)p + 1, \ell \geq 0$. Then N_k is a degree n b -Niven number. In particular, for any odd degree, there exists an infinite number of bases in which an infinite number of b -Niven numbers of that degree, with all digits different from zero, exists.

The proof of Theorem 2 is done in Section 4.

Theorems 1 and 2 have the following corollary.

Corollary 1. *For any degree there exists an infinite number of bases in which an infinite number of b -Niven numbers of that degree, with all digits different from zero, exists.*

We do not know how to answer the following question.

Question 3. Does there exist a base b for which b -Niven numbers of arbitrary high degree, with all digits different from zero, exist?

The following theorem shows that there are an infinite number of bases containing b -Niven numbers of multiple consecutive high degrees, with all digits different from zero.

Theorem 3. *Let $b \geq 2, d \geq 1$. Consider the sequence $(N_k)_{k \geq 1}$, where $N_k = [b^{2^k} - 1]_b$. Then there exists an infinite set of bases b in which the sequence $(N_k)_{k \geq 1}$ consists of b -Niven numbers of degree i for all $1 \leq i \leq d$.*

Proof. The assumptions about b needed to apply Theorems 1 and 2 for $1 \leq i \leq d$ are as follows:

1. $b \geq \binom{2^i}{i}$ if i is even;
2. $b \geq \binom{i}{\frac{i+1}{2}}$ if i is odd;
3. $b \equiv 3 \pmod{4}$;

4. $b \equiv 1 \pmod{p_i}$, where $i = 2^{\ell_i} p_i$ with p_i odd.

The first two conditions are valid for $b \geq \binom{2d}{d}$, if d is even, and for $b \geq \binom{d+1}{\frac{d+1}{2}}$, if d is odd. The system of congruences has an infinite set of solutions given by:

$$b = (4\ell + 2)p_1 p_2 \cdots p_d + 1, \ell \geq 1.$$

□

Example 2.7. Assume $b \equiv 3 \pmod{4}$ and consider the sequence $(N_k)_{k \geq 1}$, where $N_k = [b^{2^k} - 1]_b$. The numbers N_k are degree 2 b -Niven numbers and degree 3 b -Niven numbers. If $b \geq 6$, then N_k also are degree 4 b -Niven numbers. If $b \geq 10$ and $b \equiv 1 \pmod{3}$, then the N_k are also degree 5 b -Niven numbers. If $b \geq 20$ and $b \equiv 1 \pmod{3}$, then the N_k are also degree 6 b -Niven numbers. If $b \geq 35$, then the N_k are also degree 7 b -Niven numbers. If $b \geq 70$, then N_k are also degree 8 b -Niven numbers. If $b \geq 126$ and $b \equiv 1 \pmod{5}$ then the N_k are also degree 9 b -Niven numbers. If $b \geq 252$ and $b \equiv 1 \pmod{5}$ then the N_k are also degree 10 b -Niven numbers. If $b \geq 66$ and $b \equiv 1 \pmod{3}$, then the N_k are also degree 11 b -Niven numbers. If $b \geq 924$ and $b \equiv 1 \pmod{3}$, then the N_k are also degree 12 b -Niven numbers. If $b \geq 1716$ and $b \equiv 1 \pmod{7}$, then the N_k are also degree 13 b -Niven numbers. If $b \geq 3432$ and $b \equiv 1 \pmod{7}$, then the N_k are also degree 14 b -Niven numbers. If $b \geq 6435$, then the N_k are also degree 15 b -Niven numbers. Summing up, we conclude that if $b \geq 6435$ and $b = 105(4\ell + 2) + 1, \ell \geq 1$, then the numbers N_k are degree i b -Niven numbers for all $1 \leq i \leq 15$. The smallest base b that satisfies the above conditions is $b = 6511$. The base b representations for N_k and its powers can easily be derived using formulas (2) and (3) from the proofs of Theorem 1 and 2. For example:

- $N_1 = [(6510)(6510)]_{6511}$
- $N_1^2 = [(6510)(6509)(0)(1)]_{6511}$
- $N_1^3 = [(6510)(6508)(0)(2)(6510)(6510)]_{6511}$
- $N_1^4 = [(6510)(6507)(0)(5)(6510)(6507)(0)(1)]_{6511}$
- $N_1^5 = [(6510)(6506)(0)(9)(6510)(6501)(0)(4)(6510)(6510)]_{6511}$.

Question 4. Due to the condition $b \equiv 3 \pmod{4}$, the bases from Theorem 3 are all odd. Is the result in Theorem 3 true for an infinite number of even bases?

3. Proof of Theorem 1

It follows from the assumptions in Theorem 1 that b satisfies $b \equiv 3 \pmod{4}$ and $b \equiv 1 \pmod{p}$.

An equivalent representation for N_k is $[(b-1)^{\wedge 2^k}]_b$, so $s_b(N_k) = (b-1)2^k$. As b is odd, $\gcd(b, 2) = 1$. As $\phi(2^{k+1}) = 2^k$, Euler's theorem gives that 2^{k+1} divides N_k . In addition $b-1$ divides N_k . As $b \equiv 3 \pmod{4}$, $\gcd(b-1, 2) = 2$. It follows that $(b-1)2^k$ divides N_k , so N_k is a b -Niven number.

We compute $s_b(N_k^{2n})$ using the assumption that $b \geq \binom{2n}{n}$. From the binomial formula it follows that:

$$N_k^{2n} = (b^{2^k} - 1)^{2n} = \sum_{i=1}^n \left(\binom{2n}{2i} b^{2i \cdot 2^k} - \binom{2n}{2i-1} b^{(2i-1) \cdot 2^k} \right) + 1. \quad (1)$$

Each one of the differences $\binom{2n}{2i} b^{2i \cdot 2^k} - \binom{2n}{2i-1} b^{(2i-1) \cdot 2^k}$ in (1) has a base b representation given by:

$$\left[\left(\binom{2n}{2i} - 1 \right) (b-1)^{\wedge 2^k - 1} \left(b - \binom{2n}{2i-1} \right) (0)^{\wedge (2i-1) \cdot 2^k - 1} \right]_b. \quad (2)$$

As the contributions (2) to the base b representation of N_k^{2n} do not overlap, one has that:

$$\begin{aligned} s_b(N_k^{2n}) &= n(b-1)(2^k - 1) \\ &+ \left(b - \binom{2n}{1} \right) + \left(b - \binom{2n}{3} \right) + \cdots + \left(b - \binom{2n}{2n-1} \right) \\ &+ \left(\binom{2n}{2} - 1 \right) + \left(\binom{2n}{4} - 1 \right) + \cdots + \left(\binom{2n}{2n-2} - 1 \right) \\ &+ 1, \end{aligned} \quad (3)$$

where the second line in (3) has n terms and the third line has $n-1$ terms.

After rearranging the terms in (3) and using that

$$\begin{aligned} \binom{2n}{1} + \binom{2n}{3} + \cdots + \binom{2n}{2n-1} &= 2^{2n-1} \\ \binom{2n}{2} + \binom{2n}{4} + \cdots + \binom{2n}{2n-2} &= 2^{2n-1} - 2, \end{aligned}$$

formula (3) becomes:

$$\begin{aligned} s_b(N_k^{2n}) &= n(b-1)(2^k - 1) + nb - 2^{2n-1} + 2^{2n-1} - 2 - (n-1) + 1 \\ &= n(b-1)2^k. \end{aligned}$$

It remains to show that $n(b-1)2^k$ divides N_k^{2n} . From above it follows that $(b-1)2^k$ divides N_k . To finish the proof we show that n divides N_k^{2n-1} .

The assumption $b \equiv 3 \pmod{4}$ implies that 2 divides $b-1$, and the assumption $b \equiv 1 \pmod{p}$ implies that p divides $b-1$. As $\gcd(2, p) = 1$, $2p$ divides $b-1$. But $b-1$ divides N_k , so

$$2^{2n-1}p \mid N_k^{2n-1}. \quad (4)$$

Due to the fact that $\ell \leq 2^{\ell-1}$ for $\ell \geq 1$, one has that

$$2^q | 2^{2^q-1} | 2^{2^q-1} \cdot 2^{2^{q+1}p-2^q} = 2^{2^{q+1}p-1} = 2^{2n-1}. \quad (5)$$

Combining (4) and (5) and using the fact that $\gcd(p, 2^q) = 1$, one has that $n = 2^q p$ divides N_k^{2n-1} .

4. Proof of Theorem 2

It follows from the assumptions in Theorem 2 that b satisfies the equations $b \equiv 3 \pmod{4}$, $b \equiv 1 \pmod{p}$.

It follows from the first paragraph of Section 3 that N_k is a b -Niven number.

We compute $s_b(N_k^n)$ using the assumption that $b \geq \binom{n}{\frac{n+1}{2}}$. From the binomial formula, it follows that

$$N_k^n = (b^{2^k} - 1)^n = \sum_{i=1}^{\frac{n+1}{2}} \left(\binom{n}{n+2-2i} b^{(n+2-2i) \cdot 2^k} - \binom{n}{n+1-2i} b^{(n+1-2i) \cdot 2^k} \right). \quad (6)$$

Each one of the differences $\binom{n}{n+2-2i} b^{(n+2-2i) \cdot 2^k} - \binom{n}{n+1-2i} b^{(n+1-2i) \cdot 2^k}$ in (6) has a base b representation given by:

$$\left[\left(\binom{n+2-2i}{n} - 1 \right) (b-1)^{\wedge 2^k-1} \left(b - \binom{n}{n+1-2i} \right) (0)^{\wedge (n+1-2i) \cdot 2^k-1} \right]_b. \quad (7)$$

The contributions (7) to the base b representation of N_k^n do not overlap. Therefore:

$$\begin{aligned} s_b(N_k^n) &= \frac{n+1}{2} (b-1) (2^k-1) \\ &+ \left(b - \binom{n}{n-1} \right) + \left(b - \binom{n}{n-3} \right) + \cdots + \left(b - \binom{n}{0} \right) \\ &+ \left(\binom{n}{n} - 1 \right) + \left(\binom{n}{n-2} - 1 \right) + \cdots + \left(\binom{n}{1} - 1 \right), \end{aligned} \quad (8)$$

where second and third lines in (8) have $\frac{n+1}{2}$ terms.

Rearranging the terms in (8) and using that:

$$\begin{aligned} \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-1} &= 2^{n-1} - 1 \\ \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n} &= 2^{n-1} - 1, \end{aligned} \quad (9)$$

formula (8) becomes:

$$\begin{aligned} s_b(N_k^n) &= \frac{n+1}{2}(b-1)(2^k-1) + \frac{n+1}{2}b - 2^{n-1} + 2^{n-1} - \frac{n+1}{2} \\ &= \frac{n+1}{2}(b-1)2^k. \end{aligned} \quad (10)$$

It remains to show that $\frac{n+1}{2}(b-1)2^k$ divides N_k^n . From above it follows that $(b-1)2^k$ divides N_k . To finish the proof of the theorem we show that $\frac{n+1}{2}$ divides N_k^{n-1} .

The assumption $b \equiv 3 \pmod{4}$ implies that 2 divides $b-1$ and the assumption $b \equiv 1 \pmod{p}$ implies that p divides $b-1$. Overall, as $\gcd(2, p) = 1$, $2p$ divides $b-1$. As $b-1$ divide N_k we conclude that:

$$2^{n-1}p | N_k^{n-1}. \quad (11)$$

Due to the fact, easily proved by induction, that $q \leq 2^q - 1$, one has that:

$$2^q | 2^{2^q-1} | 2^{2^q-1} \cdot 2^{2^q \cdot p - 2^q} = 2^{2^q \cdot p - 1} = 2^{n-1}. \quad (12)$$

Combining (11) and (12), and using the fact that $\gcd(2^q, p) = 1$, one has that $\frac{n+1}{2} = 2^q p$ divides N_k^{n-1} .

Acknowledgements. The author would like to thank the editor and the referee for valuable comments that helped him write a better paper.

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