VALUES OF WEIGHTED DAVENPORT CONSTANTS

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Abstract

For any set $A \subseteq \mathbb{Z}$ and finite abelian group G, the *weighted Davenport constant of* G with weight A, $D_A(G)$, is the smallest positive integer s such that every sequence over G of length s contains an A-weighted zero-sum subsequence. In this work, we consider weights of the form $A = \{1, 2, \ldots, r\}$ with various finite abelian groups. We first establish upper and lower bounds for $D_A(G)$ with G an arbitrary finite abelian group, and we then determine the values of $D_A(G)$ for restricted sets of abelian groups of rank two and finite elementary abelian p-groups.

1. Introduction

Zero-sum sequences have been a focus of study for over 50 years. (See, for example, [15].) The original motivation for this study concerned applications to questions in the area of nonunique factorization, but additional applications have been developed, including to problems in Ramsey theory, coding theory, and algebraic number theory. For historical overviews of the subject, see [6, 9].

The Davenport constant, $D(G)$, of a group G is of broad interest and has been determined for a wide range of groups. Of particular interest here are the following two theorems, giving the values of the Davenport constants for all groups that are the direct sums of two finite cyclic groups, $\mathbb{Z}_h \oplus \mathbb{Z}_k$, where $h \mid k$, and for all finite abelian p-groups.

Theorem 1 (Olson [14]). *Let $h,\, k \in \mathbb{Z}^+$ with $h \mid k$. Then $D(\mathbb{Z}_h \oplus \mathbb{Z}_k) = h + k - 1$.*

Theorem 2 (Olson [13]). *Let p be prime and $m \in \mathbb{Z}^+$. For $1 \leq i \leq m$, let $e_i \in \mathbb{Z}^+$. Then $D(\mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_m}}) = 1 + \sum_{i=1}^{m}(p^{e_i} - 1)$.*

In this paper, we study a variation of $D(G)$ called the weighted Davenport constant, $D_A(G)$, where A is a nonempty subset of $\mathbb{Z}$. For a group of exponent n (avoiding the trivial case of $0 \in A$), it suffices to assume that $A \subseteq \{1, 2, \ldots, n-1\}$. For $n \geq 2$, the sets A for which the values of $D_A(\mathbb{Z}_n)$ have been determined include: each one-element subset of $\mathbb{Z}$; $\{-1, 1\}$; $\{1, 2, \ldots, n-1\}$; the set of quadratic residues (or nonresidues) modulo n; the set of primitive elements in $\mathbb{Z}_n$; the set of squares of units in $\mathbb{Z}_n$; and, for $1 \leq r \leq n-1$, $\{1, 2, \ldots, r\}$ [1, 2, 4, 7, 11, 16]. Less is known about the values of weighted Davenport constants for other groups. We refer interested readers to [3, 5, 8, 10, 16].

The result most relevant to this work is given by the following theorem.

Theorem 3 (Adhikari, David, Urroz [4]). *Let $A = \{1, 2, \ldots, r\}$, where $1 < r < n$. Then $D_A(\mathbb{Z}_n) = \lceil \frac{n}{r} \rceil$.*

In this work, we generalize Theorem 3 to additional abelian groups, in some cases restricting the values of r. In Section 2, we define terminology and notation used throughout this work. In Section 3, we present theorems providing initial upper and lower bounds for the weighted Davenport constant of arbitrary finite abelian groups. In Section 4, we specialize to groups of rank two, and in Section 5, we consider finite elementary abelian p-groups.

2. Preliminaries

Let G be an additive finite abelian group. A *sequence* over G is a finite list of elements of the group G, with repetitions allowed. As is standard, we write sequences using multiplicative notation. So, given $n \in \mathbb{Z}^+$ and $g_i \in G$, for $1 \le i \le n$, we have a sequence of *length* n,

$$S = g_1 g_2 \cdots g_n = \prod_{i=1}^{n} g_i.$$

Since all groups considered in this work are abelian, the order of the terms does not impact the properties being studied. So, as is standard, we define sequences to be unordered. For example, $g_1g_2g_3$ and $g_2g_1g_3$ represent the same sequence. A *subsequence* of a sequence S is a sequence determined by any subset $J \subseteq \{1, 2, \ldots, n\}$,

$$T = \prod_{i \in J} g_i.$$

A *zero-sum sequence* is a nonempty sequence over G whose terms sum to the zero in the group G. For any set $A \subseteq \mathbb{Z}$, an *A-weighted zero-sum sequence* is a nonempty sequence over G, $g_1g_2 \cdots g_n$, for which there are corresponding elements $a_i \in A$ such that $\sum_{i=1}^{n} a_i g_i = 0$.

The *Davenport constant* of a group G, $D(G)$, is defined to be the smallest positive integer s such that every sequence over G of length s contains a zero-sum subsequence. For a set $A \subseteq \mathbb{Z}$, the *weighted Davenport constant of G with weight A*, $D_A(G)$, is the smallest positive integer s such that every sequence over G of length s contains an A-weighted zero-sum subsequence.

It is immediate from the definitions that if $A = \{1\}$, then $D_A(G) = D(G)$, and that if A contains any multiple of the exponent of G, then $D_A(G) = 1$.

3. General Theorems

In this section, we present two theorems that apply broadly to weighted Davenport constants of finite abelian groups. These theorems will be useful throughout Sections 4 and 5.

Theorem 4 provides a lower bound for the weighted Davenport constants of a group G with weight $A = \{1, 2, \ldots, r\}$ for a positive integer r.

Theorem 4. *Let G be a finite abelian group, $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_m}$, with $m \ge 1$ and, for each i, $n_i \ge 2$. Let $r \in \mathbb{Z}^+$ and $A = \{1, 2, \ldots, r\}$. Then*

$$D_A(G) > \sum_{i=1}^{m} \left(\left\lceil \frac{n_i}{r} \right\rceil - 1 \right).$$

Proof. By [12, Lemma 3.7], we have

$$D_A(G) \ge D_A(\mathbb{Z}_{n_1}) + D_A(\mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_m}) - 1.$$

The theorem then follows from Theorem 3 and a simple induction argument. □

Theorem 5 presents an upper bound for the weighted Davenport constant of a group G with the same weight A.

Theorem 5. *Let G be a finite abelian group and let $A = \{1, 2, \ldots, r\}$ where $r \in \mathbb{Z}^+$. Then*

$$D_A(G) \leq \left\lceil \frac{D(G)}{r} \right\rceil.$$

Proof. Let $S = g_1 g_2 \cdots g_t$ be an arbitrary sequence over G of length

$$t = \left\lceil \frac{D(G)}{r} \right\rceil.$$

We show that S has an A-weighted zero-sum subsequence.

Since $S^r = g_1^r g_2^r \cdots g_t^r$ is of length

$$rt = r \left\lceil \frac{D(G)}{r} \right\rceil \geq D(G),$$

S^r has a zero-sum subsequence, say $T' = g_1^{f_1} g_2^{f_2} \cdots g_t^{f_t}$, where, for each i, $0 \leq f_i \leq r$, and at least one f_i is nonzero.

Taking the sum of the terms of T' yields $\sum_{i=1}^{t} f_i g_i = 0$. Since, for each i, $f_i \in A \cup \{0\}$, this means that the sequence

$$T = \prod_{\substack{1 \leq i \leq t \\ f_i \neq 0}} g_i$$

is an A-weighted zero-sum subsequence of S. Hence the sequence S has an A-weighted zero-sum subsequence, as required, and therefore

$$D_A(G) \leq \left\lceil \frac{D(G)}{r} \right\rceil.$$

□

4. Finite Abelian Groups of Rank Two

In this section, we fix h and $k \in \mathbb{Z}^+$ with $h \mid k$, and consider groups of the form $G \cong \mathbb{Z}_h \oplus \mathbb{Z}_k$. We determine the A-weighted Davenport constant of $\mathbb{Z}_h \oplus \mathbb{Z}_k$ with $A = \{1, 2, \ldots, r\}$, under various restrictions on h, k, and r. In particular, in Theorem 7, we determine $D_A(G)$ for all such G with $h \neq k$, and $r \in \mathbb{Z}^+$ less than each prime dividing k. At the end of the section, we consider the special case in which $h = k$.

Beginning with Theorem 1 and applying the bounds from Theorems 4 and 5, we have, for any $r \in \mathbb{Z}^+$,

$$\left\lceil \frac{h}{r} \right\rceil + \left\lceil \frac{k}{r} \right\rceil - 2 < D_A(\mathbb{Z}_h \oplus \mathbb{Z}_k) \le \left\lceil \frac{h+k-1}{r} \right\rceil. \tag{1}$$

Using these bounds, we show that there are only two possible values for $D_A(\mathbb{Z}_h \oplus \mathbb{Z}_k)$.

Theorem 6. *Let $h,\ k \in \mathbb{Z}^+$ with $h \ge 2$ and $h \mid k$. Let $A = \{1, 2, \ldots, r\}$ where $r \in \mathbb{Z}^+$. Then*

$$D_A(\mathbb{Z}_h \oplus \mathbb{Z}_k) \in \left\{ \left\lceil \frac{h}{r} \right\rceil + \left\lceil \frac{k}{r} \right\rceil - 1, \left\lceil \frac{h}{r} \right\rceil + \left\lceil \frac{k}{r} \right\rceil \right\}.$$

Proof. Using Inequality (1), it suffices to prove that

$$\left\lceil \frac{h+k-1}{r} \right\rceil \le \left\lceil \frac{h}{r} \right\rceil + \left\lceil \frac{k}{r} \right\rceil.$$

If $r = 1$, the result is trivial. So we assume that $r \ge 2$.

If $r \mid h$, then $r \mid k$, and

$$\left\lceil \frac{h+k-1}{r} \right\rceil = \frac{h}{r} + \frac{k}{r} = \left\lceil \frac{h}{r} \right\rceil + \left\lceil \frac{k}{r} \right\rceil.$$

If $r \mid k$ and $r \nmid h$, then

$$\left\lceil \frac{h+k-1}{r} \right\rceil = \frac{k}{r} + \left\lceil \frac{h-1}{r} \right\rceil \le \left\lceil \frac{h}{r} \right\rceil + \left\lceil \frac{k}{r} \right\rceil.$$

Thus, we may assume that $r \nmid k$, and so $r \nmid h$. Let $h = h_1 r + h_2$ and $k = k_1 r + k_2$, where $h_1,\ h_2,\ k_1,\ k_2 \in \mathbb{Z}_{\ge 0}$, and $1 \le h_2 \le r-1$ and $1 \le k_2 \le r-1$. Then

$$\begin{aligned} \left\lceil \frac{h+k-1}{r} \right\rceil &= h_1 + k_1 + \left\lceil \frac{h_2+k_2-1}{r} \right\rceil \le h_1 + k_1 + \left\lceil \frac{2r-3}{r} \right\rceil \\ &\le h_1 + k_1 + 2 = \left\lceil \frac{h}{r} \right\rceil + \left\lceil \frac{k}{r} \right\rceil, \end{aligned}$$

as desired. □

In the following lemma, we determine the value of $D_A(\mathbb{Z}_h \oplus \mathbb{Z}_k)$ under some specific assumptions on h, k, and r. In Theorem 7, we remove these assumptions and instead require that $h \ne k$ and that r is less than the smallest prime dividing k.

Lemma 1. *Let $h,\ k \in \mathbb{Z}^+$ with $h \ge 2$ and $h \mid k$. Let $A = \{1, 2, \ldots, r\}$ where $r \nmid k$. Let $h = h_1 r + h_2$ and $k = k_1 r + k_2$ with $1 \le h_2 \le r-1$ and $1 \le k_2 \le r-1$. If $h_2 + k_2 \le r+1$, then*

$$D_A(\mathbb{Z}_h \oplus \mathbb{Z}_k) = \left\lceil \frac{h+k-1}{r} \right\rceil = \left\lceil \frac{h}{r} \right\rceil + \left\lceil \frac{k}{r} \right\rceil - 1.$$

Proof. Since $h_2 + k_2 \leq r + 1$,

$$D_A(\mathbb{Z}_h \oplus \mathbb{Z}_k) \leq \left\lceil \frac{h+k-1}{r} \right\rceil = h_1 + k_1 + \left\lceil \frac{h_2+k_2-1}{r} \right\rceil$$
$$= h_1 + k_1 + 1 = \left\lceil \frac{h}{r} \right\rceil + \left\lceil \frac{k}{r} \right\rceil - 1.$$

The lemma now follows from Theorem 6. □

We use Lemma 1 to prove the following theorem, which applies to all groups of the form $\mathbb{Z}_h \oplus \mathbb{Z}_k$ with $h \mid k$ and $h \neq k$.

Theorem 7. *Let h, $k \in \mathbb{Z}^+$ with $h \geq 2$, $h \mid k$, and $h \neq k$. Let $A = \{1, 2, \ldots, r\}$, with $r \in \mathbb{Z}^+$ less than each prime dividing k. Then*

$$D_A(\mathbb{Z}_h \oplus \mathbb{Z}_k) = \left\lceil \frac{h+k-1}{r} \right\rceil.$$

Proof. The case where $r = 1$ is given in Theorem 1, so we may assume that $r \geq 2$. Let $h = h_1 r + h_2$ and $k = k_1 r + k_2$, where h_1, h_2, k_1, $k_2 \in \mathbb{Z}_{\geq 0}$ and, since $r \nmid k$, $1 \leq h_2 \leq r - 1$, $1 \leq k_2 \leq r - 1$. The case in which $h_2 + k_2 \leq r + 1$ is given in Lemma 1.

For the case $h_2 + k_2 \geq r + 2$, we have

$$\left\lceil \frac{h+k-1}{r} \right\rceil = h_1 + k_1 + \left\lceil \frac{h_2+k_2-1}{r} \right\rceil = h_1 + k_1 + 2.$$

By Inequality (1), it suffices to show that there exists a sequence of length h_1+k_1+1 with no A-weighted zero-sum subsequence.

Let $d = h_2+k_2 \geq r+2$. Since r is relatively prime to h, there exists some $s \in \mathbb{Z}^+$ satisfying $sh \equiv 1 \pmod{r}$ and $1 \leq s \leq r - 1$. Note that, since $h \neq k$ and $h \mid k$, $k \geq (r+1)h$, and so $h+k-sh-d+1 \geq (r-s+2)h-d+1 \geq 3h-2(r-1)+1 > 0$. Set

$$S = (1,0)(1,1)^{\frac{h+k-sh-d+1}{r}}(0,1)^{\frac{sh-1}{r}}.$$

Then S is a sequence of length $1 + \frac{h+k-d}{r} = h_1 + k_1 + 1$.

Suppose that S has an A-weighted zero-sum subsequence. Then S^r contains a zero-sum subsequence, say,

$$T = (1,0)^a(1,1)^b(0,1)^c,$$

where $a \leq r$, $b \leq h + k - sh - d + 1$, $c \leq sh - 1$, and not all of a, b, and c are zero.

These bounds imply that, since T is a zero-sum subsequence, $b + c = k$ and $a + b \equiv 0 \pmod{h}$. From the former, $b \geq k - sh + 1$ and, therefore, $k - sh + 1 \leq a + b \leq r + h + k - sh - d + 1$. Dividing this by h, we find that

$$\frac{k}{h} - s + \frac{1}{h} \leq \frac{a+b}{h} \leq 1 + \frac{k}{h} - s + \frac{r-d+1}{h} \leq \frac{k}{h} - s + \frac{h-1}{h},$$

since $d \geq r+2$. This is a contradiction, since $a+b$ is a multiple of h, but there is no integer between the two bounds. Hence, $D_A(\mathbb{Z}_h \oplus \mathbb{Z}_k) > h_1 + k_1 + 1$, and so

$$D_A(\mathbb{Z}_h \oplus \mathbb{Z}_k) = \left\lceil \frac{h+k-1}{r} \right\rceil,$$

as desired. □

We now focus on the special case where $h = k$. From Theorem 6, we have that

$$D_A(\mathbb{Z}_h \oplus \mathbb{Z}_h) \in \left\{ 2\left\lceil \frac{h}{r} \right\rceil - 1, 2\left\lceil \frac{h}{r} \right\rceil \right\},$$

and from Lemma 1, we have the following.

Corollary 1. *Let $h \geq 2$ and $A = \{1, 2, \ldots, r\}$ where $r \nmid h$. Let $h = h_1 r + h_2$ with $1 \leq h_2 \leq r-1$. If $2h_2 \leq r+1$, then*

$$D_A(\mathbb{Z}_h \oplus \mathbb{Z}_h) = 2\left\lceil \frac{h}{r} \right\rceil - 1.$$

Although we have been unable to extend Corollary 1 to cover the case where $2h_2 \geq r+2$, we have implemented a Sage computer program to examine many examples with $h < 40$. The program checks all possible forms of sequences in order to verify the value of $D_A(\mathbb{Z}_h \oplus \mathbb{Z}_h)$ for many, but certainly not all, such examples. Supported by these results, we offer the following conjecture.

Conjecture 1. Let $h \geq 2$ and $A = \{1, 2, \ldots, r\}$ where $r \nmid h$. Then

$$D_A(\mathbb{Z}_h \oplus \mathbb{Z}_h) = 2\left\lceil \frac{h}{r} \right\rceil - 1.$$

5. Finite Elementary Abelian p-Groups

We now consider the case of finite abelian p-groups and, in particular, finite elementary abelian p-groups. We note some overlap between the following and relevant results in [16]. In particular, the upper bound in Inequality (2) could instead be derived from [16, Theorem 1], and [16, Corollary 1.2(a)(f)] are special cases of Theorem 8, below, as described in Corollary 3. (We note that [16, Corollary 1.2(g)] is incorrect without the hypotheses of Theorem 8.)

By Theorems 2, 4, and 5, we have that for

$$G = \mathbb{Z}_{p^{e_1}} \oplus \mathbb{Z}_{p^{e_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_m}},$$

with m and $e_i \in \mathbb{Z}^+$,

$$\sum_{i=1}^{m}\left(\left\lceil \frac{p^{e_i}}{r} \right\rceil - 1\right) < D_A(G) \le \left\lceil \frac{1+\sum_{i=1}^{m}(p^{e_i}-1)}{r} \right\rceil. \quad (2)$$

For finite elementary abelian p-groups, we have the following corollary.

Corollary 2. *Let p be a prime, let*

$$G = \bigoplus_{i=1}^{m} \mathbb{Z}_p,$$

with $m \in \mathbb{Z}^+$, and let $A = \{1, 2, \ldots, r\}$, with $r \in \mathbb{Z}^+$. Then

$$m\left(\left\lceil \frac{p}{r} \right\rceil - 1\right) < D_A(G) \le \left\lceil \frac{1+m(p-1)}{r} \right\rceil.$$

Clearly, when the upper bound in Corollary 2 is one more than the lower bound, the value of the weighted Davenport constant is determined. This leads to the following.

Theorem 8. *Let p be a prime, let*

$$G = \bigoplus_{i=1}^{m} \mathbb{Z}_p,$$

with $m \in \mathbb{Z}^+$, and let $A = \{1, 2, \ldots, r\}$, with $r \in \mathbb{Z}^+$. Let p_1, $p_2 \in \mathbb{Z}$ with $0 \le p_2 \le r-1$ and $p = p_1 r + p_2$. If $m(p_2 - 1) \le r - 1$, then

$$D_A(G) = 1 + m\left(\left\lceil \frac{p}{r} \right\rceil - 1\right).$$

Proof. If $r = 1$, then the result is immediate from Theorem 2. If $r \ge p$, then $p \in A$, and so $D_A(G) = 1$, proving the result. Thus, we can assume that $1 < r < p$ and, therefore, $p_1 \ge 1$. Since p is prime, it follows that $p_2 \ge 1$.

Assume that $m(p_2 - 1) \le r - 1$. Using Corollary 2, we have

$$\begin{aligned} D_A(G) &\le \left\lceil \frac{1+m(p-1)}{r} \right\rceil = mp_1 + \left\lceil \frac{mp_2 - m + 1}{r} \right\rceil = mp_1 + 1 \\ &= m\left(\left\lceil \frac{p}{r} \right\rceil - 1\right) + 1 < D_A(G) + 1. \end{aligned}$$

Hence, since each expression is an integer, the initial inequality is an equality, completing the proof. □

The following is immediate.

Corollary 3. *Let p be a prime, let*

$$G = \bigoplus_{i=1}^{m} \mathbb{Z}_p,$$

with $m \in \mathbb{Z}^+$, and let $A = \{1, 2, \ldots, r\}$, with $r \in \mathbb{Z}^+$.

1. *If $r = p - 1$, then $D_A(G) = 1 + m$.*
2. *If $r = (p - 1)/2$, then $D_A(G) = 1 + 2m$.*
3. *If $p \equiv 1 \pmod{r}$, then $D_A(G) = 1 + m(p - 1)/r$.*

Although we have completely determined $D_A(G)$ for many cases of finite elementary abelian p-groups, there remains the case $m(p_2 - 1) \geq r$ for future study. The search for a general formula continues.

Open Problem. Let p be a prime, let

$$G = \bigoplus_{i=1}^{m} \mathbb{Z}_p,$$

with $m \in \mathbb{Z}^+$, and let $A = \{1, 2, \ldots, r\}$, with $r \in \mathbb{Z}^+$. Is there a general formula for $D_A(G)$ that holds in all cases, including when $m(p_2 - 1) \geq r$?

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