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THREE IMPRIMITIVE CHARACTER SUMS

Brad Isaacson [1]
Department of Mathematics, New York City College of Technology, Brooklyn, New York
bisaacson@citytech.cuny.edu

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Abstract

We express three imprimitive character sums in terms of generalized Bernoulli numbers. These sums are generalizations of sums introduced and studied by Arakawa, Berndt, Ibukiyama, Kaneko and Ramanujan in the context of modular forms and theta function identities. As a corollary, we obtain a formula for cotangent power sums considered by Apostol.

– Dedicated to Robert Sczech on the occasion of his retirement

1. Introduction

Let χ be a Dirichlet character modulo m, and h be any positive integer prime to m. We put $\zeta = exp(2\pi i/m)$. Let $\tau(n, \chi)$ denote the Gaussian sum $\tau(n,\chi) = \sum_{j=0}^{m-1} \chi(j)\zeta^{jn}$. Let a, b be nonnegative integers. In this paper, we obtain formulas expressing the closely related character sums

$$M_{a,b}(h,\chi) = \sum_{j=1}^{m-1} \frac{\chi(j)}{(\zeta^{hj}-1)^a(\zeta^j-1)^b},$$

$$S_{a,b}(h,\chi;e_1,\dots,e_{a+b}) = \sum_{j_1,\cdots,j_{m+n}=1}^{m-1} \tau\left(h\sum_{k=1}^{a} j_k + \sum_{l=1}^{b} j_{a+l}, \chi\right) j_1^{e_1}\cdots j_{a+b}^{e_{a+b}},$$

$$c_{a,b}(h,\chi) = \sum_{j=1}^{m-1} \cot^a\left(\frac{h\pi j}{m}\right)\cot^b\left(\frac{\pi j}{m}\right)\chi(j)$$

by generalized Bernoulli numbers. As a corollary, we obtain a formula for cotangent power sums $c_{a,0}(1,\chi)$ considered by Apostol [1]. In [15], we obtained formulas for these sums in the case of primitive character χ. Here, we extend our results to treat the imprimitive case.

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As discussed in [15], the sums $M_{a,b}(h,\chi)$ and $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$ are natural generalizations of sums introduced and studied by Berndt [6] and Arakawa-Ibukiyama-Kaneko [3] in the context of the theory of modular forms. For examples, we refer the interested reader to [2, 3, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19]. The sum $c_{a,b}(h,\chi)$ is a variation of trigonometric sums first investigated by Ramanujan in connection to certain theta function identities. For examples, we refer the interested reader to [4, 5, 7].

The layout of this paper is as follows. In Section 2, we introduce two kinds of generalized Bernoulli functions $\mathscr{B}_{k,\chi}(x)$, $B_{k,\chi}(x)$ and derive some of their properties, such as their finite sum representations and multiplication formulas. In Section 3, we express both $\mathscr{B}_{k,\chi}(x)$, $B_{k,\chi}(x)$ by generalized Bernoulli numbers. In Section 4, we express the sums $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$, $c_{a,b}(h,\chi)$ in terms of generalized Bernoulli functions $\mathscr{B}_{k,\chi}(x)$. Since the $\mathscr{B}_{k,\chi}(x)$ are expressible by generalized Bernoulli numbers, so are the sums $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$, and $c_{a,b}(h,\chi)$. In Section 5, we give examples expressing the sums $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$, and $c_{a,b}(h,\chi)$ by generalized Bernoulli numbers.

2. The Generalized Bernoulli Functions $\mathscr{B}_{k,\chi}(x)$, $B_{k,\chi}(x)$

We now fix the notation. Let χ be a Dirichlet character modulo m, and h be any positive integer prime to m. We put $\zeta = exp(2\pi i/m)$. We denote the conductor of χ by f and the corresponding primitive character by ψ. We put

$$q = \prod_{\substack{p \mid m \\ p \nmid f \\ p \text{ prime}}} p, \quad R = \frac{m}{fq}.$$

For integers r, s with s prime to m, we define the Gaussian sum

$$\tau(r/s,\chi) = \sum_{j(m)} \chi(j)\zeta^{jrs^{-1}},$$

where s^{-1} is regarded as an element of $\mathbb{Z}/m\mathbb{Z}$ such that $ss^{-1} \equiv 1 \,(\mathrm{mod}\ m)$, and j runs over a complete residue system modulo m. We will write $\tau(\chi)$ for $\tau(1,\chi)$. We also extend the definition of χ by multiplicativity by defining $\chi(r/s) = \chi(rs^{-1})$.

For $x \in \mathbb{Q}$ with denominator prime to m, we define two types of generalized Bernoulli functions $\mathscr{B}_{k,\chi}(x), B_{k,\chi}(x)$ by the generating functions

$$\sum_{j=0}^{m-1} \frac{\tau(j+x,\chi)te^{(j+x)t}}{e^{mt}-1} = \sum_{k=0}^{\infty} \mathscr{B}_{k,\chi}(x)\frac{t^k}{k!},$$

$$\sum_{j=0}^{m-1} \frac{\chi(j+x)te^{(j+x)t}}{e^{mt}-1} = \sum_{k=0}^{\infty} B_{k,\chi}(x)\frac{t^k}{k!}.$$

Note that if χ is primitive, then $\mathscr{B}_{k,\chi}(x) = \tau(\chi)B_{k,\overline{\chi}}(x)$, and if χ is trivial, then $\mathscr{B}_{k,\chi}(x), B_{k,\chi}(x)$ both reduce to the ordinary Bernoulli polynomials $B_k(x)$. These definitions lead to the formulas

$$\mathscr{B}_{k,\chi}(x) = m^{k-1}\sum_{j=0}^{m-1} B_k\left(\frac{j+x}{m}\right)\tau(j+x,\chi),$$
$$B_{k,\chi}(x) = m^{k-1}\sum_{j=0}^{m-1} B_k\left(\frac{j+x}{m}\right)\chi(j+x),$$

or equivalently, for any natural number s prime to m and any integer r, we have

$$\mathscr{B}_{k,\chi}\left(\frac{r}{s}\right) = \chi(s)m^{k-1}\sum_{j=0}^{m-1} B_k\left(\frac{sj+r}{sm}\right)\tau(sj+r,\chi),$$
$$B_{k,\chi}\left(\frac{r}{s}\right) = \overline{\chi}(s)m^{k-1}\sum_{j=0}^{m-1} B_k\left(\frac{sj+r}{hm}\right)\chi(sj+r).$$

We define the generalized Bernoulli numbers $\mathscr{B}_{k,\chi}$, $B_{k,\chi}$ by $\mathscr{B}_{k,\chi} = \mathscr{B}_{k,\chi}(0)$ and $B_{k,\chi} = B_{k,\chi}(0)$.

Corresponding to the generalized Bernoulli functions $\mathscr{B}_{k,\chi}(x), B_{k,\chi}(x)$, we have the periodic generalized Bernoulli functions $\overline{\mathscr{B}}_{k,\chi}(x), \overline{B}_{k,\chi}(x)$ given by $\overline{\mathscr{B}}_{k,\chi}(x) = \mathscr{B}_{k,\chi}(x-[x])$, $\overline{B}_{k,\chi}(x) = B_{k,\chi}(x-[x])$, where $[x]$ denotes the greatest integer not exceeding x. We give the following useful formulations for $\overline{\mathscr{B}}_{k,\chi}(r/s)$, $\overline{B}_{k,\chi}(r/s)$:

$$\begin{aligned}\overline{\mathscr{B}}_{k,\chi}\left(\frac{r}{s}\right) &= \chi(s)m^{k-1}\sum_{j(m)} \overline{B}_k\left(\frac{sj+r}{sm}\right)\tau(sj+r,\chi),\\ \overline{B}_{k,\chi}\left(\frac{r}{s}\right) &= \overline{\chi}(s)m^{k-1}\sum_{j(m)} \overline{B}_k\left(\frac{sj+r}{sm}\right)\chi(sj+r).\end{aligned} \tag{1}$$

We state the multiplication formula for periodic Bernoulli functions which follows from Raabe's multiplication formula for ordinary Bernoulli polynomials:

$$\overline{B}_k(nx) = n^{k-1}\sum_{j(n)} \overline{B}_k\left(x+\frac{j}{n}\right). \tag{2}$$

As a natural generalization of Equation (2), we have the multiplication formula for periodic generalized Bernoulli functions.

Lemma 2.1. *Let χ, m be as above. Let $n \in \mathbb{N}$ with $(n,m) = 1$ and $x \in \mathbb{Q}$ with* *denominator relatively prime to m. Then, we have*

$$\overline{\mathscr{B}}_{k,\chi}(nx) = n^{k-1}\overline{\chi}(n)\sum_{j(n)}\overline{\mathscr{B}}_{k,\chi}\left(x+\frac{j}{n}\right),$$

$$\overline{B}_{k,\chi}(nx) = n^{k-1}\chi(n)\sum_{j(n)}\overline{B}_{k,\chi}\left(x+\frac{j}{n}\right).$$

Proof. We only prove the case for $\overline{\mathscr{B}}_{k,\chi}$ as the case for $\overline{B}_{k,\chi}$ can be proved similarly. By definition, we have

$$\overline{\mathscr{B}}_{k,\chi}(nx) = m^{k-1}\sum_{l(m)}\overline{B}_k\left(\frac{l+nx}{m}\right)\tau(l+nx,\chi).$$

Replacing l by nl and applying the multiplication formula Equation (2), we get

$$\overline{\mathscr{B}}_{k,\chi}(nx) = n^{k-1}\overline{\chi}(n)m^{k-1}\sum_{l(m)}\sum_{j(n)}\overline{B}_k\left(\frac{l+x+\frac{mj}{n}}{m}\right)\tau(l+x,\chi).$$

Since $\tau(l+x,\chi) = \tau(l+x+mj/n,\chi)$, we have

$$\overline{\mathscr{B}}_{k,\chi}(nx) = n^{k-1}\overline{\chi}(n)\sum_{j(n)}\overline{\mathscr{B}}_{k,\chi}\left(x+\frac{mj}{n}\right).$$

Thus the assertion of the lemma follows by replacing j by $m^{-1}j$, where $mm^{-1} \equiv 1 \pmod{n}$. □

We note that when χ is trivial, Lemma 2.1 reduces to Equation (2).

3. The Evaluation of $\overline{\mathscr{B}}_{k,\chi}(x)$, $\overline{B}_{k,\chi}(x)$

We keep the notation used previously. In this section, we express the periodic generalized Bernoulli functions $\overline{\mathscr{B}}_{k,\chi}(x)$, $\overline{B}_{k,\chi}(x)$ by generalized Bernoulli numbers.

Let μ and ϕ denote the Möbius and Euler phi functions, respectively. We state a useful result about Gaussian sums proved in [16].

Theorem 3.1 ([16]). *Let χ, m, ψ, f, q, R be as above. Let the multiplicative function g be defined by $g(n) = \mu((n,q))\phi((n,q))\overline{\psi}(n)$. Then, we have*

$$\begin{cases} \tau(\alpha,\chi) = 0 & \text{if } R \nmid \alpha, \\ \tau(Rn,\chi) = R\mu(q)\psi(q)\tau(\psi)g(n) & \text{for } n = 1,2,\ldots. \end{cases}$$

The following theorem expresses $\overline{\mathscr{B}}_{k,\chi}(x)$ as a linear combination of periodic generalized Bernoulli functions $\overline{B}_{k,\overline{\psi}}(x)$.

Theorem 3.2. *Let $\chi, m, \psi, f, h, q, R$ be as above. For any integer c, we have*

$$\overline{\mathscr{B}}_{k,\chi}\left(\frac{c}{h}\right) = R^k q^{k-1}\mu(q)\tau(\psi)\sum_{e(q)}\mu((he+c_0,q))\phi((he+c_0,q))\overline{B}_{k,\overline{\psi}}\left(\frac{he+c_0}{hq}\right),$$

where c_0 denotes an integer such that $Rc_0 \equiv c \pmod{h}$.

Proof. Let c_0 be as in the theorem. From Equation (1) and applying Theorem 3.1, we have

$$\begin{aligned}\overline{\mathscr{B}}_{k,\chi}\left(\frac{c}{h}\right) &= \chi(h)m^{k-1}\sum_{j(fq)}\tau(R(hj+c_0),\chi)\overline{B}_k\left(\frac{R(hj+c_0)}{hm}\right)\\ &= \chi(h)m^{k-1}R\mu(q)\psi(q)\tau(\psi)\\ &\quad\times\sum_{j(fq)}\mu((hj+c_0,q))\phi((hj+c_0,q))\overline{\psi}(hj+c_0)\overline{B}_k\left(\frac{hj+c_0}{hfq}\right).\end{aligned}$$

Writing $j = dq + e$ where d and e run over complete residue systems modulo f and q, respectively, together with Equation (1), we obtain

$$\begin{aligned}\overline{\mathscr{B}}_{k,\chi}\left(\frac{c}{h}\right) &= \chi(h)m^{k-1}R\mu(q)\psi(q)\tau(\psi)\\ &\quad\times\sum_{\substack{d(f)\\ e(q)}}\mu((he+c_0,q))\phi((he+c_0,q))\overline{\psi}((hq)d+(he+c_0))\overline{B}_k\left(\frac{(hq)d+(he+c_0)}{hfq}\right)\\ &= R^k q^{k-1}\mu(q)\tau(\psi)\sum_{e(q)}\mu((he+c_0,q))\phi((he+c_0,q))\overline{B}_{k,\overline{\psi}}\left(\frac{he+c_0}{hq}\right).\end{aligned}$$

□

For any integer u with $u \mid h$, denote by h_u the u-primary part of h, that is, the maximum integer which divides h and is prime to u. For any natural number n, we denote by $Y(n)$ the set of primitive Dirichlet characters modulo n. The next theorem expresses generalized Bernoulli functions $\overline{B}_{k,\chi}(c/h)$ by generalized Bernoulli numbers. This was proved in [13] and is a generalization of Ibukiyama's Theorem 2 in [9].

Theorem 3.3 ([13]). *Let χ, m, h be as above. For any integer c prime to h, we have*

$$\overline{B}_{k,\chi}\left(\frac{c}{h}\right) = \frac{\overline{\chi}(h)}{\phi(h)h^{k-1}}\sum_{u\mid h}\sum_{\delta\in Y(u)}\left(\bar{\delta}(c)B_{k,\delta\chi}\prod_{\substack{q\mid h_u\\ q\ prime}}(1-q^{k-1}\chi(q)\delta(q))\right).$$

In view of the above two theorems, it follows that we can also express $\overline{\mathscr{B}}_{k,\chi}(c/h)$ by generalized Bernoulli numbers.

4. The Evaluation of $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$, and $c_{a,b}(h,\chi)$

We remind the reader of the notation. Let χ be a Dirichlet character modulo m, and h be any positive integer prime to m. Let a, b be nonnegative integers. We assume that $m > 1$ and $a + b \geq 1$ to exclude the trivial cases. Without a loss of generality, we further assume that $a \geq 1$. In this section, we express the sums $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$ and $c_{a,b}(h,\chi)$ by periodic generalized Bernoulli functions $\overline{\mathscr{B}}_{k,\chi}(x)$. These are generalizations of Theorems 3.3, 4.1, 5.1 in [15], respectively, so we omit the proofs.

For positive integers n and k, we denote by $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$ the Stirling numbers of the first and second kind, respectively. That is, Stirling's cycle numbers $\left[{n \atop k}\right]$ denote the number of permutations of n letters (elements of the symmetric group of degree n) that consist of k disjoint cycles, and Stirling's subset numbers $\left\{{n \atop k}\right\}$ denote the number of ways to divide a set of n elements into k nonempty sets.

Theorem 4.1. *Let χ, m, h, a, b be as above. We have*

$$M_{a,b}(h,\chi) = \frac{(-1)^{a+b-1}\overline{\chi}(h)}{(a+b-1)!} \times \sum_{j=1}^{a+b} \sum_{c=0}^{b(h-1)} C_h(b,c) \left(\sum_{r=0}^{a+b-j} \frac{(-1)^r}{r+j} \left[{a+b \atop r+j}\right] \binom{r+j}{j} \left(\frac{c}{h}\right)^r \right) \overline{\mathscr{B}}_{j,\chi}\left(\frac{c}{h}\right),$$

where

$$C_h(b,c) = \begin{cases} 1 & \text{if } b = 0, \\ \displaystyle\sum_{k=0}^{[c/h]} (-1)^k \binom{b}{k} \binom{b-1+c-hk}{b-1} & \text{if } b \geq 1. \end{cases}$$

Theorem 4.2. *Let χ, m, h, a, b be as above. We have*

$$S_{a,b}(h,\chi;e_1,\ldots,e_{a+b}) = (-1)^{e_1+\cdots+e_{a+b}} \times \sum_{\substack{1\leq k_j \leq e_j \\ 1 \leq l_j \leq e_j - k_j + 1 \\ 1 \leq j \leq a+b}} \left(\prod_{i=1}^{a+b} (-m)^{l_i} \binom{e_i}{l_i} \left\{{e_i - l_i + 1 \atop k_i}\right\} (k_i - 1)! \right) \times M_{k_1+\cdots+k_a, k_{a+1}+\cdots+k_{a+b}}(h,\chi).$$

Remark. In the case where $e_j = 1$ for $j = 1, \ldots, a+b$, we get $S_{a,b}(h,\chi;1,\ldots,1) = m^{a+b} M_{a,b}(h,\chi)$.

Theorem 4.3. *Let χ, m, h, a, b be as above. We have*

$$c_{a,b}(h,\chi) = i^{a+b} \sum_{j=0}^{a} \sum_{k=0}^{b} 2^{j+k} \binom{a}{j} \binom{b}{k} M_{j,k}(h,\chi).$$

Since the sums $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$, and $c_{a,b}(h,\chi)$ can be expressed by periodic generalized Bernoulli functions $\overline{\mathscr{B}}_{k,\chi}(x)$, and $\overline{\mathscr{B}}_{k,\chi}(x)$ are expressible by generalized Bernoulli numbers by virtue of Theorems 3.2 and 3.3, so are the sums $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$, and $c_{a,b}(h,\chi)$. As a corollary, we obtain a formula for cotangent power sums considered by Apostol [1], Berndt [6] and others.

Corollary 4.4. *Let χ, m, a be as above. We have*

$$\sum_{j=1}^{m-1} \cot^a\left(\frac{\pi j}{m}\right)\chi(j)$$
$$= -i^a \sum_{k=1}^{a} \frac{1}{k} \left(\sum_{j=k}^{a} \frac{(-2)^j \binom{a}{j} \left[j \atop k \right]}{(j-1)!} \right) \mathscr{B}_{k,\chi} + \begin{cases} i^a \phi(m) & \text{if } \chi \text{ is principal} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. By Theorem 4.3, we have

$$c_{a,0}(1,\chi) = i^a \sum_{j=0}^{a} 2^j \binom{a}{j} M_{j,0}(1,\chi).$$

Clearly,

$$M_{0,0}(1,\chi) = \begin{cases} \phi(m) & \text{if } \chi \text{ is principal} \\ 0 & \text{otherwise} \end{cases},$$

and from Theorem 4.1 for $j \geq 1$, we have

$$M_{j,0}(1,\chi) = \frac{(-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} \frac{1}{k} \left[j \atop k \right] \mathscr{B}_{k,\chi}.$$

Plugging these values into the expression for $c_{a,0}(1,\chi)$ given above and interchanging the order of summation, we get the assertion of the corollary. □

5. Examples

We keep the notation used previously. In this section, we give examples expressing $M_{a,b}(h,\chi)$, $S_{a,b}(h,\chi;e_1,\ldots,e_{a+b})$, and $c_{a,b}(h,\chi)$ by generalized Bernoulli numbers. For examples where χ is a primitive character, see [15]. We note the following useful fact that follows from the multiplication formula Lemma 2.1.

$$\sum_{c(h)} \overline{\mathscr{B}}_{k,\chi}(c/h) = \frac{\chi(h)}{h^{k-1}} \mathscr{B}_{k,\chi}. \tag{3}$$

We consider the case where χ is a Dirichlet character modulo $m = f \cdot 3^n$ for some positive integer n with $(f, 3) = 1$. Note that χ is necessarily imprimitive.

Proposition 5.1. *Let $\chi, m, \psi, f, q, R, a$ be as above. Suppose $f > 1$ with $(f, 12) = 1$ and $m = f \cdot 3^n$ for some positive integer n. Let δ be the unique primitive character modulo 4. We have*

$$(i) \quad M_{1,1}(4,\chi) = \sum_{j=1}^{m-1} \frac{\chi(j)}{(\zeta^{4j}-1)(\zeta^j-1)}$$
$$= \tau(\psi)\left\{ \frac{(\psi(3)-3)(\overline{\psi}(4)+1)R}{2} B_{1,\overline{\psi}} + (-1)^{n+1}\frac{(\psi(3)+3)R}{4} B_{1,\delta\overline{\psi}} + \frac{(\psi(3)-9)R^2}{8} B_{2,\overline{\psi}} \right\},$$

$$(ii) \quad S_{1,1}(4,\chi;1,1) = \sum_{j,k=1}^{m-1} \tau(4j+k,\chi) jk$$
$$= m^2\tau(\psi)\left\{ \frac{(\psi(3)-3)(\overline{\psi}(4)+1)R}{2} B_{1,\overline{\psi}} + (-1)^{n+1}\frac{(\psi(3)+3)R}{4} B_{1,\delta\overline{\psi}} + \frac{(\psi(3)-9)R^2}{8} B_{2,\overline{\psi}} \right\},$$

$$(iii) \quad c_{1,1}(4,\chi) = \sum_{j=1}^{m-1} \cot\left(\frac{4\pi j}{m}\right)\cot\left(\frac{\pi j}{m}\right)\chi(j)$$
$$= -\tau(\psi)\left\{ (-1)^{n+1}(\psi(3)+3)RB_{1,\delta\overline{\psi}} + \frac{(\psi(3)-9)R^2}{2} B_{2,\overline{\psi}} \right\},$$

$$(iv) \quad c_{a,0}(1,\chi) = \sum_{j=1}^{m-1} \cot^a\left(\frac{\pi j}{m}\right)\chi(j)$$
$$= i^a\tau(\psi)\sum_{k=1}^{a} \frac{R^k(\psi(3)-3^k)}{k}\left(\sum_{j=k}^{a} \frac{(-2)^j\binom{a}{j}\left[{j \atop k}\right]}{(j-1)!}\right) B_{k,\overline{\psi}}.$$

Proof. By Theorem 4.1 together with the help of Equation (3), we have

$$M_{1,1}(4,\chi) = \tau(\chi)\left\{ -\mathscr{B}_{1,\chi} - \frac{1}{8}\mathscr{B}_{2,\chi} + \frac{\overline{\chi}(4)}{4}\sum_{c=1}^{3} c\,\overline{\mathscr{B}}_{1,\chi}(c/4) \right\}.$$

Since $m = f \cdot 3^n$ with $(f,3) = 1$, we have $q = 3$ and $R = 3^{n-1}$. By Theorem 3.2, noting that $R^{-1} \equiv R \pmod 4$ and $c_0 \equiv (-1)^{n-1}c \pmod 4$, we get

$$\begin{aligned}
\overline{\mathscr{B}}_{j,\chi} = \overline{\mathscr{B}}_{j,\chi}(0) &= -R^j q^{j-1}\tau(\psi)\left\{-2B_{j,\overline{\psi}} + \overline{B}_{j,\overline{\psi}}\left(\frac{1}{3}\right) + \overline{B}_{j,\overline{\psi}}\left(\frac{2}{3}\right)\right\},\\
\overline{\mathscr{B}}_{j,\chi}(1/4) &= -R^j q^{j-1}\tau(\psi)\left\{\overline{B}_{j,\overline{\psi}}\left(\frac{(-1)^n 11}{12}\right) + \overline{B}_{j,\overline{\psi}}\left(\frac{(-1)^n 7}{12}\right) - 2\overline{B}_{j,\overline{\psi}}\left(\frac{(-1)^n}{4}\right)\right\},\\
\overline{\mathscr{B}}_{j,\chi}(2/4) = \overline{\mathscr{B}}_{j,\overline{\chi}}(1/2) &= -R^j q^{j-1}\tau(\psi)\left\{\overline{B}_{j,\overline{\psi}}\left(\frac{1}{6}\right) - 2\overline{B}_{j,\overline{\psi}}\left(\frac{1}{2}\right) + \overline{B}_{j,\overline{\psi}}\left(\frac{5}{6}\right)\right\},\\
\overline{\mathscr{B}}_{j,\chi}(3/4) &= -R^j q^{j-1}\tau(\psi)\left\{-2\overline{B}_{j,\overline{\psi}}\left(\frac{(-1)^n 3}{4}\right) + \overline{B}_{j,\overline{\psi}}\left(\frac{(-1)^n 5}{12}\right) + \overline{B}_{j,\overline{\psi}}\left(\frac{(-1)^n}{12}\right)\right\}.
\end{aligned}$$

Let $\alpha_j(d) = \frac{\psi(d)}{\phi(d)d^{j-1}}$ and δ_3 be the unique primitive character modulo 3. By Theorem 3.3, for c relatively prime to its denominator below, we have

$$\begin{aligned}
\overline{B}_{j,\overline{\psi}}(c/2) &= \alpha_j(2)\left(1 - 2^{j-1}\overline{\psi}(2)\right)B_{j,\overline{\psi}},\\
\overline{B}_{j,\overline{\psi}}(c/3) &= \alpha_j(3)\left\{\left(1 - 3^{j-1}\overline{\psi}(3)\right)B_{j,\overline{\psi}} + \overline{\delta}_3(c)B_{j,\delta_3\overline{\psi}}\right\},\\
\overline{B}_{j,\overline{\psi}}(c/4) &= \alpha_j(4)\left\{\left(1 - 2^{j-1}\overline{\psi}(2)\right)B_{j,\overline{\psi}} + \overline{\delta}(c)B_{j,\delta\overline{\psi}}\right\},\\
\overline{B}_{j,\overline{\psi}}(c/6) &= \alpha_j(6)\left\{\left(1 - 2^{j-1}\overline{\psi}(2)\right)\left(1 - 3^{j-1}\overline{\psi}(3)\right)B_{j,\overline{\psi}} + \overline{\delta}_3(c)(1 + 2^{j-1}\overline{\psi}(2))B_{j,\delta_3\overline{\psi}}\right\},\\
\overline{B}_{j,\overline{\psi}}(c/12) &= \alpha_j(12)\Big\{\left(1 - 2^{j-1}\overline{\psi}(2)\right)\left(1 - 3^{k-1}\overline{\psi}(3)\right)B_{j,\overline{\psi}} + \overline{\delta}_3(c)(1 + 2^{j-1}\overline{\psi}(2))B_{j,\delta_3\overline{\psi}}\\
&\qquad + \overline{\delta}(c)(1 + 3^{j-1}\overline{\psi}(3))B_{j,\delta\overline{\psi}}\Big\}.
\end{aligned}$$

Plugging these values into the expressions for $\overline{\mathscr{B}}_{j,\chi}(c/4)$ $(0 \le c \le 3)$, then those values into the expression for $M_{1,1}(4,\chi)$ given above, we obtain the assertion (i).

By Theorem 4.2, we have

$$S_{1,1}(4,\chi;1,1) = m^2 M_{1,1}(4,\chi).$$

Thus the assertion (ii) follows from (i).

From Theorem 4.3, we have

$$c_{1,1}(4,\chi) = -2\{M_{1,0}(4,\chi) + M_{0,1}(4,\chi) + 2M_{1,1}(4,\chi)\}.$$

By Theorem 4.1, we get

$$\begin{aligned}
M_{1,0}(4,\chi) &= \overline{\chi}(4)\mathscr{B}_{1,\chi} = -\overline{\chi}(4)(\psi(3) - 3)\tau(\psi)RB_{1,\overline{\psi}},\\
M_{0,1}(4,\chi) &= \chi(4)M_{1,0}(4,\chi) = -(\psi(3) - 3)\tau(\psi)RB_{1,\overline{\psi}}.
\end{aligned}$$

Thus the assertion (iii) follows from (i).

Since $\mathscr{B}_{k,\chi} = -\tau(\psi)R^k(\psi(3) - 3^k)B_{k,\overline{\psi}}$, the assertion (iv) follows from Corollary 4.4. □

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