MULTIDIMENSIONAL SMALL DIVISOR FUNCTIONS

Andreas Mono
Department of Mathematics and Computer Science, Division of Mathematics, University of Cologne, Cologne, Germany
amono@math.uni-koeln.de

Received: 10/4/20, Revised: 9/15/21, Accepted: 10/21/21, Published: 11/8/21

Abstract

We construct polar harmonic Maaß forms of non-positive integral weight utilizing a technique based on holomorphic projection. This generalizes recent work due to Mertens, Ono, Rolen, and due to Males, Rolen, and the author to higher even dimensions, except for dimension 2. We provide explicit examples in dimension 4, 6, 8, and 10.

1. Introduction - One-dimensional Case

In a recent paper [2], Mertens, Ono, and Rolen defined and investigated a new type of mock modular form, whose coefficients are given by a *small divisor function*. We summarize their approach. As usual, we let $\tau = u + iv \in \mathbb{H}$ and $q := \mathrm{e}^{2\pi i\tau}$. Let $P_\ell\left(\frac{n}{d}, d\right) \in \mathbb{Q}[X,Y]$, and ψ, χ be Dirichlet characters of moduli M_ψ, M_χ respectively. We denote by χ_{-4} the unique odd Dirichlet character of modulus 4, and we define

$$D_n := \left\{ d \mid n \,:\, 1 \leq d \leq \frac{n}{d} \text{ and } d \equiv \frac{n}{d} \pmod{2} \right\},$$
$$\sigma_\ell^{\mathrm{sm}}(n) := \sum_{d \in D_n} \chi\left(\frac{\frac{n}{d} - d}{2}\right) \psi\left(\frac{\frac{n}{d} + d}{2}\right) P_\ell\left(\frac{n}{d}, d\right).$$

Additionally, we require *Shimura's theta-function*

$$\theta_\psi(\tau) := \frac{1}{2} \sum_{n \in \mathbb{Z}} \psi(n) n^{\lambda_\psi} q^{n^2}, \qquad \lambda_\psi := \frac{1 - \psi(-1)}{2},$$

and recall that

$$\theta_\psi \in \begin{cases} M_{\frac{1}{2}}(\Gamma_0(4M_\psi^2), \psi) & \text{if } \lambda_\psi = 0, \\ S_{\frac{3}{2}}(\Gamma_0(4M_\psi^2), \psi \cdot \chi_{-4}) & \text{if } \lambda_\psi = 1. \end{cases} \tag{1}$$

Furthermore, we recall the definition of a harmonic Maaß form[1].

Definition 1.1. Let $k \in \frac{1}{2}\mathbb{Z}$, and choose $N \in \mathbb{N}$ such that $4 \mid N$ whenever $k \notin \mathbb{Z}$. Let ϕ be a Dirichlet character of modulus N.

(i) A *weight k harmonic Maaß form on a subgroup $\Gamma_0(N)$ with Nebentypus ϕ* is any smooth function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying the following three properties:

(a) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and all $\tau \in \mathbb{H}$ we have

$$f(\tau) = (f|_k\gamma)(\tau) := \begin{cases} \phi(d)^{-1}(c\tau+d)^{-k} f(\gamma\tau) & \text{if } k \in \mathbb{Z}, \\ \phi(d)^{-1}\left(\frac{c}{d}\right)\varepsilon_d^{2k}(c\tau+d)^{-k} f(\gamma\tau) & \text{if } k \in \frac{1}{2}+\mathbb{Z}, \end{cases}$$

where $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol, and

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

for odd integers d.

(b) The function f is harmonic with respect to the weight k hyperbolic Laplacian on $\mathbb{H}$, especially

$$0 = \Delta_k f := \left(-v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\right) f.$$

(c) The function f has at most linear exponential growth at all cusps.

(ii) A *polar harmonic Maaß form* is a harmonic Maaß form with isolated poles on the upper half plane.

Then the main result of [2] reads as follows.

Theorem 1.2 ([2, Theorem 1.1])**.** *Suppose that $\psi = \chi \neq \mathbb{1}$, and that $P_2\left(\frac{n}{d}, d\right) = d$. Denote the corresponding small divisor function by σ_1^{sm}, and by E_2 the Eisenstein series*

$$E_2(\tau) := 1 - 24\sum_{n\geq 1}\Big(\sum_{d|n} d\Big)q^n.$$

Define

$$\mathcal{E}^+(\tau) := \frac{1}{\theta_\psi(\tau)}\left(\alpha_\psi E_2(\tau) + \sum_{n\geq 1}\sigma_1^{\mathrm{sm}}(n)q^n\right),$$

$$\mathcal{E}^-(\tau) := (-1)^{\lambda_\psi}\frac{(2\pi)^{\lambda_\psi - \frac{1}{2}}i}{8\Gamma\left(\frac{1}{2}+\lambda_\psi\right)}\int_{-\overline{\tau}}^{i\infty}\frac{\theta_{\overline{\psi}}(w)}{(-i(w+\tau))^{\frac{3}{2}-\lambda_\psi}}dw,$$

[1] Be aware that there is no overall convention which terminology encodes which growth condition.

where α_ψ is an implicit constant depending only on ψ to ensure a certain growth condition. Then the function $\mathcal{E}^+ + \mathcal{E}^-$ is a polar harmonic Maaß form of weight $\frac{3}{2} - \lambda_\psi$ on $\Gamma_0\left(4M_\psi^2\right)$ with Nebentypus $\overline{\psi} \cdot \chi_{-4}^{\lambda_\psi}$.

In analogy to the classical divisor sums $\sigma_k(n)$, Mertens, Ono, and Rolen called their function $\mathcal{E}^+$ a *mock modular Eisenstein series with Nebentypus*. Furthermore, they related their result to partition functions for special choices of ψ, and proved a p-adic property of $\mathcal{E}^+$, compare [2, Corollary 1.3, Theorem 1.4].

In [1], Males, Rolen, and the author discovered another example of a polar harmonic Maaß form adapting the construction from [2].

Theorem 1.3 ([1, Theorem 1.1, Theorem 1.3])**.** *Suppose that ψ is odd, χ is even, and that $P_2\left(\frac{n}{d}, d\right) = d^2$. Denote the corresponding small divisor function by σ_2^{sm}, and define*

$$\mathcal{F}^+(\tau) := \frac{1}{\theta_\psi(\tau)} \cdot \begin{cases} \sum_{n\geq 1} \sigma_2^{\mathrm{sm}}(n) q^n & \text{if } \chi \neq \mathbb{1}, \\ \frac{1}{2}\sum_{n\geq 1} \psi(n) n^2 q^{n^2} + \sum_{n\geq 1} \sigma_2^{\mathrm{sm}}(n) q^n & \text{if } \chi = \mathbb{1}, \end{cases}$$

$$\mathcal{F}^-(\tau) := \frac{i}{\pi\sqrt{2}} \int_{-\overline{\tau}}^{i\infty} \frac{\theta_\chi(w)}{(-i(w+\tau))^{\frac{3}{2}}} dw.$$

(i) If $\chi \neq \mathbb{1}$ then the function $\mathcal{F}^+ + \mathcal{F}^-$ is a polar harmonic Maaß form of weight $\frac{3}{2}$ on $\Gamma_0\left(4M_\chi^2\right) \cap \Gamma_0\left(4M_\psi^2\right)$ with Nebentypus $\overline{\chi} \cdot (\psi \cdot \chi_{-4})^{-1}$.

(ii) If $\chi = \mathbb{1}$ then the function $\mathcal{F}^+ + \mathcal{F}^-$ is a polar harmonic Maaß form of weight $\frac{3}{2}$ on $\Gamma_0\left(4M_\psi^2\right)$ with Nebentypus $(\psi \cdot \chi_{-4})^{-1}$.

Moreover, if $\psi = \chi_{-4}$, $\chi = \mathbb{1}$, Males, Rolen and the author related $\mathcal{F}^+$ to Hurwitz class numbers, and proved a p-adic property of $\mathcal{F}^+$ in both cases of χ as well, compare [1, Corollary 1.6, Theorem 1.8].

The proof of Theorem 1.2 and 1.3 is performed in three main steps. To describe them, we let

$$\Gamma(s, z) := \int_z^\infty t^{s-1} \mathrm{e}^{-t} dt,$$

be the *incomplete Gamma function*, which is defined for $\mathrm{Re}(s) > 0$ and $z \in \mathbb{C}$. It can be analytically continued in s via the functional equation

$$\Gamma(s+1, z) = s\Gamma(s, z) + z^s \mathrm{e}^{-z},$$

and has the asymptotic behavior

$$\Gamma(s, v) \sim v^{s-1} \mathrm{e}^{-v}, \quad |v| \to \infty$$

for $v \in \mathbb{R}$. In addition, let

$$\xi_\kappa := 2iv^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}} = iv^\kappa \overline{\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)}$$

be the *Bruinier–Funke operator of weight* κ, and

$$\pi_\kappa f(\tau) := \frac{(\kappa-1)(2i)^\kappa}{4\pi} \int_{\mathbb{H}} \frac{f\,(x+iy)\,y^k}{(\tau - x + iy)^\kappa} \frac{dxdy}{y^2},$$

be the *weight* κ *holomorphic projection operator*, whenever f is translation invariant, and the integral converges absolutely.

Moreover, we let

$$g(\tau) := \sum_{n\geq 1} \beta\,(n)\,q^n, \qquad f^+(\tau) := \frac{1}{g(\tau)} \sum_{n\geq 1} \sigma_\ell^{\mathrm{sm}}(n) q^n,$$
$$f^-(\tau) := \sum_{m\geq 1} \alpha\,(m)\,m^{k_f-1}\Gamma\,(1-k_f, 4\pi m v)\,q^{-m}, \qquad f(\tau) := (f^+ + f^-)(\tau).$$

Then we proceed as follows.

(I) Show that

$$\pi_\kappa\,(fg)\,(\tau) = 0.$$

To this end, we rewrite the definition of the given non-holomorphic part (see [1, Lemma 4.1] for instance), and next we utilize the following result. Here and throughout, $\mathcal{P}_r^{(a,b)}$ denotes the *Jacobi polynomial* of degree r and parameter a, b, which we introduce in Section 4.1.

Proposition 1.4 ([1, Proposition 1.7, Corollary 4.2])**.** *Let* $k_f \in \mathbb{R} \setminus \mathbb{N}$*,* $k_g \in \mathbb{R} \setminus (-\mathbb{N})$*, such that* $\kappa := k_f + k_g \in \mathbb{Z}_{\geq 2}$*. Let* $\alpha(m)$*,* $\beta(n)$ *be two complex sequences, and define the functions* f*,* g *as above. Suppose that*

(a) the function $(fg)(r+iv)$ *grows at most polynomially as* $v \searrow 0$*, where* $r \in \mathbb{Q}$*, and that*

(b) the function $(fg)(iv)$ *grows at most polynomially as* $v \nearrow \infty$*.*

Then the weight κ *holomorphic projection of* f^-g *is given by*

$$\pi_\kappa\left(f^-g\right)(\tau) = -\Gamma(1-k_f) \sum_{m\geq 1} \sum_{n-m\geq 1} \alpha\,(m)\,\beta\,(n) \times \left(n^{k_f-1}\mathcal{P}_{\kappa-2}^{(1-k_f,1-\kappa)}\left(1-2\frac{m}{n}\right) - m^{k_f-1}\right) q^{n-m}.$$

Furthermore, it holds that $\pi_\kappa\,(f^+g)\,(\tau) = (f^+g)\,(\tau)$*.*

In addition, the holomorphic part f^+g has to be rewritten as well, see the proof of Theorem 1.2 in [1, Section 4].

(II) We compute

$$\xi_\kappa\left(fg\right)\left(\tau\right) = -(4\pi)^{1-k_f} v^{k_g} \left(\sum_{m\geq 1} \overline{\alpha(m)} q^m\right) \overline{g(\tau)},$$

and choose the coefficients $\alpha\left(m\right)$, $\beta\left(n\right)$, such that this function is modular of weight $2-\kappa$.

(III) Conclude that fg is modular of weight κ by the following result.

Proposition 1.5 ([2, Proposition 2.3])**.** *Let $h\colon \mathbb{H} \to \mathbb{C}$ be a translation invariant function such that $|h(\tau)|v^\delta$ is bounded on $\mathbb{H}$ for some $\delta > 0$. If the weight k holomorphic projection of h vanishes identically for some $k > \delta + 1$ and $\xi_k h$ is modular of weight $2-k$ for some subgroup $\Gamma < \mathrm{SL}_2(\mathbb{Z})$, then h is modular of weight k for Γ.*

The subtle growth conditions are required to include the case π_2, and are clearly satisfied if we deal with higher weight holomorphic projections, in which case the integral defining π_k converges absolutely.

Lastly, verify harmonicity and the growth property towards the cusps required by the definition of a harmonic Maaß form.

Finally, we mention one remark from [1, p. 5], which states that there are more choices of half integral parameters k_f, k_g, which lead to other choices of polynomials $P_\ell\left(\frac{n}{d}, d\right)$ in the definition of $\sigma_\ell^{\mathrm{sm}}$, such that step (I) above works.

We refer to the first two sections of [1] for more details, and for overall preliminaries introducing the aforementioned objects together with their key properties.

2. Statement of the Result

We define the function f_ℓ in equation (2) based on the objects[2] introduced at the beginning of Section 3. We apply the outlined steps from Section 1 to f_ℓ, which leads to several intermediate results during Section 3. Combining these results, we arrive at the following theorem.

[2]In short, we choose ℓ first, which leads to the weights k_{f_ℓ}, and κ. In turn, this defines the polynomial P_ℓ via Corollay 3.2, which yields $\sigma_\ell^{\mathrm{sm}}$ eventually.

Theorem 2.1. *Let ψ be an odd Dirichlet character, and χ be an even and non-trivial Dirichlet character. Let $\ell \in 2\mathbb{N}+2$. Define P_ℓ as indicated in Corollay 3.2, obtaining the corresponding small divisor function $\sigma_\ell^{\mathrm{sm}}$. Then the resulting function f_ℓ is a polar harmonic Maaß form of weight $2-\frac{\ell}{2} \in -\mathbb{N}_0$ on $\Gamma_0(4M_\chi^2) \cap \Gamma_0(4M_\psi^2)$ with Nebentypus $\overline{\chi} \cdot (\psi \cdot \chi_{-4})^{-1}$. Its shadow is given by a non-zero constant multiple of $\theta_{\overline{\chi}}^{\ell}$.*

In other words, the technique presented in [1], [2] applies straightforward in higher even dimensions, except for dimension two. We plan to find and investigate applications of f_ℓ to other areas of number theory, such as combinatorics, as in the one–dimensional case [2, Corollary 1.3].

3. Multidimensional Case

We fix $\ell \in \mathbb{N}$ throughout. Let $\vec{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$. We recall the usual multi-index conventions

$$\vec{n}! \coloneqq n_1 n_2 \cdots n_\ell, \quad |\vec{n}| \coloneqq n_1 + \ldots + n_\ell, \quad \|\vec{n}\| \coloneqq \sqrt{n_1^2 + \ldots + n_\ell^2}.$$

We let $\psi \neq \mathbb{1}$, and consider

$$\theta_\psi(\tau)^\ell = \sum_{\vec{n} \in \mathbb{N}^\ell} \psi(\vec{n}!) (\vec{n}!)^{\lambda_\psi} q^{\|\vec{n}\|^2}.$$

Moreover, we relax our assumption to $P_\ell \in \mathbb{Q}(X,Y)$, and we let

$$\begin{aligned}
\mathcal{D}_{\vec{n}} &\coloneqq \bigtimes_{j=1}^{\ell} D_{n_j} \\
&= \left\{ \vec{d} \in \mathbb{N}^\ell : d_j \mid n_j,\ 1 \leq d_j \leq \frac{n_j}{d_j}, \text{ and } d_j \equiv \frac{n_j}{d_j} \pmod 2 \quad \forall\, 1 \leq j \leq \ell \right\} \\
\sigma_\ell^{\mathrm{sm}}(\vec{n}) &\coloneqq \sum_{\vec{d} \in \mathcal{D}_{\vec{n}}} \left\{ \prod_{j=1}^{\ell} \chi\left(\frac{\frac{n_j}{d_j} - d_j}{2}\right) \psi\left(\frac{\frac{n_j}{d_j} + d_j}{2}\right) \left(\frac{\frac{n_j}{d_j} - d_j}{2}\right)^{\lambda_\chi} \left(\frac{\frac{n_j}{d_j} + d_j}{2}\right)^{\lambda_\psi} \right\} \\
&\qquad \times P_\ell\left(\|(n_j/d_j)_{1 \leq j \leq \ell}\|^2, \|\vec{d}\|^2 \right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
f_\ell^+(\tau) &\coloneqq \frac{1}{\theta_\psi(\tau)^\ell} \sum_{\vec{n} \in \mathbb{N}^\ell} \sigma_\ell^{\mathrm{sm}}(\vec{n})\, q^{|\vec{n}|}, \\
f_\ell^-(\tau) &\coloneqq \frac{1}{\Gamma(1-k_{f_\ell})} \sum_{\vec{m} \in \mathbb{N}^\ell} \chi(\vec{m}!) (\vec{m}!)^{\lambda_\chi} \|\vec{m}\|^{2(k_{f_\ell}-1)} \Gamma(1-k_{f_\ell}, 4\pi \|\vec{m}\|^2 v) q^{-\|\vec{m}\|^2}, \\
f_\ell(\tau) &\coloneqq (f_\ell^+ + f_\ell^-)(\tau). \qquad (2)
\end{aligned}$$

We insert this setting into the constructive method described in the first section, and devote a subsection to each step.

3.1. First Step

We verify that the first step continues to hold due to exactly the same proofs as in [1, Section 3]. We have to be careful regarding the summation conditions, which are determined one step after the application of the Lipschitz summantion formula. Explicitly, we obtain

$$\pi_\kappa\left(f_\ell^- \theta_\psi^\ell\right)(\tau) = -\sum_{r\geq 1} \sum_{\substack{\vec{m},\vec{n}\in\mathbb{N}^\ell \\ \|\vec{n}\|^2-\|\vec{m}\|^2=r}} \chi\left(\vec{m}!\right)\left(\vec{m}!\right)^{\lambda_\chi} \psi\left(\vec{n}!\right)\left(\vec{n}!\right)^{\lambda_\psi} \times \left(\|\vec{n}\|^{2(k_{f_\ell}-1)} \mathcal{P}_{\kappa-2}^{(1-k_{f_\ell},1-\kappa)}\left(1-2\frac{\|\vec{m}\|^2}{\|\vec{n}\|^2}\right) - \|\vec{m}\|^{2(k_{f_\ell}-1)}\right) q^r.$$

To match this expression with $f_\ell^+ g$, we rewrite the small divisor function. We substitute

$$\vec{a} := \left(\frac{\frac{n_1}{d_1}+d_1}{2},\ldots,\frac{\frac{n_\ell}{d_\ell}+d_\ell}{2}\right), \qquad \vec{b} := \left(\frac{\frac{n_1}{d_1}-d_1}{2},\ldots,\frac{\frac{n_\ell}{d_\ell}-d_\ell}{2}\right),$$

from which we deduce

$$\vec{d} = \vec{a}-\vec{b}, \qquad \vec{a}+\vec{b} = \left(n_j/d_j\right)_{1\leq j\leq \ell}, \qquad |n| = \|\vec{a}\|^2 - \|\vec{b}\|^2.$$

Thus,

$$f_\ell^+ \theta_\psi^\ell(\tau) = \sum_{\vec{b}\in\mathbb{N}^\ell} \sum_{\vec{a}-\vec{b}\in\mathbb{N}^\ell} \chi\left(\vec{b}!\right)\left(\vec{b}!\right)^{\lambda_\chi} \psi\left(\vec{a}!\right)\left(\vec{a}!\right)^{\lambda_\psi} \; P_\ell\left(\|\vec{a}+\vec{b}\|, \|\vec{a}-\vec{b}\|\right) q^{\|\vec{a}\|^2-\|\vec{b}\|^2}.$$

We transform the summation condition.

Lemma 3.1. *We have*

$$f_\ell^+ \theta_\psi^\ell(\tau) = \sum_{r\geq 1} \sum_{\substack{\vec{m},\vec{n}\in\mathbb{N}^\ell \\ \|\vec{n}\|^2-\|\vec{m}\|^2=r}} \chi\left(\vec{m}!\right)\left(\vec{m}!\right)^{\lambda_\chi} \psi\left(\vec{n}!\right)\left(\vec{n}!\right)^{\lambda_\psi} \; P_\ell\left(\|\vec{m}+\vec{n}\|, \|\vec{m}-\vec{n}\|\right) q^r.$$

Proof. Note that if $\vec{a}-\vec{b}\in\mathbb{N}^\ell$, then

$$\|\vec{a}\|^2 - \|\vec{b}\|^2 = \sum_{j=1}^{\ell} (a_j+b_j)(a_j-b_j) \geq 1.$$

Conversely, suppose $\|\vec{a}\|^2 - \|\vec{b}\|^2 \geq 1$. Recall that $n_j = (a_j+b_j)(a_j-b_j) \in \mathbb{N}$ for every $1 \leq j \leq \ell$ by definition of f^+, and $a_j + b_j$ is always positive. Thus, $(a_j - b_j) \geq 1$ for every $1 \leq j \leq \ell$, which proves the lemma. □

Hence, we achieve the following result by virtue of Proposition 1.4.

Corollary 3.2. *If P_ℓ is defined by the condition*

$$\|\vec{b}\|^{2(k_{f_\ell}-1)}\mathcal{P}_{\kappa-2}^{(1-k_{f_\ell},1-\kappa)}\left(1-2\frac{\|\vec{a}\|^2}{\|\vec{b}\|^2}\right)-\|\vec{a}\|^{2(k_{f_\ell}-1)}=P_\ell\left(\|\vec{a}+\vec{b}\|,\|\vec{a}-\vec{b}\|\right),$$

then we have $\pi_\kappa\left(f_\ell\theta_\psi^\ell\right)(\tau)=0$.

3.2. Second Step

We summarize the result of a standard calcualtion.

Lemma 3.3. *We have*

$$\xi_\kappa\left(f_\ell\theta_\psi^\ell\right)(\tau)=-\frac{(4\pi)^{1-k_{f_\ell}}}{\Gamma\left(1-k_{f_\ell}\right)}v^{k_{\theta_\psi^\ell}}\theta_{\overline{\chi}}(\tau)^\ell\frac{|\theta_\psi(\tau)|^{2\ell}}{\theta_\psi(\tau)^\ell}$$

away from the zeros of θ_ψ.

Proof. By definition and linearity of ξ_κ, it holds that

$$\xi_\kappa\left(f_\ell^-\theta_\psi^\ell\right)(\tau)=(\xi_\kappa f_\ell^-)(\tau)\cdot\overline{\theta_\psi(\tau)^\ell}+\overline{f_\ell^-(\tau)}\left(\xi_\kappa\theta_\psi^\ell\right)(\tau)=(\xi_\kappa f_\ell^-)(\tau)\cdot\overline{\theta_\psi(\tau)^\ell},$$

where the last step uses that θ_ψ^ℓ is holomorphic. Next, one computes[3]

$$(\xi_\kappa f_\ell^-)(\tau)=-\frac{(4\pi)^{1-k_{f_\ell}}}{\Gamma\left(1-k_{f_\ell}\right)}v^{k_{\theta_\psi^\ell}}\sum_{\vec{m}\in\mathbb{N}^\ell}\overline{\chi\left(\vec{m}!\right)}\left(\vec{m}!\right)^{\lambda_\chi}q^{\|\vec{m}\|^2},$$

from which we infer the claim. □

Combining the previous result with the modularity of Shimura's theta function (see equation (1)), and the fact that

$$\mathrm{Im}\left(\gamma\tau\right)=\frac{v}{|c\tau+d|^2}$$

for every $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ and every $\tau\in\mathbb{H}$, we obtain the following corollary.

Corollary 3.4. *If $\chi\neq\mathbb{1}$ then $\xi_\kappa\left(f_\ell\theta_\psi^\ell\right)$ is modular of weight*

$$\ell\left(\frac{1}{2}+\lambda_{\overline{\chi}}\right)-\ell\left(\frac{1}{2}+\lambda_\psi\right)$$

on $\Gamma_0(4M_\chi^2)\cap\Gamma_0(4M_\psi^2)$ *with Nebentypus* $\overline{\chi}\cdot\left(\psi\cdot\chi_{-4}\right)^{-1}$.

Thus, we stipulate ψ to be odd, and χ to be even and non-trivial, getting

$$\kappa=2-(-\ell)\in\mathbb{Z}_{\geq 2},\qquad k_{f_\ell}=2-\frac{\ell}{2},$$

as desired.

[3] Compare the proof of [1, Lemma 2.12] for some intermediate steps.

3.3. Third Step

We verify the two remaining conditions of a polar harmonic Maaß form.

Lemma 3.5. *Let $\tau \in \mathbb{H}$ with $\theta_\psi(\tau) \neq 0$. Then, the function $f_\ell = f_\ell^+ + f_\ell^-$ satisfies*

$$0 = \Delta_{k_{f_\ell}} f_\ell,$$

and has the required growth property of a polar harmonic Maaß form.

Proof. The first assertion follows by construction of f_ℓ. Since θ_ψ^ℓ is of exponential decay towards all cusps, the function f_ℓ^+ admits at most linear exponential growth towards all cusps. In particular, the cusp $i\infty$ is a removable singularity of f^+, because both numerator and denominator vanish at $i\infty$ of order ℓ. In addition, the function f_ℓ^- decays exponentially towards $i\infty$, since the incomplete Gamma function does (and it dominates the powers of q). The transformation behaviour of θ_χ under the full modular group $\mathrm{SL}_2(\mathbb{Z})$ implies that f_ℓ^- is of at most moderate growth towards all cusps. Indeed, choosing suitable scaling matrices yields additional factors of polynomial growth inside the Fourier expansion of f_ℓ^-. This establishes the second assertion. □

3.4. Conclusion

We justify the application of Proposition 1.4, which proves Theorem 2.1.

Proof of Theorem 2.1. By definition, the Fourier coefficients of $\theta_\psi^\ell f_\ell^+$ expanded at $i\infty$ are of moderate growth, whence the growth of $\theta_\psi^\ell f_\ell^+$ towards any cusp has to be moderate. Consequently, the growth of $\theta_\psi^\ell f_\ell$ towards any cusp is moderate according to the proof of Lemma 3.5. Thus, the assumptions in Proposition 1.4 are satisfied by $\theta_\psi^\ell f_\ell$. Performing the outlined steps concludes the proof of Theorem 2.1. □

4. Numerical Examples

4.1. An Interlude on Jacobi Polynomials

The Jacobi polynomials $\mathcal{P}_r^{(a,b)}$ admit a representation in terms of of Gauß' hypergeometric function ${}_2F_1$, namely

$$\mathcal{P}_r^{(a,b)}(z) = \frac{\Gamma(a+r+1)}{r!\,\Gamma(a+1)}\,{}_2F_1\left(-r, a+b+r+1, a+1, \frac{1-z}{2}\right),$$

for any $r \in \mathbb{N}$. This yields many identities between Jacobi polynomials of "neighboring" degree r and parameters a, b, that is $r \in \{r-1, r, r+1\}$ and analogously for a, b. For instance, one could use Gauß contiguous relations to obtain such identities.

In particular, this leads to a recursive characterization of the Jacobi polynomials. More precisely, we have

$$\mathcal{P}_0^{(a,b)}(z) = 1, \qquad \mathcal{P}_1^{(a,b)}(z) = \frac{1}{2}\left(a - b + (a+b+2)z\right),$$
$$c_1(j)\mathcal{P}_{j+1}^{(a,b)}(z) = \left(c_2(j) + c_3(j)z\right)\mathcal{P}_j^{(a,b)}(z) - c_4(j)\mathcal{P}_{j-1}^{(a,b)}(z),$$

where

$$c_1(j) = 2(j+1)(j+a+b+1)(2j+a+b), \qquad c_2(j) = (2j+a+b+1)\left(a^2 - b^2\right),$$
$$c_3(j) = (2j+a+b)(2j+a+b+1)(2j+a+b+2),$$
$$c_4(j) = 2(j+a)(j+b)(2j+a+b+2).$$

4.2. Explicit Examples

Note that the parallelogram law and the fact $|n| = \|\vec{a}+\vec{b}\|\|\vec{a}-\vec{b}\|$ yield

$$\|\vec{a}\|^2 = \frac{\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2}{4} + \frac{\|\vec{a}+\vec{b}\|\|\vec{a}-\vec{b}\|}{2},$$
$$\|\vec{b}\|^2 = \frac{\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2}{4} - \frac{\|\vec{a}+\vec{b}\|\|\vec{a}-\vec{b}\|}{2}.$$

The case $\ell = 2$ has to be excluded since $k_{f_\ell} \neq 1$.

4.2.1. Higher Even Dimensions

On one hand, if $\ell = 4$ for instance, we have

$$\kappa = 6, \qquad k_{f_4} = 0, \qquad \frac{\mathcal{P}_4^{(1,-5)}\left(1 - 2\frac{\|\vec{a}\|^2}{\|\vec{b}\|^2}\right)}{\|\vec{b}\|^2} - \frac{1}{\|\vec{a}\|^2} = \frac{\left(\|\vec{a}\|^2 - \|\vec{b}\|^2\right)^5}{\|\vec{a}\|^2\|\vec{b}\|^{10}},$$

and thus, we choose the function P_4 as

$$P_4\left(\|\vec{a}+\vec{b}\|, \|\vec{a}-\vec{b}\|\right)$$
$$= \frac{\|\vec{a}-\vec{b}\|^5\|\vec{a}+\vec{b}\|^5}{\left(\frac{\|\vec{a}+\vec{b}\|^2+\|\vec{a}-\vec{b}\|^2}{4} + \frac{\|\vec{a}+\vec{b}\|\|\vec{a}-\vec{b}\|}{2}\right)\left(\frac{\|\vec{a}+\vec{b}\|^2+\|\vec{a}-\vec{b}\|^2}{4} - \frac{\|\vec{a}+\vec{b}\|\|\vec{a}-\vec{b}\|}{2}\right)^5}.$$

Similarly, we compute (with $x \coloneqq \|\vec{a}\|$, $y \coloneqq \|\vec{b}\|$)

$$y^{-4}\ \mathcal{P}_6^{(2,-7)}\left(1-2\frac{x^2}{y^2}\right)-x^{-4}=\frac{\left(x^2-y^2\right)^7}{x^4y^{16}}\left(7x^2+y^2\right),$$
$$y^{-6}\ \mathcal{P}_8^{(3,-9)}\left(1-2\frac{x^2}{y^2}\right)-x^{-6}=\frac{\left(x^2-y^2\right)^9}{x^6y^{22}}\left(45x^4+9x^2y^2+y^4\right),$$
$$y^{-8}\ \mathcal{P}_{10}^{(4,-11)}\left(1-2\frac{x^2}{y^2}\right)-x^{-8}=\frac{\left(x^2-y^2\right)^{11}}{x^8y^{28}}\left(286x^6+66x^4y^2+11x^2y^4+y^6\right),$$

from which we read off the corresponding definitions of P_ℓ.

Because of the aforementioned recursive nature of the Jacobi polynomials, the indicated pattern continues to hold for every even dimension $\ell \in 2\mathbb{N}+2$ by induction.

4.2.2. Higher Odd Dimensions

On the other hand, the case of dimension $\ell \in 2\mathbb{Z}_{\geq 2}-1$ produces more complicated functions P_ℓ. For example, if $\ell = 3$ we have

$$\kappa = 5, \qquad k_{f_3} = \frac{1}{2},$$
$$\frac{\mathcal{P}_3^{\left(\frac{1}{2},-4\right)}\left(1-2\frac{\|\vec{a}\|^2}{\|\vec{b}\|^2}\right)}{\|\vec{b}\|}-\frac{1}{\|\vec{a}\|}=-\frac{\left(\|\vec{a}\|-\|\vec{b}\|\right)^4\left(5\|\vec{a}\|^3+20\|\vec{a}\|^2\|\vec{b}\|+29\|\vec{a}\|\|\vec{b}\|^2+16\|\vec{b}\|^3\right)}{16\|\vec{a}\|\|\vec{b}\|^7},$$

and if $\ell = 5$, we have

$$\kappa = 7, \qquad k_{f_5} = -\frac{1}{2},$$
$$\frac{\mathcal{P}_5^{\left(\frac{3}{2},-6\right)}\left(1-2\frac{\|\vec{a}\|^2}{\|\vec{b}\|^2}\right)}{\|\vec{b}\|^3}-\frac{1}{\|\vec{a}\|^3}$$
$$=\frac{-693\|\vec{a}\|^{13}+4095\|\vec{a}\|^{11}\|\vec{b}\|^2-10010\|\vec{a}\|^9\|\vec{b}\|^4+12870\|\vec{a}\|^7\|\vec{b}\|^6-9009\|\vec{a}\|^5\|\vec{b}\|^8+3003\|\vec{a}\|^3\|\vec{b}\|^{10}-256\|\vec{b}\|^{13}}{256\|\vec{a}\|^3\|\vec{b}\|^{13}}.$$

We observe that we are left with odd powers of $\|\vec{a}\|$, $\|\vec{b}\|$ in both odd-dimensional cases. If we keep the dependence of P_ℓ on $\|\vec{a} \pm \vec{b}\|$, which ultimately justifies the terminology "divisor function", then odd powers obstruct a definition of P_ℓ via the parallelogram law in these cases of ℓ. Once more, an inductive argument via the recursive characterization of the Jacobi polynomials extends this phenomenon to all odd dimensions $\ell \in 2\mathbb{N}+1$.

Acknowledgement. We would like to thank the anonymous referee for many valuable comments on an earlier version of this paper.

References

[1] J. Males, A. Mono and L. Rolen, Polar harmonic Maaß forms and holomorphic projection, https://arxiv.org/abs/2009.04955v2 (2020), preprint.

[2] M. H. Mertens, K. Ono and L. Rolen, Mock modular Eisenstein series with Nebentypus, *Int. J. Number Theory* **17** (2021), no. 3, 683–697.