ABUNDANCE OF ARITHMETIC PROGRESSIONS IN SOME COMBINATORIALLY RICH SETS BY ELEMENTARY MEANS

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Abstract

H. Furstenberg and E. Glasner proved that for an arbitrary $k \in \mathbb{N}$, any piecewise syndetic set of integers contains a k-term arithmetic progression and the collection of such progressions is itself piecewise syndetic in $\mathbb{Z}$. The above result was extended for arbitrary semigroups by V. Bergelson and N. Hindman, using the algebra of the Stone-Čech compactification of discrete semigroups. In a recent work, N. Hindman and D. Strauss proved a more general result on the set of natural numbers. They provide an abundance of various types of large sets in correspondence to each image partition regular matrix. Recently they have further extended those results in more abstract settings. In these works they have used the complex machinery of the algebra of the Stone-Čech compactification of discrete semigroups. In this work we will investigate the abundance of the arithmetic progressions in quasi-central sets, J-sets and C-sets in an elementary way.

1. Introduction

A subset S of $\mathbb{Z}$ is called *syndetic* if there exists $r \in \mathbb{N}$ such that $\bigcup_{i=1}^{r}(S-i) = \mathbb{Z}$, and is called *thick* if it contains arbitrarily large intervals. Sets which can be expressed as the intersection of thick and syndetic sets are called *piecewise syndetic*. Let $(S,+)$ be a semigroup. For any $A \subseteq S$ and $t \in S$, let $-t + A$ denote the set $\{s \in S : t + s \in A\}$. Now we can state the definitions of large sets for general semigroups. For a countable semigroup $(S,+)$, a set $A \subseteq S$ is called *syndetic* in $(S,+)$ if there exists a finite set $F \subset S$ such that $\bigcup_{t \in F} -t + A = S$, and is called *thick* if for every finite set $E \subset S$, there exists an element $x \in S$ such that $E+x \subset A$. A set $A \subseteq S$ is a *piecewise syndetic* set if there exists a finite set $F \subset S$ such that $\bigcup_{t\in F} -t + A$ is thick in S [8, Definition 4.38, page 101]. It can be proved that a piecewise syndetic set can be interpreted as the intersection of a thick set and a syndetic set [8, Theorem 4.49, page 105].

One of the famous Ramsey theoretic results is van der Waerden's Theorem [11], which states that at least one cell of any finite partition $\{C_1, C_2, \ldots, C_r\}$ of $\mathbb{N}$, contains arithmetic progressions of arbitrary length. Since arithmetic progressions are invariant under shifts, it follows that every piecewise syndetic set contains arbitrarily long arithmetic progressions. The following theorem was proved algebraically by H. Furstenberg and E. Glasner in [6], and combinatorially by M. Beigelböck in [1].

Theorem 1.1. *Let $k \in \mathbb{N}$ and assume that $S \subseteq \mathbb{Z}$ is piecewise syndetic. Then $\{(a,d) : \{a, a+d, \ldots, a+kd\} \subset S\}$ is piecewise syndetic in $\mathbb{Z}^2$.*

The technique of the above theorem can be lifted easily to the set of natural numbers $\mathbb{N}$ to prove the following theorem; hence we omit the proof.

Theorem 1.2. *Let $k \in \mathbb{N}$ and assume that $S \subseteq \mathbb{N}$ is piecewise syndetic. Then $\{(a,d) : \{a, a+d, \ldots, a+kd\} \subset S\}$ is piecewise syndetic in $\mathbb{N}^2$.*

In [9], N. Hindman and D. Strauss recently showed that if A is an $u \times v$ matrix on $\mathbb{N}$ which is image partition regular over $\mathbb{N}$, and Ψ is a large subset of $\mathbb{N}$, then the set $\{x \in \mathbb{N}^v : Ax \in C^u\}$, where C is a Ψ-set, is itself a Ψ set in $\mathbb{N}^v$. In [10], they have the above result over countable commutative semigroups. Their proof is completely algebraic in nature. In this article, we consider the study of arithmetic progressions and this is the case in which A is a $(l+1) \times 2$ matrix.

Let us now recall some prerequisites of the algebra of the Stone-Čech compactification of discrete semigroups. For details, the readers are invited to read [8].

Let $(S, \cdot)$ be a discrete semigroup and βS be the set of ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S$. Given $A \subseteq S$ let us define

$$\overline{A} = \{p \in \beta S : A \in p\}.$$

Then the set $\{\overline{A} : A \subseteq S\}$ is a basis for a topology on βS. The operation $\cdot$ on S can be extended to the Stone-Čech compactification βS of S so that $(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$, is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$, is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. Also note that every compact right topological semigroup contains idempotent elements.

A nonempty subset I of a semigroup $(S,\cdot)$ is called a left ideal of S if $S\cdot I\subset I$, a right ideal of S if $I\cdot S\subset I$, and a two-sided ideal (or simply an ideal) of S if it is both a left and right ideal. A minimal left ideal is the left ideal that does not contain any proper left ideal. Similarly, we can define minimal right ideals and the smallest ideal.

Any compact Hausdorff right topological semigroup $(S,\cdot)$ has the smallest two-sided ideal

$$\begin{aligned} K(S) &= \bigcup\{L : L \text{ is a minimal left ideal of } S\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } S\}. \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R, $L\cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(S)$ is called a minimal idempotent. If p and q are idempotents in S, we write $p\le q$ if and only if $p\cdot q=q\cdot p=p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal.

Definition 1.3. A set $A\subseteq S$ in a semigroup $(S,\cdot)$ is said to be an *IP*-set if and only if there exists an infinite set X such that all the finite products of distinct elements from X belong to A, where the product is taken in an increasing order of indices.

It can be proved that A is IP-set if and only if A belongs to some idempotent of βS. A set $E\subset S$ is called an IP*-set if and only if it meets non-trivially with every IP-set. One can show that E is an IP*-set if and only if E is contained in every idempotent in βS.

The notion of *central* set was introduced by H. Furstenberg in [5] in terms of topological dynamics and the definition makes sense in any semigroup. In [2], that definition was shown to be equivalent to a much simpler algebraic characterization. It is this algebraic characterization which we take as the definition for all semigroups.

Definition 1.4. Let $(S,\cdot)$ be a semigroup and let $A\subseteq S$. Then A is *central* if and only if there is some minimal idempotent $p\in\beta S$ with $p\in\overline{A}$.

Note that, the two-sided ideal $K(\beta S)$ is not closed in βS and any member of the idempotents in the closure of $K(\beta S)$, say $cl(K(\beta S)$, is called *quasi-central* set.

Definition 1.5 ([7])**.** Let $(S,\cdot)$ be a semigroup and let $A\subseteq S$. Then A is *quasi-central* if and only if there is some idempotent $p\in cl(K(\beta S))$ with $p\in\overline{A}$.

For countable commutative semigroups, quasi-central sets have a nice combinatorial characterization as follows.

Theorem 1.6 ([7])**.** *For a countable semigroup $(S,\cdot)$, $A\subseteq S$ is quasi-central if and only if there is a decreasing sequence $\langle C_n\rangle_{n=1}^{\infty}$ of subsets of A such that,*

(1) *for each* $n \in \mathbb{N}$, *and each* $x \in C_n$, *there exists* $m \in \mathbb{N}$ *such that* $C_m \subseteq x^{-1}C_n$ *and*

(2) *for each* $n \in \mathbb{N}$, C_n *is piecewise syndetic.*

The importance of the quasi-central sets is that, they are very close to central sets and they enjoy a close combinatorial property of those central sets. For any set X, define $\mathcal{P}_f(X) = \{A \subseteq X : |A| < \infty\}$. The following theorem is called the Central Sets Theorem for countable commutative semigroups.

Theorem 1.7 ([8, Theorem 14.11])**.** *Let* $(S, +)$ *be a discrete commutative semigroup. Let* A *be a central set in* S, *and for each* $l \in \mathbb{N}$, *let* $\langle y_{l,n} \rangle_{n=1}^{\infty}$ *be a sequence in* S. *There exist a sequence* $\langle a_n \rangle_{n=1}^{\infty}$ *in* S *and a sequence* $\langle H_n \rangle_{n=1}^{\infty}$ *in* $\mathcal{P}_f(\mathbb{N})$ *such that* $\max H_n < \min H_{n+1}$ *for each* $n \in \mathbb{N}$ *and such that for each* $f \in \Phi$,

$$FS\left(\left\langle a_n + \sum_{t \in H_n} y_{f(n),t} \right\rangle_{n=1}^{\infty}\right) \subseteq A,$$

where Φ *is the set of all functions* $f : \mathbb{N} \to \mathbb{N}$ *for which* $f(n) \leq n$ *for all* $n \in \mathbb{N}$.

A strengthening of the above Central Sets Theorem was proved by D. De, N. Hindman and D. Strauss in [4]. The following version for commutative semigroups is simpler to state.

Theorem 1.8. *Let* $(S, +)$ *be a commutative semigroup. Let* C *be a central subset of* S. *Then there exist functions* $\alpha : \mathcal{P}_f\left(S^{\mathbb{N}}\right) \to S$ *and* $H : \mathcal{P}_f\left(S^{\mathbb{N}}\right) \to \mathcal{P}_f(\mathbb{N})$ *such that:*

1. *if* $F, G \in \mathcal{P}_f\left(S^{\mathbb{N}}\right)$ *and* $F \subsetneq G$, *then* $\max H(F) < \min H(G)$;
2. *whenever* $r \in \mathbb{N}$, $G_1, G_2, ..., G_r \in \mathcal{P}_f\left(S^{\mathbb{N}}\right)$ *such that* $G_1 \subsetneq G_2 \subsetneq ..... \subsetneq G_r$ *and for each* $i \in \{1, 2, ..., r\}$, $f_i \in G_i$ *one has*

$$\sum_{i=1}^{r}\left(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\right) \in C.$$

Another important set, which is known as a J*-set*, is defined as follows.

Definition 1.9. Let $(S, +)$ be a commutative semigroup. A set $A \subseteq S$ is a J-set if and only if whenever $F \in \mathcal{P}_f\left(S^{\mathbb{N}}\right)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + \sum_{t \in H} f(t) \in A$.

It can be shown that a piecewise syndetic set is a J-set [8, Theorem 14.8.3, page 336]. The set $J(S) = \{p \in \beta S : \text{for all } A \in p, A \text{ is a } J\text{-set}\}$ is a compact two-sided ideal of βS.

Definition 1.10. $A \subseteq S$ is a C-set if and only if there exists an idempotent $p \in J(S)$ such that $A \in p$.

It can be shown that A is a C-set if and only if it satisfies the conclusion of the Central Sets Theorem [4, Theorem 2.2]. It can also be shown that, for countable commutative semigroups, C-sets satisfy a nice combinatorial characterization in terms of J-sets which is similar to Theorem 1.6.

Theorem 1.11 ([8, Theorem 14.27, page 358])**.** *For a countable semigroup $(S, \cdot)$, $A \subseteq S$ is a C-set if and only if there is a decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of subsets of A such that,*

1. *for each $n \in \mathbb{N}$, and each $x \in C_n$, there exists $m \in \mathbb{N}$ such that $C_m \subseteq x^{-1}C_n$ and*
2. *for each $n \in \mathbb{N}$, C_n is a J-set.*

2. Proof of the Main Theorems

Theorem 2.1. *Let $A \subseteq \mathbb{N}$ be a quasi-central set in $\mathbb{N}$ and $l \in \mathbb{N}$. Then the collection $\{(a, b) : \{a, a + b, a + 2b, \ldots, a + lb\} \subset A\}$ is quasi-central in $(\mathbb{N} \times \mathbb{N}, +)$.*

Proof. As A is quasi-central, there is a decreasing sequence of piecewise syndetic sets of A, say $\{A_n : n \in \mathbb{N}\}$, satisfying Theorem 1.6. So,

$$A \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots.$$

It follows from Theorem 1.2 that $B = \{(a, b) : \{a, a + b, a + 2b, \ldots, a + lb\} \subset A\}$ is piecewise syndetic in $\mathbb{N} \times \mathbb{N}$. As A_i is piecewise syndetic for all $i \in \mathbb{N}$; it follows from Theorem 1.2 that

$$B_i = \{(a, b) \in \mathbb{N} \times \mathbb{N} : \{a, a + b, a + 2b, \ldots, a + lb\} \subset A_i\}$$

is piecewise syndetic in $\mathbb{N} \times \mathbb{N}$ for all $i \in \mathbb{N}$.

Clearly,

$$B \supseteq B_1 \supseteq B_2 \supseteq \ldots \supseteq B_n \supseteq \ldots.$$

If $n \in \mathbb{N}$, and $(a, b) \in B_n$, then $\{a, a + b, a + 2b, \ldots, a + lb\} \subset A_n$. From Theorem 1.6, there exists $N \in \mathbb{N}$ such that

$$A_N \subseteq \bigcap_{i=0}^{l} (-(a + ib) + A_n).$$

Now $(a_1, b_1) \in B_N$ implies

$$\begin{aligned}\{a_1, a_1+b_1, a_1+2b_1, \ldots, a_1+lb_1\} &\subseteq A_N \\ &\subseteq \bigcap_{i=0}^{l}(-(a+ib)+A_n).\end{aligned}$$

This implies that $(a_1+a)+i\cdot(b_1+b) \in A_n$ for all $i \in \{0,1,2,\ldots,l\}$. Hence, $(a_1,b_1) \in -(a,b)+B_n$, which implies $B_N \subseteq -(a,b)+B_n$.

Therefore, for any $(a,b) \in B_n$, there exists $N \in \mathbb{N}$ such that $B_N \subseteq -(a,b)+B_n$, and it follows from Theorem 1.6 that B is quasi-central in $(\mathbb{N}\times\mathbb{N},+)$. □

Now we will prove the following result, which is analogous to Theorem 2.1 and also gives an abundance of J-sets.

Theorem 2.2. *Let $A \subseteq \mathbb{N}$ be a J-set in $\mathbb{N}$ and $l \in \mathbb{N}$. Then the collection $\{(a,b) : \{a, a+b, a+2b, \ldots, a+lb\} \subset A\}$ is a J-set in $(\mathbb{N}\times\mathbb{N},+)$.*

Proof. Let

$$C = \{(a,d) : \{a, a+d, ..., a+ld\} \subseteq A\}.$$

Our goal is to show C is a J-set in $(\mathbb{N}\times\mathbb{N},+)$. Any $f \in \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is of the form $f=(f_1,f_2)$, where $f_1, f_2 : \mathbb{N} \longrightarrow \mathbb{N}$. Let us choose $F \in P_f\left((\mathbb{N}\times\mathbb{N})^{\mathbb{N}}\right)$. Then we have to show that there exist $(a_1,a_2) \in \mathbb{N}\times\mathbb{N}$ and $H_1 \in P_f(\mathbb{N})$, such that for all $f \in F$, $(a_1,a_2)+\sum_{t\in H_1} f(t) \in C$. Let us assume that $F=\{f_1, f_2, \ldots, f_m\}$ for some $m \in \mathbb{N}$, where $f_i = (g_{2i-1}, g_{2i})$, for $i = 1,2,\ldots,m$. Choose any $b \in \mathbb{N}$, and consider the set $G \in P_f\left(\mathbb{N}^{\mathbb{N}}\right)$, where

$$G = \{g_{2i-1} + j\,(b+g_{2i}) : i = 1,\ldots,m \,\text{and}\, j = 0,\ldots,l\}.$$

As A is a J-set, we have $a \in \mathbb{N}$, $H \in P_f(\mathbb{N})$, such that $a+\sum_{t\in H} h(t) \in A$, for all $h \in G$, i.e.,

$$a + \sum_{t\in H}\left(g_{2i-1}+j\,(b+g_{2i})\right)(t) \in A,$$

where $i = 1,2,\ldots,m$ and $j = 0,1,....,l$.

Hence $a+\sum_{t\in H} g_{2i-1}(t)+j\cdot\left(b\,|H|+\sum_{t\in H} g_{2i}(t)\right) \in A$, i.e., $\left(a+\sum_{t\in H} g_{2i-1}(t),\, b\,|H|+\sum_{t\in H} g_{2i}(t)\right) \in C$,(from definition of C) i.e.,$(a, b\,|H|)+\sum_{t\in H}(g_{2i-1}, g_{2i})(t) \in C$, i.e., $(a, b\,|H|)+\sum_{t\in H} f_i(t) \in C$, where $i = 1,2,\ldots,m$. So, for any F in $P_f\left((\mathbb{N}\times\mathbb{N})^{\mathbb{N}}\right)$, there exist $(a, b\,|H|) \in \mathbb{N}\times\mathbb{N}$ and $H \in P_f(\mathbb{N})$, such that $(a, b\,|H|)+\sum_{t\in H} f_i(t) \in C$, $f_i \in F$, $i = 1,2,3,\ldots,m$. Hence C is a J set in $(\mathbb{N}\times\mathbb{N},+)$ and this proves the theorem. □

The following theorem is an analogous version of Theorem 2.1.

Theorem 2.3. *Let $A \subseteq \mathbb{N}$ be a C-set in $\mathbb{N}$ and $l \in \mathbb{N}$. Then the collection $\{(a, b) : \{a, a + b, a + 2b, \ldots, a + lb\} \subset A\}$ is a C-set in* $(\mathbb{N} \times \mathbb{N}, +)$.

Proof. The proof of this Theorem is the same as the proof of Theorem 2.1, except that one has to use Theorem 2.2 instead of Theorem 2.1. So we leave it to the reader. $\square$

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