#A106 INTEGERS 21 (2021)

ON (YET ANOTHER) FAMILY OF STIRLING POLYNOMIALS OF THE SECOND KIND

Helena Cobo
Palma de Mallorca, Spain
helenacobo@gmail.com

Received: 2/18/21, Revised: 7/6/21, Accepted: 10/20/21, Published: 11/8/21

Abstract
We define a family of multivariate polynomials generalizing the Stirling numbers of the second kind and some of their multiple generalizations existent in the literature. We use simple techniques like generating functions or polynomial sequence transforms to deduce recurrences for these polynomials.

1. Introduction

The Stirling numbers are very well-known combinatorial numbers present in many branches of mathematics. They appear naturally in combinatorics, number theory and probability, but also in fields such as algebraic topology or non-commutative algebra.

There are many ways to define them (see for instance [14] and [17]) and there are (too) many different notations in the literature (a historical compilation can be found in [21]). We choose the generating functions to define these combinatorial numbers, and will use one of the most standard notations (though maybe not the best one according to [19]).

There are two types of Stirling numbers: those of the first kind, denoted by $s(\ell, k)$, can be defined by

$$\frac{\big(\ln(1+x)\big)^k}{k!} = \sum_{\ell=k}^{\infty} s(\ell, k)\frac{x^\ell}{\ell!}, \tag{1}$$

and the Stirling numbers of the second kind, denoted by $S(\ell, k)$, defined by

$$\frac{(e^x-1)^k}{k!} = \sum_{\ell=k}^{\infty} S(\ell, k)\frac{x^\ell}{\ell!}. \tag{2}$$

The formulas above may suggest that the Stirling numbers of the first and the second kind are somehow inverse to each other. This is indeed the case, in the following sense:

$$\sum_{k=m}^{d} s(k,m)S(d,k) = \sum_{k=m}^{d} s(d,k)S(k,m) = \delta_{md}, \tag{3}$$

where δ_{md} denotes the Kronecker delta.

The Stirling numbers can be interpreted combinatorially as follows. The number $(-1)^{\ell-k}s(\ell,k)$ is the number of permutations of ℓ symbols which have exactly k cycles, while the Stirling number of the second kind $S(\ell,k)$ counts the number of ways to partition a set of ℓ elements into k nonempty subsets.

There are many generalizations of these numbers (some of them are unified in [18]), not only to generalized Stirling numbers but also to polynomials. See [15], [24], [20] or [23] for some Stirling polynomials seemingly unrelated with the object of study of this paper.

The usefulness of the Stirling polynomials studied in this paper resides in their appearance in the Weyl algebra. More precisely, when computing an invariant attached to holonomic ideals in the Weyl algebra, the so-called b-function with respect to weights, we are led to solve certain linear systems of equations defined in terms of the Stirling polynomials. In fact, the two-dimensional version of the Stirling polynomials of the second kind has already appeared in [11] in connection with the study of certain polynomial systems appearing in the computation of b-functions with respect to weights.

2. The Stirling Polynomials of the Second Kind

In this section we define a family of Stirling polynomials which can be seen as generalizations of the weighted Stirling numbers of the second kind. There are many different generalizations of the Stirling numbers in the literature (see [18] and the references therein). One way to generalize the Stirling numbers is to refine their combinatorial definition, like the so-called r-Stirling numbers (see [5]). There are also generalizations of the following recurrences:

$$\begin{aligned} s(n+1,k) &= s(n,k-1) - ns(n,k), \\ S(n+1,k) &= S(n,k-1) + kS(n,k), \end{aligned} \tag{4}$$

that the classical Stirling numbers satisfy. This is the case of the q-Stirling numbers (see [8]).

We have first met the Stirling polynomials object of study of this note, not in a combinatorial context, but in the Weyl algebra. Hence we will define them in this context and proceed later to give a combinatorial definition in terms of generating functions.

We denote by D_n the ring $\mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ subject to the relations

$$\partial_i x_j = x_j \partial_i + \delta_{ij},\ x_i x_j = x_j x_i,\ \partial_i \partial_j = \partial_j \partial_i,$$

for $1 \le i, j \le n$. It is called the n-th Weyl algebra over the field $\mathbb{C}$.

As in any non-commutative ring, we are interested in writing an operator $P \in D_n$ in its so-called *normally ordered expression*:

$$P = \sum_{\alpha, \beta} c_{\alpha\beta} \mathbf{x}^{\alpha} \partial^{\beta}.$$

And here the Stirling numbers naturally appear. In the first Weyl algebra $D_1 \subseteq \mathbb{C}[x, \partial_x]$, the normally ordered expression of $(x\partial_x)^{\ell}$ is given by

$$\big(x\partial_x\big)^{\ell} = \sum_{k=0}^{\ell} S(\ell, k) x^k \partial_x^k, \tag{5}$$

which can be seen as yet another possible definition of the Stirling numbers of the second kind. The normally ordered expression of other operators in D_1 give rise to generalizations of the Stirling numbers. For instance, (5) was generalized in [6] as

$$\big(x^r \partial_x^s\big)^n = x^{n(r-s)} \sum_{k=1}^{n} S_{r,s}(n, k) x^k \partial_x^k,$$

where $r \ge s$ and the $S_{r,s}(n,k)$ are one of the many *generalized Stirling numbers of the second kind* existent in the literature. In fact, the expansion of $\big(x^r \partial_x^s\big)^n$ was studied much earlier (see [7]).

It is interesting to note that in [9] the authors study another expansions, non-normally ordered, of operators like $(x + x\partial_x)^n$ and $(ax + x(1+x)\partial_x)^n$ in terms of the polynomials

$$S_n(x) = \sum_{k=0}^{n} S(n, k) x^k,$$

which are mostly known in the literature as the *Bell polynomials*.

In this paper we deal with a very direct generalization of (5). More precisely with an analogous formula in the n-th Weyl algebra D_n, the normally ordered expression of a power of an Euler operator.

Definition 1. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{C}^{n+1}$ and $\ell \in \mathbb{Z}_{>0}$. The *Stirling polynomials of the second kind* $S_{\mathbf{k}}^{(\ell)}(x_0, x_1, \ldots, x_n)$ (or simply $S_{\mathbf{k}}^{(\ell)}$) are defined as

$$\big(\alpha_0 + \alpha_1 x_1 \partial_1 + \cdots + \alpha_n x_n \partial_n\big)^{\ell} = \sum_{\mathbf{k} \in \mathbb{Z}^n_{\ge 0},\ |\mathbf{k}| \le \ell} S_{\mathbf{k}}^{(\ell)}(\alpha) x_1^{k_1} \cdots x_n^{k_n} \partial_1^{k_1} \cdots \partial_n^{k_n}.$$

Notation. For any $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$ we use the standard notations

$$|\mathbf{k}| = k_1 + \cdots + k_n,$$

$$\mathbf{k}! = k_1! \cdots k_n!.$$

Given two vectors $\mathbf{k}, \mathbf{i} \in \mathbb{Z}^n_{\geq 0}$, by $\mathbf{i} \leq \mathbf{k}$ we mean

$$0 \leq i_j \leq k_j \text{ for } 1 \leq j \leq n,$$

and by $\mathbf{i} < \mathbf{k}$ we mean $\mathbf{i} \leq \mathbf{k}$ and $\mathbf{i} \neq \mathbf{k}$. We set

$$\binom{\mathbf{k}}{\mathbf{i}} = \binom{k_1}{i_1} \cdots \binom{k_n}{i_n}$$

and

$$\delta_{\mathbf{ki}} = \delta_{k_1 i_1} \cdots \delta_{k_n i_n} = \begin{cases} 1 & \text{if } \mathbf{k} = \mathbf{i} \\ 0 & \text{otherwise} \end{cases}$$

Moreover, for a positive integer ℓ,

$$\binom{\ell}{\mathbf{i}} = \binom{\ell}{i_1}\binom{\ell - i_1}{i_2} \cdots \binom{\ell - i_1 - \cdots - i_{n-1}}{i_n}.$$

We denote $\mathbf{x} = (x_0, \dots, x_n)$ and hence $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_0, \dots, x_n]$. The set $\{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ denotes the standard bases of $\mathbb{R}^n$.

Let us see how these polynomials can alternatively be defined in terms of generating functions. The formula (2) can be written as

$$S(\ell, k) = \left\langle \tfrac{t^\ell}{\ell!} \right\rangle \tfrac{(e^t - 1)^k}{k!}, \tag{6}$$

where $\left\langle \frac{t^\ell}{\ell!} \right\rangle f(t)$ denotes the coefficient of $\frac{t^\ell}{\ell!}$ in the expansion of $f(t)$. One of the first generalizations of the Stirling numbers of the second kind was proposed by Carlitz in [10], the so-called weighted Stirling numbers of the second kind

$$S(\ell, k, \lambda) = \left\langle \tfrac{t^\ell}{\ell!} \right\rangle e^{\lambda t} \tfrac{(e^t - 1)^k}{k!}.$$

Notice how subtle the difference between the generalized Stirling numbers and the Stirling polynomials is: if we do not consider λ as a parameter, but as a variable, we could equally talk of the weighted Stirling polynomials

$$S(\ell, k)(x) = \left\langle \frac{t^\ell}{\ell!} \right\rangle e^{xt} \frac{(e^t - 1)^k}{k!}.$$

Generalizing to any number of variables we end up with the polynomials object of study in this paper.

Lemma 1. *Given a point* $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$ *and a positive integer* $\ell \in \mathbb{Z}_{>0}$, *the Stirling polynomial of the second kind,* $S^{(\ell)}_{\mathbf{k}}(\mathbf{x})$, *is*

$$S^{(\ell)}_{\mathbf{k}}(\mathbf{x}) = \left\langle \frac{t^\ell}{\ell!} \right\rangle \frac{1}{\mathbf{k}!} e^{x_0 t} \prod_{j=1}^{n} (e^{x_j t} - 1)^{k_j}.$$

Proof. By the multinomial theorem we have

$$\big(\alpha_0 + \alpha_1 x_1 \partial_1 + \cdots + \alpha_n x_n \partial_n\big)^\ell = \sum_{i_0 + \cdots + i_n = \ell} \binom{\ell}{i_0, \dots, i_n} \alpha_0^{i_0} \cdots \alpha_n^{i_n} (x_1 \partial_1)^{i_1} \cdots (x_n \partial_n)^{i_n},$$

where $\binom{\ell}{i_0,\dots,i_n} = \frac{\ell!}{i_0! \cdots i_n!}$, or in other words,

$$\big(\alpha_0 + \alpha_1 x_1 \partial_1 + \cdots + \alpha_n x_n \partial_n\big)^\ell = \sum_{\mathbf{i} \in \mathbb{Z}^n_{\geq 0},\ |\mathbf{i}| \leq \ell} \binom{\ell}{\mathbf{i}} \alpha_0^{\ell - |\mathbf{i}|} \alpha_1^{i_1} \cdots \alpha_n^{i_n} (x_1 \partial_1)^{i_1} \cdots (x_n \partial_n)^{i_n}.$$

The result follows by using the identities in (5) and (2). □

In other words, given $\mathbf{k}$, the polynomial sequence $\{S^{(\ell)}_{\mathbf{k}}\}_\ell$ is defined by means of the following generating function:

$$\frac{1}{\mathbf{k}!} e^{x_0 t} \prod_{j=1}^{n} \left(e^{x_j t} - 1\right)^{k_j} = \sum_{\ell=0}^{\infty} S^{(\ell)}_{\mathbf{k}}(\mathbf{x}) \frac{t^\ell}{\ell!}.$$

Remark 1. When $\mathbf{k} = k\mathbf{e_i}$ with $1 \leq i \leq n$ we have

$$S^{(\ell)}_{k\mathbf{e_i}} = \sum_{j=k}^{\ell} \binom{\ell}{j} S(j,k) x_0^{\ell - j} x_i^j.$$

The polynomials $S^{(\ell)}_{k\mathbf{e_i}}$ specialize to well-known generalizations of the Stirling numbers.

(i) First, it is clear that under the specialization $x_0 = 0$ and $x_i = 1$ we obtain

$$S^{(\ell)}_{k\mathbf{e_i}}(x_0 = 0, x_i = 1) = S(\ell, k),$$

the classical Stirling numbers.

(ii) As we have explained above, specializing $x_i = 1$ and considering $x_0 = \lambda$ as a parameter, leads to the weighted Stirling numbers of the second kind defined by Carlitz in [10]. When λ is an integer, the weighted Stirling numbers form a very interesting sequence. See for instance [26], where the author studies the period of the sequence modulo a prime p.

(iii) These polynomials also specialize to the r-Stirling numbers defined in [5] (denoted by $S_r(\ell,k)$), which are equivalent to the weighted Stirling numbers of the second kind. Indeed, as it is proved in Theorem 16 [5], the r-Stirling numbers can be defined by the following generating function

$$\frac{1}{k!}e^{rx}(e^x-1)^k=\sum_{n=k}^{\infty}S_r(n+k,k+r)\frac{x^n}{n!}.$$

(iv) Moreover, these polynomials specialize to the generalized Stirling numbers introduced in [22] and defined as follows

$$S^{(\alpha)}(n,k,r)=\frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\alpha+rj)^n.$$

Indeed, by Corollary 2 we deduce that

$$S^{(\ell)}_{k\mathbf{e_i}}=S^{(x_0)}(\ell,k,x_i).$$

The interest of these polynomials comes from its very definition, since they appear when computing b-functions with respect to weights of holonomic ideals in the Weyl algebra (see [25]). In [11] it already appeared the two-dimensional version of the Stirling polynomials, besides the study of a linear system defined by the Stirling polynomials, that has to be solved to compute the b-function. The results in [11] have been applied in [12] and [13] to compute the b-function of certain families of hypergeometric ideals.

In order to generalize in a forthcoming paper the results in [11] concerning the linear systems of equations defined by the polynomials $S^{(\ell)}_{\mathbf{k}}$, we study in the next section the Stirling polynomials.

3. Properties of the Stirling Polynomials

First we will derive a closed formula for the Stirling polynomials. By Lemma 1 we deduce that

$$\begin{aligned} S^{(\ell)}_{\mathbf{k}}(\mathbf{x}) &= \tfrac{\ell!}{\mathbf{k}!}\left\langle t^{\ell}\right\rangle e^{x_0t}(e^{x_1t}-1)^{k_1}\cdots(e^{x_nt}-1)^{k_n}\\ &= \tfrac{\ell!}{\mathbf{k}!}\textstyle\sum_{\mathbf{i}\in\mathbb{Z}^n_{\geq 0},\ |\mathbf{i}|\leq\ell}\Big(\left\langle t^{\ell-|\mathbf{i}|}\right\rangle e^{x_0t}\Big)\prod_{j=1}^{n}\Big(\left\langle t^{i_j}\right\rangle(e^{x_jt}-1)^{k_j}\Big). \end{aligned}$$

By the identity (6) we have $\left\langle t^{\ell}\right\rangle(e^{xt}-1)^k=\frac{k!}{\ell!}x^{\ell}S(\ell,k)$, and since $\left\langle t^{\ell-|\mathbf{i}|}\right\rangle e^{x_0t}=\frac{x_0^{\ell-|\mathbf{i}|}}{(\ell-|\mathbf{i}|)!}$, we straightforwardly deduce

$$S^{(\ell)}_{\mathbf{k}}(\mathbf{x})=\sum_{\mathbf{i}\geq\mathbf{k},\ |\mathbf{i}|\leq\ell}\binom{\ell}{\mathbf{i}}\Big(\prod_{j=1}^{n}S(i_j,k_j)\Big)x_0^{\ell-|\mathbf{i}|}x_1^{i_1}\cdots x_n^{i_n}\in\mathbb{Z}[x_0,\ldots,x_n]. \tag{7}$$

Hence it is clear that for any $\mathbf{k} \in \mathbb{Z}_{\geq 0}^n$ and $\ell \in \mathbb{Z}_{\geq 0}$, the Stirling polynomial $S_{\mathbf{k}}^{(\ell)}$ is a homogeneous polynomial of degree ℓ in $n+1$ variables. Moreover we have that

$$S_{\mathbf{k}}^{(\ell)}(\mathbf{x}) = 0 \text{ whenever } \ell < |\mathbf{k}|,$$

$$S_{\mathbf{k}}^{(|\mathbf{k}|)}(\mathbf{x}) = \binom{|\mathbf{k}|}{\mathbf{k}} x_1^{k_1} \cdots x_n^{k_n}.$$

And the monomial $x_1^{k_1} \cdots x_n^{k_n}$ divides $S_{\mathbf{k}}^{(\ell)}(\mathbf{x})$ for any $\mathbf{k} \in \mathbb{Z}_{\geq 0}^n$ such that $\ell > |\mathbf{k}|$.

Example 1. We give a few examples of these Stirling polynomials.

- If $\mathbf{k} = (0, \ldots, 0)$ then for any $\ell \in \mathbb{Z}_{\geq 0}$ we have

$$S_{\mathbf{0}}^{(\ell)} = x_0^{\ell}.$$

- If $\ell = 1$ and $\mathbf{k} = \mathbf{e_i}$ with $1 \leq i \leq n$, we have

$$S_{\mathbf{e_i}}^{(1)} = x_i.$$

- If $\mathbf{k} = (1, \ldots, 1)$ we have

$$S_{\mathbf{k}}^{(n)} = n! x_1 \cdots x_n,$$

$$S_{\mathbf{k}}^{(n+1)} = (n+1)! \Big(x_0 x_1 \cdots x_n + \frac{1}{2} \sum_{j=1}^{n} x_1 \cdots x_{j-1} x_j^2 x_{j+1} \cdots x_n \Big).$$

- If $n = 1$ the polynomials are of the form $S_k^{(\ell)} = \sum_{i=k}^{\ell} \binom{\ell}{i} S(i,k) x_0^{\ell-i} x_1^i$. In particular, when $k = 1$, it can be written as

$$S_1^{(\ell)} = (x_0 + x_1)^{\ell} - x_0^{\ell}.$$

This property will be generalized in Corollary 2 to $n > 1$ and any $\mathbf{k}$.

- Some polynomials of the form $S_{(3,0)}^{(\ell)}$ are

$$S_{(3,0)}^{(3)} = x_1^3,$$

$$S_{(3,0)}^{(4)} = 4x_0 x_1^3 + 6x_1^4,$$

$$S_{(3,0)}^{(5)} = 10x_0^2 x_1^3 + 30x_0 x_1^4 + 25x_1^5.$$

- Some polynomials of the form $S_{(1,2)}^{(\ell)}$ are:

$$S_{(1,2)}^{(3)} = 3x_1 x_2^2,$$

$$S_{(1,2)}^{(4)} = 12x_0 x_1 x_2^2 + 12x_1 x_2^3 + 6x_1^2 x_2^2,$$

$$S_{(1,2)}^{(5)} = 30x_0^2 x_1 x_2^2 + 60x_0 x_1 x_2^3 + 35x_1 x_2^4 + 30x_0 x_1^2 x_2^2 + 10x_1^3 x_2^2 + 30x_1^2 x_2^3.$$

- Some polynomials of the family $S^{(\ell)}_{(1,2,3)}$:

$$S^{(6)}_{(1,2,3)} = 60x_1x_2^2x_3^3,$$

$$S^{(7)}_{(1,2,3)} = 420x_0x_1x_2^2x_3^3 + 630x_1x_2^2x_3^4 + 420x_1x_2^3x_3^3 + 210x_1^2x_2^2x_3^3.$$

3.1. Looking for Recurrences

In combinatorics, the method of sequence transforms has been proven very useful to deduce combinatorial and polynomial identities (see [16] and [2]). Among the more popular are the binomial transform and the Stirling transform (see [4]). We proceed to define them.

(i) The *binomial transform*. Given an integer sequence $\{a_n\}_n$, its binomial transform is another sequence $\{b_n\}_n$ defined by $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. We have that

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad \text{if and only if} \quad a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k.$$

(ii) The *Stirling transform*. Given an integer sequence $\{f_n\}_n$ its *Stirling transform of the second kind* is the sequence $\{g_n\}_n$ defined by

$$g_n = \sum_{k=0}^{n} S(n,k) f_k.$$

It is clear, by the orthogonality property in (3), that this is equivalent to

$$f_n = \sum_{k=0}^{n} s(n,k) g_k.$$

Combinatorial properties of an integer sequence $\{a_n\}$ may transform into combinatorial properties of the corresponding sequence transform (see [2]).

Remark 2. Exactly with the same definitions as above we can consider transforms of polynomial sequences. A simple example is given by one of the many ways to define the Stirling numbers. Indeed, we have the following (horizontal) generating series

$$x(x-1)\cdots(x-n+1) = \textstyle\sum_{k=0}^{n} s(n,k)x^k,$$

$$x^n = \textstyle\sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1).$$

Hence the polynomial sequence $\{x^n\}$ is the Stirling transform of the sequence of falling factorials $\{x(x-1)\cdots(x-n+1)\}$.

Moreover, by definition of the Bell polynomials

$$B_n(x) = \sum_{k=0}^{n} S(n,k)x^k,$$

the Stirling transform of the sequence $\{x^n\}$ is the sequence of Bell polynomials $\{B_n(x)\}$.

Not only polynomials, but we can even consider operators. For instance, the identity (5) implies that the sequence of operators $\{(x\partial)^n\}_n$ can be seen as the Stirling transform of the sequence $\{x^n\partial^n\}_n$. Hence we recover the *dual* of Equation (5), as

$$x^n\partial^n = \sum_{k=1}^{n} s(n,k)(x\partial)^k.$$

There are generalizations of these transforms in the literature. See for instance [1], where the author defines the Stirling transforms as above but in terms of generalized Stirling numbers. Here we are going to use multidimensional versions of both the binomial and the Stirling transforms to derive interesting identities related with the Stirling polynomials of the second kind.

If we consider sequences $\{a_{\mathbf{k}}\}_{\mathbf{k}}$ (integral or polynomial) depending on a parameter $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$, the obvious generalization of the binomial transform is the following.

Definition 2. The *binomial transform* of a sequence $\{a_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^n_{\geq 0}}$ is the sequence

$$b_{\mathbf{k}} = \sum_{\mathbf{i}\leq\mathbf{k}} \binom{\mathbf{k}}{\mathbf{i}} a_{\mathbf{i}}.$$

Lemma 2. *Given two sequences $\{a_{\mathbf{k}}\}$ and $\{b_{\mathbf{k}}\}$ we have*

$$b_{\mathbf{k}} = \sum_{\mathbf{i}\leq\mathbf{k}} \binom{\mathbf{k}}{\mathbf{i}} a_{\mathbf{i}} \quad \textit{if and only if} \quad a_{\mathbf{k}} = \sum_{\mathbf{i}\leq\mathbf{k}} \binom{\mathbf{k}}{\mathbf{i}} (-1)^{|\mathbf{k}|-|\mathbf{i}|} b_{\mathbf{i}}.$$

Proof. The result follows since

$$\sum_{\mathbf{j}\leq\mathbf{i}\leq\mathbf{k}} \binom{\mathbf{k}}{\mathbf{i}}\binom{\mathbf{i}}{\mathbf{j}} (-1)^{|\mathbf{i}|-|\mathbf{j}|} = \delta_{\mathbf{jk}}.$$

□

The obvious generalization of the Stirling transform of a sequence $\{f_{\mathbf{k}}\}$ depending on a vector parameter would be

$$g_{\mathbf{k}} = \sum_{\mathbf{i}\leq\mathbf{k}} \big(\prod_{r=1}^{n} S(k_r, i_r)\big) f_{\mathbf{i}}.$$

But we are interested in presenting our Stirling polynomials as a Stirling transform of certain polynomial sequence, and these polynomials depend on a parameter vector $\mathbf{k}$ and a positive integer ℓ. Hence we consider here an n-dimensional variant of the Stirling transform as follows.

Definition 3. Given a polynomial sequence $\{f_{\mathbf{k},\ell}\}$ with $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$ and $\ell \in \mathbb{Z}_{\geq 0}$, we define the *modified Stirling transform* as

$$g_{\mathbf{k},\ell} = \sum_{\mathbf{i}\geq\mathbf{k},\ |\mathbf{i}|\leq\ell} \Big(\prod_{r=1}^{n} S(i_r, k_r)\Big) f_{\mathbf{i},\ell}.$$

Remark 3. The Stirling polynomials $S^{(\ell)}_{\mathbf{k}}$ are the modified Stirling transforms of the polynomials of the sequence $\{f_{\mathbf{k},\ell}\}$ defined as

$$f_{\mathbf{k},\ell} = \binom{\ell}{\mathbf{k}} x_0^{\ell-|\mathbf{k}|} x_1^{k_1} \cdots x_n^{k_n}.$$

Next we prove that this definition makes sense, since the transform is easily inverted.

Lemma 3. *A sequence $\{g_{\mathbf{k},\ell}\}$ is the modified Stirling transform of the sequence $\{f_{\mathbf{k},\ell}\}$ if and only if*

$$f_{\mathbf{k},\ell} = \sum_{\mathbf{i}\geq\mathbf{k},\ |\mathbf{i}|\leq\ell} \Big(\prod_{j=1}^{n} s(i_j, k_j)\Big) g_{\mathbf{i},\ell}.$$

Proof. If $\{g_{\mathbf{k},\ell}\}$ is the modified Stirling transform of the sequence $\{f_{\mathbf{k},\ell}\}$, then

$$\begin{aligned}
\sum_{\mathbf{i}\geq\mathbf{k},\ |\mathbf{i}|\leq\ell} \Big(\prod_{r=1}^{n} s(i_r, k_r)\Big) g_{\mathbf{i},\ell} &= \sum_{\mathbf{i}\geq\mathbf{k},\ |\mathbf{i}|\leq\ell} \Big(\prod_{r} s(i_r, k_r)\Big) \sum_{\mathbf{j}\geq\mathbf{i},\ |\mathbf{j}|\leq\ell} \Big(\prod_{r} S(j_r, i_r)\Big) f_{\mathbf{j},\ell} \\
&= \sum_{\mathbf{j}\geq\mathbf{i}\geq\mathbf{k},\ |\mathbf{j}|\leq\ell} \Big(\prod_{r} s(i_r, k_r) S(j_r, i_r)\Big) f_{\mathbf{j},\ell} \\
&= \sum_{\mathbf{j}\geq\mathbf{k},\ |\mathbf{j}|\leq\ell} \Big(\sum_{\mathbf{j}\geq\mathbf{i}\geq\mathbf{k}} \prod_{r} s(i_r, k_r) S(j_r, i_r)\Big) f_{\mathbf{j},\ell}.
\end{aligned}$$

And the result follows since, by (3),

$$\begin{aligned}
\sum_{\mathbf{j}\geq\mathbf{i}\geq\mathbf{k}} \prod_{r} s(i_r, k_r) S(j_r, i_r) &= \sum_{i_1=j_1}^{k_1} s(i_1, k_1) S(j_1, i_1) \cdots \sum_{i_n=j_n}^{k_n} s(i_n, k_n) S(j_n, i_n) \\
&= \delta_{j_1,k_1} \cdots \delta_{j_n,k_n} \\
&= \delta_{\mathbf{j},\mathbf{k}}.
\end{aligned}$$

□

As a consequence we obtain a first relation for the Stirling polynomials of the second kind.

Corollary 1. *Given* $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$ *and a positive integer* $\ell \geq |\mathbf{k}|$,

$$\sum_{\mathbf{i}\geq\mathbf{k},\ |\mathbf{i}|\leq\ell} \Big(\prod_{r=1}^{n} s(i_r,k_r)\Big) S_{\mathbf{i}}^{(\ell)} = \binom{\ell}{\mathbf{k}} x_0^{\ell-|\mathbf{k}|} x_1^{k_1}\cdots x_n^{k_n}.$$

Definition 4. For any $\mathbf{k} \in \mathbb{Z}^n$ we define the linear form

$$A_{\mathbf{k}} = x_0 + k_1x_1 + \cdots + k_nx_n \in \mathbb{Z}[x_0,\ldots,x_n].$$

Next result is a rephrasing of Lemma 3.9 in [11], generalized to any number of variables.

Proposition 1. *For any* $\ell \in \mathbb{Z}_{\geq 0}$, *the polynomial sequence* $\{A_{\mathbf{k}}^{\ell}\}_{\mathbf{k}}$ *is the binomial transform of the sequence* $\{\mathbf{k}!S_{\mathbf{k}}^{(\ell)}\}_{\mathbf{k}}$.

Proof. We have to prove that for all $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$ and $\ell \geq 0$ we have

$$A_{\mathbf{k}}^{\ell} = \sum_{\mathbf{i}\leq\mathbf{k}} \binom{\mathbf{k}}{\mathbf{i}} \mathbf{i}! S_{\mathbf{i}}^{(\ell)}. \tag{8}$$

Recall that by $\mathbf{i} \leq \mathbf{k}$ we mean $i_j \leq k_j$ for $1 \leq j \leq n$ and by $\mathbf{i} < \mathbf{k}$ we mean $\mathbf{i} \leq \mathbf{k}$ and $\mathbf{i} \neq \mathbf{k}$. By Lemma 3

$$\sum_{\mathbf{i}\geq\mathbf{j},\ |\mathbf{i}|\leq\ell} \Big(\prod_{r} s(i_r,j_r)\Big) S_{\mathbf{i}}^{(\ell)} = \binom{\ell}{\mathbf{j}} x_0^{\ell-|\mathbf{j}|} x_1^{j_1}\cdots x_n^{j_n}.$$

Then, for $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$ we have

$$k_1^{j_1}\cdots k_n^{j_n} \sum_{\mathbf{i}\geq\mathbf{j},\ |\mathbf{i}|\leq\ell} \Big(\prod_{r} s(i_r,j_r)\Big) S_{\mathbf{i}}^{(\ell)} = \binom{\ell}{\mathbf{j}} x_0^{\ell-|\mathbf{j}|} (k_1x_1)^{j_1}\cdots (k_nx_n)^{j_n}.$$

Summing this equality in $\mathbf{j}$ such that $\mathbf{j} \in \mathbb{Z}^n_{\geq 0}$ with $|\mathbf{j}| \leq \ell$, we have on the right-hand side $A_{\mathbf{k}}^{\ell}$. The left-hand side gives

$$\sum_{\mathbf{j}\in\mathbb{Z}^n_{\geq 0},\ |\mathbf{j}|\leq\ell}\ \sum_{\mathbf{i}\geq\mathbf{j},\ |\mathbf{i}|\leq\ell} k_1^{j_1}\cdots k_n^{j_n} \prod_{r} s(i_r,j_r) S_{\mathbf{i}}^{(\ell)} =$$

$$= \sum_{\mathbf{i}\in\mathbb{Z}^{\mathbf{n}}_{\geq\mathbf{0}},\ |\mathbf{i}|\leq\ell} \Big(\sum_{\mathbf{j}\leq\mathbf{i}} k_1^{j_1}\cdots k_n^{j_n} \prod_{r} s(i_r,j_r)\Big) S_{\mathbf{i}}^{(\ell)}.$$

And the result follows, since

$$\begin{aligned}\sum_{\mathbf{j}\le\mathbf{i}} k_1^{j_1}\cdots k_n^{j_n}\prod_r s(i_r,j_r) &= \Big(\sum_{j_1=0}^{i_1} k_1^{j_1} s(i_1,j_1)\Big)\cdots\Big(\sum_{j_n=0}^{i_n} k_n^{j_n} s(i_n,j_n)\Big)\\ &= \tbinom{k_1}{i_1} i_1!\cdots\tbinom{k_n}{i_n} i_n!,\end{aligned}$$

because $\sum_{j=0}^{i} s(i,j)x^j = x(x-1)\cdots(x-i+1)$. □

Corollary 2. *For any* $\mathbf{k}\in\mathbb{Z}_{\ge0}^n$ *and any* $\ell\in\mathbb{Z}_{\ge0}$,

$$\mathbf{k}!S_{\mathbf{k}}^{(\ell)} = \sum_{\mathbf{i}\le\mathbf{k}} \binom{\mathbf{k}}{\mathbf{i}}(-1)^{|\mathbf{k}|-|\mathbf{i}|}A_{\mathbf{i}}^{\ell}.$$

Moreover we can deduce from Proposition 1 some recurrences among the polynomials $S_{\mathbf{k}}^{(\ell)}$.

Lemma 4. *For all* $\mathbf{k}\in\mathbb{Z}_{\ge0}^n$ *and* $\ell>0$ *we have*

$$S_{\mathbf{k}}^{(\ell+1)} = x_0 S_{\mathbf{k}}^{(\ell)} + \sum_{j=1}^{n}\sum_{c=0}^{\ell}\binom{\ell}{c}x_j^{\ell-c+1}S_{\mathbf{k}-\mathbf{e_j}}^{(c)}.$$

Proof. By Corollary 2

$$\begin{aligned}S_{\mathbf{k}}^{(\ell+1)} &= \tfrac{1}{\mathbf{k}!}\sum_{\mathbf{i}\le\mathbf{k}}\tbinom{\mathbf{k}}{\mathbf{i}}(-1)^{|\mathbf{k}|-|\mathbf{i}|}A_{\mathbf{i}}^{\ell+1}\\ &= \tfrac{1}{\mathbf{k}!}\sum_{\mathbf{i}\le\mathbf{k}}\tbinom{\mathbf{k}}{\mathbf{i}}(-1)^{|\mathbf{k}|-|\mathbf{i}|}\big(x_0+i_1x_1+\cdots+i_nx_n\big)A_{\mathbf{i}}^{\ell}\\ &= x_0S_{\mathbf{k}}^{(\ell)} + \tfrac{1}{\mathbf{k}!}\sum_{j=1}^{n}\sum_{\mathbf{i}\le\mathbf{k}} i_jx_j\tbinom{\mathbf{k}}{\mathbf{i}}(-1)^{|\mathbf{k}|-|\mathbf{i}|}A_{\mathbf{i}}^{\ell}.\end{aligned}$$

Notice that the sum $\sum_{\mathbf{i}\le\mathbf{k}} i_j\binom{\mathbf{k}}{\mathbf{i}}(-1)^{|\mathbf{k}|-|\mathbf{i}|}A_{\mathbf{i}}^{\ell}$ runs over $\mathbf{i}$ such that $i_j\neq0$ and we have

$$\sum_{\mathbf{i}\le\mathbf{k}} i_j\binom{\mathbf{k}}{\mathbf{i}}(-1)^{|\mathbf{k}|-|\mathbf{i}|}A_{\mathbf{i}}^{\ell} = k_j\sum_{\mathbf{i}\le\mathbf{k}}\binom{\mathbf{k}-\mathbf{e_j}}{\mathbf{i}-\mathbf{e_j}}(-1)^{|\mathbf{k}|-|\mathbf{i}|}A_{\mathbf{i}}^{\ell},$$

where $k_j\neq0$. We can write it as

$$\sum_{\mathbf{i}\le\mathbf{k}} i_j\binom{\mathbf{k}}{\mathbf{i}}(-1)^{|\mathbf{k}|-|\mathbf{i}|}A_{\mathbf{i}}^{\ell} = k_j\sum_{\mathbf{r}\le\mathbf{k}-\mathbf{e_j}}\binom{\mathbf{k}-\mathbf{e_j}}{\mathbf{r}}(-1)^{|\mathbf{k}-\mathbf{e_j}|-|\mathbf{r}|}A_{\mathbf{r}+\mathbf{e_j}}^{\ell},$$

and since $A_{\mathbf{r}+\mathbf{e_j}} = A_{\mathbf{r}}+x_j$ we deduce that

$$k_j\sum_{\mathbf{r}\le\mathbf{k}-\mathbf{e_j}}\binom{\mathbf{k}-\mathbf{e_j}}{\mathbf{r}}(-1)^{|\mathbf{k}-\mathbf{e_j}|-|\mathbf{r}|}A_{\mathbf{r}+\mathbf{e_j}}^{\ell} = k_j\sum_{c=0}^{\ell}\binom{\ell}{c}x_j^{\ell-c}\sum_{\mathbf{r}\le\mathbf{k}-\mathbf{e_j}}c_{\mathbf{r}}A_r^c,$$

where we denote $c_{\mathbf{r}} = \binom{\mathbf{k}-\mathbf{e_j}}{\mathbf{r}}(-1)^{|\mathbf{k}-\mathbf{e_j}|-|\mathbf{r}|}$. Again by the identity in Corollary 2 we deduce the recurrence we want to prove. $\square$

This recurrence does not specialize to (4), defining the Stirling numbers of the second kind, but to the following recurrence.

Corollary 3. *We recover recurrences for the classical Stirling numbers of the second kind*

(i) $S(\ell+1,k) = \sum_{c=0}^{\ell}\binom{\ell}{c}S(c,k-1)$,

(ii) $kS(\ell,k) = \sum_{c=k-1}^{\ell-1}\binom{\ell}{c}S(c,k-1)$.

Proof. The first identity follows directly from Lemma 4 under the specialization as explained in Remark 1 (i). Identity (ii) follows from (i) and the recurrence in (4) on classical Stirling numbers of the second kind. $\square$

Now we study another recurrence deduced from Proposition 1. By Corollary 2, it follows that under conditions of the type $A_{\mathbf{k}} = A_{\mathbf{k}'}$, there exist recurrences of the form

$$\sum_{\mathbf{i}\leq\mathbf{k}}\binom{\mathbf{k}}{\mathbf{i}}\mathbf{i}!S_{\mathbf{i}}^{(\ell)} = \sum_{\mathbf{i}\leq\mathbf{k}'}\binom{\mathbf{k}'}{\mathbf{i}}\mathbf{i}!S_{\mathbf{i}}^{(\ell)}.$$

But there are many cancelations on this recurrence. To work it out we introduce some notation. The condition $A_{\mathbf{k}} = A_{\mathbf{k}'}$ can be written as:

$$\alpha_1x_1 + \cdots + \alpha_nx_n = 0, \tag{9}$$

with $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{P}^{n-1}$. In fact, it is enough to consider $\alpha \in \mathbb{Z}^n$ such that $\alpha_1 + \cdots + \alpha_n \geq 0$. Then we can write it in a unique way as

$$\alpha = \alpha^+ - \alpha^-,$$

with $\alpha^+, \alpha^- \in \mathbb{Z}^n_{\geq 0}$ and disjoint support. Moreover we can assume that $\alpha^+ \neq \mathbf{0}$.

The next result unifies Lemma 4.5 and Lemma 4.12 in [11], generalized to any number of variables.

Proposition 2. *Let $\alpha \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ with $|\alpha| \geq 0$ and let $\mathbf{k} \in \mathbb{Z}^n_{\geq 0}$ such that $\mathbf{k} \geq \alpha^-$. If $\alpha_1x_1 + \cdots + \alpha_nx_n = 0$ we have the following identities for any $\ell \geq 0$:*

$$\sum_{\mathbf{i}\leq\alpha^+}\frac{\alpha^+!}{(\alpha^+-\mathbf{i})!}\binom{\mathbf{k}+\mathbf{i}}{\mathbf{k}}S^{(\ell)}_{\mathbf{k}+\mathbf{i}-\alpha^-} = \sum_{\mathbf{i}\leq\alpha^-}\frac{\alpha^-!}{(\alpha^--\mathbf{i})!}\binom{\mathbf{k}-\alpha^-+\mathbf{i}}{\mathbf{k}-\alpha^-}S^{(\ell)}_{\mathbf{k}+\mathbf{i}-\alpha^-}.$$

Proof. We abuse of notation by denoting $S_{\mathbf{i}}^{(\ell)}$ the polynomial under the condition $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$.

One way to prove the equality of the statement is by using the powerful tool of generating functions (see [27]). We compute the generating functions of the left and right-hand side and check that they are equal under the condition $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$. The left-hand side gives

$$\sum_{\ell \geq 0} \frac{w^\ell}{\ell!} \sum_{\mathbf{i} \leq \alpha^+} \frac{\alpha^+!}{(\alpha^+ - \mathbf{i})!} \binom{\mathbf{k}+\mathbf{i}}{\mathbf{k}} \left\langle \frac{t^\ell}{\ell!} \right\rangle \frac{1}{(\mathbf{k}+\mathbf{i}-\alpha^-)!} e^{x_0 t} \prod_{j=1}^{n} (e^{x_j t} - 1)^{k_j + i_j - \alpha_j^-}$$

$$= \sum_{\mathbf{i} \leq \alpha^+} \frac{\alpha^+!}{(\alpha^+ - \mathbf{i})!} \binom{\mathbf{k}+\mathbf{i}}{\mathbf{k}} \frac{1}{(\mathbf{k}+\mathbf{i}-\alpha^-)!} \sum_{\ell \geq 0} \frac{w^\ell}{\ell!} \left\langle \frac{t^\ell}{\ell!} \right\rangle e^{x_0 t} \prod_{j=1}^{n} (e^{x_j t} - 1)^{k_j + i_j - \alpha_j^-}$$

$$= \sum_{\mathbf{i} \leq \alpha^+} \frac{\alpha^+!}{(\alpha^+ - \mathbf{i})!} \binom{\mathbf{k}+\mathbf{i}}{\mathbf{k}} \frac{1}{(\mathbf{k}+\mathbf{i}-\alpha^-)!} e^{x_0 w} \prod_{j=1}^{n} (e^{x_j w} - 1)^{k_j + i_j - \alpha_j^-}$$

$$= e^{x_0 w} \prod_{j=1}^{n} (e^{x_j w} - 1)^{k_j - \alpha_j^-} \sum_{\mathbf{i} \leq \alpha^+} \binom{\alpha^+}{\mathbf{i}} \frac{(\mathbf{k}+\mathbf{i})!}{\mathbf{k}!(\mathbf{k}+\mathbf{i}-\alpha^-)!} \prod_{j=1}^{n} (e^{x_j w} - 1)^{i_j}$$

$$= \frac{e^{x_0 w} \prod_{j=1}^{n} (e^{x_j w} - 1)^{k_j - \alpha_j^-}}{\mathbf{k}!} \Big(\sum_{i_1=0}^{\alpha_1^+} \binom{\alpha_1^+}{i_1} \frac{(k_1+i_1)!}{(k_1+i_1-\alpha_1^-)!} (e^{x_1 w} - 1)^{i_1} \Big) \cdots$$

$$\cdots \Big(\sum_{i_n=0}^{\alpha_n^+} \binom{\alpha_n^+}{i_n} \frac{(k_n+i_n)!}{(k_n+i_n-\alpha_n^-)!} (e^{x_n w} - 1)^{i_n} \Big).$$

Notice that if $\alpha_j^+ = 0$ then

$$\sum_{i_j=0}^{\alpha_j^+} \binom{\alpha_j^+}{i_j} \frac{(k_j+i_j)!}{(k_j+i_j-\alpha_j^-)!} (e^{x_j w} - 1)^{i_j} = \frac{k_j!}{(k_j - \alpha_j^-)!} = \frac{k_j!}{(k_j - \alpha_j^-)!} e^{x_j w \alpha_j^+},$$

while, if $\alpha_j^+ > 0$ then $\alpha_j^- = 0$ and hence

$$\sum_{i_j=0}^{\alpha_j^+} \binom{\alpha_j^+}{i_j} \frac{(k_j+i_j)!}{(k_j+i_j-\alpha_j^-)!} (e^{x_j w} - 1)^{i_j} = e^{x_j w \alpha_j^+} = \frac{k_j!}{(k_j - \alpha_j^-)!} e^{x_j w \alpha_j^+}.$$

Therefore we conclude that the generating function of the left-hand side equals

$$\frac{1}{(\mathbf{k}-\alpha^-)!} e^{w(x_0 + x_1 \alpha_1^+ + \cdots + x_n \alpha_n^+)} \prod_{j=1}^{n} (e^{x_j w} - 1)^{k_j - \alpha_j^-}.$$

The other generating function is

$$\sum_{\ell\geq 0}\frac{w^\ell}{\ell!}\sum_{\mathbf{i}\leq\alpha^-}\frac{\alpha^-!}{(\alpha^--\mathbf{i})!}\binom{\mathbf{k}-\alpha^-+\mathbf{i}}{\mathbf{k}-\alpha^-}\left\langle\frac{t^\ell}{\ell!}\right\rangle\frac{1}{(\mathbf{k}+\mathbf{i}-\alpha^-)!}e^{x_0t}\prod_{j=1}^n(e^{x_jt}-1)^{k_j+i_j-\alpha_j^-}$$

$$=\sum_{\mathbf{i}\leq\alpha^-}\frac{\alpha^-!}{(\alpha^--\mathbf{i})!}\frac{1}{(\mathbf{k}-\alpha^-)!\mathbf{i}!}\sum_{\ell\geq 0}\frac{w^\ell}{\ell!}\left\langle\frac{t^\ell}{\ell!}\right\rangle e^{x_0t}\prod_{j=1}^n(e^{x_jt}-1)^{k_j+i_j-\alpha_j^-}$$

$$=\sum_{\mathbf{i}\leq\alpha^-}\binom{\alpha^-}{\mathbf{i}}\frac{1}{(\mathbf{k}-\alpha^-)!}e^{x_0w}\prod_{j=1}^n(e^{x_jw}-1)^{k_j+i_j-\alpha_j^-}$$

$$=e^{x_0w}\prod_{j=1}^n(e^{x_jw}-1)^{k_j-\alpha_j^-}\frac{1}{(\mathbf{k}-\alpha^-)!}\Big(\sum_{i_1=0}^{\alpha_1^-}\binom{\alpha_1^-}{i_1}(e^{x_1w}-1)^{i_1}\Big)\cdots$$
$$\cdots\Big(\sum_{i_n=0}^{\alpha_n^-}\binom{\alpha_n^-}{i_n}(e^{x_nw}-1)^{i_n}\Big)$$

$$=e^{x_0w}\prod_{j=1}^n(e^{x_jw}-1)^{k_j-\alpha_j^-}\frac{1}{(\mathbf{k}-\alpha^-)!}e^{w(x_1\alpha_1^-+\cdots+x_n\alpha_n^-)}.$$

We are done, since $\alpha_1x_1+\cdots+\alpha_nx_n=0$ is equivalent to

$$\alpha_1^+x_1+\cdots+\alpha_n^+x_n=\alpha_1^-x_1+\cdots+\alpha_n^-x_n.$$

□

4. Some Combinatorial Applications

The Stirling numbers have a vast literature and there is a lot of work done to state relations among these combinatorial numbers, of which [21] is a good example. In this section we highlight some relations on the Stirling numbers of the second kind that can be deduced from the results developed in the previous section.

A first consequence is a generalization of the well-known formula (see [3])

$$\sum_{k=0}^n\binom{n}{k}(-1)^k\big(xk+y\big)^m=(-1)^nn!\sum_{j=n}^m\binom{m}{j}x^jy^{m-j}S(j,n),\qquad(10)$$

by simply rewriting the equation in Corollary 2.

Corollary 4. *For any $\mathbf{k}\in\mathbb{Z}_{\geq 0}^n$ we have the polynomial identity:*

$$\sum_{\mathbf{i}\leq\mathbf{k}}\frac{(-1)^{|\mathbf{k}|-|\mathbf{i}|}}{\mathbf{i}!(\mathbf{k}-\mathbf{i})!}\big(x_0+i_1x_1+\cdots+i_nx_n\big)^\ell=\sum_{\mathbf{i}\geq\mathbf{k},\ |\mathbf{i}|\leq\ell}\binom{\ell}{\mathbf{i}}\Big(\prod_{j=1}^nS(i_j,k_j)\Big)x_0^{\ell-|\mathbf{i}|}x_1^{i_1}\cdots x_n^{i_n}.$$

Proposition 2 contains many combinatorial identities. Indeed, proper specializations of the Stirling polynomials together with a clever use of Proposition 2 (that is, a right choice of the parameter α), can be useful to obtain (hopefully interesting) relations among the classical Stirling numbers of the second kind.

The first example illustrates how we can recover a well-known generalized convolution formula.

Lemma 5. *Given* $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}$ *and* $\ell \in \mathbb{Z}_{\geq 0}$

$$\binom{k_1+k_2}{k_1} \cdots \binom{k_1+\cdots+k_n}{k_n} S(\ell, k_1+\cdots+k_n) =$$

$$= \sum_{i_1+\cdots+i_n=\ell,\ \mathbf{i}\geq\mathbf{k}} \binom{\ell}{\mathbf{i}} S(i_1,k_1)\cdots S(i_n,k_n).$$

Proof. Under the condition

$$x_0 = 0, \text{ and } x_1 = x_2 = \cdots = x_n,$$

we have for any $\mathbf{k}$ and any ℓ,

$$S^{(\ell)}_{\mathbf{k}} = x^\ell \sum_{i_1+\cdots+i_n=\ell,\ \mathbf{i}\geq\mathbf{k}} \binom{\ell}{\mathbf{i}} \prod_{j=1}^{n} S(i_j,k_j), \tag{11}$$

where we are denoting $x = x_i$ for $1 \leq i \leq n$. Recall that by $\mathbf{i} \geq \mathbf{k}$ we mean $i_j \geq k_j$ for any $1 \leq j \leq n$.

Let us apply Proposition 2 with $\alpha = \mathbf{e_i} - \mathbf{e_j}$ for $1 \leq i < j \leq n$ (then $\alpha^+ = \mathbf{e_i}$ and $\alpha^- = \mathbf{e_j}$). We deduce that, if $x_i = x_j$, then

$$(k_i+1)S^{(\ell)}_{\mathbf{k}+\mathbf{e_i}-\mathbf{e_j}} = k_j S^{(\ell)}_{\mathbf{k}}.$$

Therefore, when $x_1 = x_2 = \cdots = x_n$, repeating this argument for $x_1 = x_j$ with $2 \leq j \leq n$, we have

$$\begin{aligned} S^{(\ell)}_{\mathbf{k}} &= \cdots = \tbinom{k_1+k_2}{k_1} S^{(\ell)}_{\mathbf{k}+k_2\mathbf{e_1}-k_2\mathbf{e_2}} \\ &= \cdots \\ &= \tbinom{k_1+k_2}{k_1}\tbinom{k_1+k_2+k_3}{k_3}\cdots\tbinom{k_1+\cdots+k_n}{k_n} S^{(\ell)}_{|\mathbf{k}|\mathbf{e_1}}, \end{aligned}$$

for any $\mathbf{k}$ and any ℓ. Then

$$S^{(\ell)}_{\mathbf{k}} = \binom{k_1+k_2}{k_1}\binom{k_1+k_2+k_3}{k_3}\cdots\binom{k_1+\cdots+k_n}{k_n} \sum_{i=|\mathbf{k}|}^{\ell} \binom{\ell}{i} S(i,|\mathbf{k}|) x_0^{\ell-i} x_1^i,$$

which reduces to

$$S_{\mathbf{k}}^{(\ell)} = \binom{k_1+k_2}{k_1}\binom{k_1+k_2+k_3}{k_3}\cdots\binom{k_1+\cdots+k_n}{k_n}S(\ell,|\mathbf{k}|)x^{\ell},$$

under the condition $x_0 = 0$. The result follows from this equation together with Equation (11). □

As a corollary we recover the following closed formula for the Stirling numbers of the second kind.

Corollary 5. *For any two integers $\ell \geq n > 0$, we have*

$$S(\ell,n) = \frac{1}{n!}\sum_{i_1+\cdots+i_n=\ell,\ i_j\geq 1}\binom{\ell}{i_1}\binom{\ell-i_1}{i_2}\cdots\binom{\ell-i_1-\cdots-i_{n-1}}{i_n}.$$

Another example is the following result, where we give a family of relations on the Stirling numbers of the second kind.

Lemma 6. *Given two positive integers a, b, we have*

$$\sum_{i=0}^{a}\frac{a!}{(a-i)!}\sum_{j=i}^{\ell}\binom{\ell}{j}S(j,i)x_0^{\ell-j}x_1^{j} = \sum_{i=0}^{b}\frac{b!}{(b-i)!}\sum_{j=i}^{\ell}\binom{\ell}{j}S(j,i)(\frac{a}{b})^{j}x_0^{\ell-j}x_1^{j}.$$

Proof. It follows by Proposition 2 with $\alpha = a\mathbf{e_1} - b\mathbf{e_j}$ and $\mathbf{k} = b\mathbf{e_j}$ with $1 < j \leq n$. □

Corollary 6. *Given two positive integers a and b,*

$$\sum_{i=0}^{a}\frac{a!}{(a-i)!}\sum_{j=i}^{\ell}\binom{\ell}{j}S(j,i)b^{j}x^{\ell-j} = \sum_{i=0}^{b}\frac{b!}{(b-i)!}\sum_{j=i}^{\ell}\binom{\ell}{j}S(j,i)a^{j}x^{\ell-j},$$

and hence

$$b^{\ell}\sum_{i=0}^{a}\frac{a!}{(a-i)!}S(\ell,i) = a^{\ell}\sum_{i=0}^{b}\frac{b!}{(b-i)!}S(\ell,i).$$

5. Final Remarks

The main results of this paper are Proposition 1 and Proposition 2. They will be crucial in studying the linear systems generalizing the work in [11]. Moreover we believe that both may have other applications. Proposition 2 encodes many recurrences on the Stirling numbers of the second kind and on some of their generalizations, as we have briefly illustrated in Section 4. As for Proposition 1 we believe it may have other applications as a combinatorial tool.

We finish by pointing out that it may be worth investigating an analogous version of these polynomials, namely the Stirling polynomials of the first kind, satisfying orthogonality relations with the polynomials defined in this paper, as well as studying a kind of *generalized Bell polynomials* defined by

$$B_{\mathbf{k}}^{(\ell)} = \sum_{j=0}^{\ell} S_{\mathbf{k}}^{(j)}.$$

Acknowledgments. I want to thank the Universidad Complutense de Madrid for still keeping in use my student e-mail address. The free access to Mathscinet that it provides has been essential to look for bibliography.

References

[1] E.T. Bell, Generalized Stirling transform of sequences, *Amer. J. Math* **61**, no 1 (1939), 89-101.

[2] M. Bernstein and N.J.A. Sloane, Some canonical sequences of integers, *Lin. Alg. and its Appl.* **226-228**, 57-72 (1995).

[3] K.N. Boyadzhiev, Close encounters with the Stirling numbers of the second kind, *Math. Mag.* **85** (2012), 252-266.

[4] K.N. Boyadzhiev, *Notes on the Binomial Transform. Theory and Table with Appendix on Stirling Transform*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.

[5] A.Z. Broder, The r-Stirling numbers, *Discrete Math.* **49** (1984), 241-259.

[6] P. Blasiak, K.A. Penson and A.I. Solomon, The general boson normal ordering problem, *Phys. Lett. A* **309** (2003), 198-205.

[7] L. Carlitz, On arrays of numbers, *Amer. J. Math.* **54** (1932), 739-752.

[8] L. Carlitz, q-Bernoulli numbers and polynomials, *Duke Math. J.* **15** (1948), 987-1000.

[9] L. Carlitz and M.S. Klamkin, Stirling operators, *Collect. Math.* **25**, no. 2 (1974), 185-212.

[10] L. Carlitz, Weighted Stirling numbers of the first and second kind I, *Fibonacci Quart.* **18** (1980) no. 2, 147-162.

[11] F.J. Castro-Jiménez and H. Cobo Pablos, On certain polynomial systems involving Stirling numbers of second kind, *J. Symbolic Comput.* (Special issue on the occasion of MEGA 2019) **109** (2022), 325-350.

[12] F.J. Castro-Jiménez and H. Cobo Pablos, The b-function with respect to weights of a family of hypergeometric ideals in the third Weyl algebra, preprint.

[13] H. Cobo, The b-function with respect to weights of hypergeometric ideals of codimension one, preprint.

[14] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Reidel, Dordrecht and Boston, 1974.

[15] I. Gessel and R.P. Stanley, Stirling polynomials, *J. Comb. Theory*, Series A, **24** 24-33 (1978).

[16] H.W. Gould, Series transformations for finding recurrences, *Fibonacci Quart.* **28** (1990), 166-171.

[17] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Second Edition, Addison-Wesley, 1998.

[18] L.C. Hsu and P.J. Shiue, A Unified Approach to Generalized Stirling Numbers, *Advances in Applied Mathematics* **20** (1998), 366-384.

[19] D.E. Knuth, Two notes on notation, *The American Math. Monthly* **99**, no. 5 (1992), 403-422.

[20] F. Qi and B.N. Guo, A closed form for the Stirling polynomials in terms of the Stirling numbers, *Tbisili Math. J.*, **10**, no. 4 (2017), 153-158.

[21] J. Quaintance and H.W. Gould, *Combinatorial Identities for Stirling Numbers. The Unpublished Notes of H.W. Gould*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2016.

[22] R.P. Singh, On generalized Truesdell polynomials, *Riv. Mat. Univ. Parma* (2) **8** (1967), 345-353.

[23] R.C. Singh Chandel, Generalized Stirling numbers and polynomials, *Publ. Inst. Math.* (Beograd) (N.S.) **22** (36) (1977), 43-48.

[24] A. Schreiber, Multivariate Stirling polynomials of the first and second kind, *Discrete Math.* **338** (2015), 2464-2484.

[25] M. Saito, B. Sturmfels and N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, in *Algorithms and Computation in Mathematics*, 6. Springer-Verlag, Berlin, 2000.

[26] Z.W. Sun, Combinatorial congruences and Stirling numbers, *Acta Arith.* **126**, no. 4 (2007), 387-398.

[27] H.S. Wilf, *Generatingfunctionology*, Academic Press Inc., Boston MA, 1994.