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WZ PROOFS FOR LEMNISCATE-LIKE CONSTANT EVALUATIONS

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Abstract

We show how a variant of Zeilberger's famous WZ proof of a series for $\frac{1}{\pi}$ due to Ramanujan may be applied to construct a WZ proof for an explicit, nontrivial evaluation for a lemniscate-like constant that was recently established through the use of fractional calculus and Fourier–Legendre theory. This lemniscate-like expression is defined by a naturally occurring series that involves squared central binomial coefficients, but this series is remarkably recalcitrant in that current computer algebra systems such as Maple 2021 and Mathematica 11 cannot evaluate this summation. Furthermore, we introduce a new and simplified proof, using a clever application of the WZ method, for a lemniscate-like constant evaluation that had been proved previously using a Watson-type almost poised ${}_3F_2(1)$-identity. WZ proofs are also given for other lemniscate/lemniscate-like constants.

1. Introduction

Out of all of the beautiful formulas due to Ramanujan, an especially famous such formula is the infinite series evaluation whereby

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \left(-\frac{1}{64}\right)^n \binom{2n}{n}^3 (4n+1), \tag{1}$$

as included in Ramanujan's initial letter to Hardy and in Ramanujan's second notebook. This formula was introduced in 1859 by Bauer, who proved (1) via a Fourier–Legendre (FL) expansion [2], and this same formula was rediscovered by Glaisher in 1905 [13]. In 1994 [12], Zeilberger employed the Wilf–Zeilberger (WZ) method [23] so as to formulate a remarkably simplified proof of (1). In this article, we make use of a variant of this proof due to Zeilberger, in order to construct a WZ proof of a summation formula for a lemniscate-like constant [8] that cannot be evaluated by current Computer Algebra System (CAS) software and that was, quite recently, symbolically computed using Fourier–Legendre theory and fractional calculus [7]. This constant is given as an infinite sum in (2) below. We also offer new and simplified proofs, via the WZ method, for a number of closely related series.

The mathematical constant in (1) is indexed in the On-line Encyclopedia of Integer Sequences [22], according to its base-10 expansion, as entry A060294 and is referred to as *Buffon's constant*, in reference to the famous Buffon needle problem. One of the main results in this article, as in Section 3 below, is our new proof of an explicit symbolic evaluation, which involves Buffon's constant, for the difficult [7] infinite series

$$\sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \binom{2n}{n}^2 \frac{1}{4n+3} \tag{2}$$

using a variant of Zeilberger's proof of the series for $\frac{2}{\pi}$ in (1) [12].

As suggested above, the concept of a *lemniscate-like constant* [8] is central to our article; this leads us to recall the classical lemniscate constants

$$\int_0^1 \frac{1}{\sqrt{1-t^4}}\, dt = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \binom{2n}{n} \frac{1}{4n+1} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}} \tag{3}$$

and

$$\int_0^1 \frac{t^2}{\sqrt{1-t^4}}\, dt = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \binom{2n}{n} \frac{1}{4n+3} = \frac{\sqrt{2\pi^3}}{\Gamma^2\left(\frac{1}{4}\right)}, \tag{4}$$

as indexed, according to the decimal digit expansions of these famous constants, in the OEIS as A085565 and A076390, referring the interested reader to the texts cited in these OEIS entries for classic references in which the importance of the lemniscate constants in the history of mathematics is considered [3, pp. 412–417] [5, §1]. With regard to the similarity between the summands of the series in (2) and (4), series as in

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \binom{2n}{n} \frac{f_n}{4n+1} \quad \text{and} \quad \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \binom{2n}{n} \frac{f_n}{4n+3} \tag{5}$$

were the main subject of study in our recent article [8], in which we had referred to such series as "lemniscate-like constants" for suitable sequences $(f_n : n \in \mathbb{N}_0)$ [8], and had evaluated such series for the cases whereby f_n is equal to either O_n or O_{2n} for all $n \in \mathbb{N}_0$, letting $O_m = 1 + \frac{1}{3} + \cdots + \frac{1}{2m-1}$ denote the m^{th} odd harmonic number. We see that the expression shown in (2) is a lemniscate-like constant for the case whereby $(f_n)_{n\in\mathbb{N}_0}$ is the normalized version of $\left(\binom{2n}{n} : n \in \mathbb{N}_0\right)$. Much of our interest in this constant is due how surprising it is that such a "natural-looking" rational series as in (2) cannot be evaluated by state-of-the-art symbolic computation software [7], together with how our new WZ proof for an evaluation of this series greatly simplifies our previous proof of this result [7].

As we had previously expressed [7], the infinite sum in (2) cannot be evaluated in elementary ways, e.g., by using elementary generating functions, index shifts, Thomae-type transforms, etc. Inputting the infinite series from (2) into Mathematica 11, this CAS is only able to output

$$\frac{1}{3}\,{}_3F_2\!\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{3}{4}\\ 1,\frac{7}{4}\end{matrix}\,\middle|\,1\right], \tag{6}$$

and is not able to explicitly evaluate this hypergeometric expression via commands such as `FunctionExpand`, referring the reader to the upcoming Section 2 for preliminary material on hypergeometric series. Inputting the same series from (2) into Maple 2021, this CAS is only able to compute the aforementioned series as an expression that involves the Meijer G-function and that is trivially equivalent to (6). We had previously asserted that it is astonishing [7] that no widely available CAS is able to evaluate the series in (2), which greatly motivates our WZ proof of the explicit evaluation that we provide. It seems that the only previously published proof of this same result is based on rather sophisticated methods [7] compared to our simplified proof in Section 3.

In Section 4 below, we cleverly employ, via the WZ method, an experimental approach toward our formulating a new and simplified proof of a particularly difficult lemniscate-like constant that we had employed in our previous work on binomial-harmonic sums [6, 8]. The main results in this article are our proofs of Theorems 1 and 2, but we also introduce WZ proofs for the lemniscate-like series evaluations shown in (3), (4), (14), and (15). In Section 5, we conclude by briefly considering some future areas of research based on the main material in this article, and we show how variants of our methods may be used to obtain WZ proofs for rational series for $\frac{\sqrt{2}}{\pi}$.

2. Zeilberger's Proof

The WZ method constitutes an important part of many areas in combinatorics and number theory. Since the concept of a *WZ pair* and that of a *WZ proof* are central to our article, we find it worthwhile to succinctly review these concepts.

A hypergeometric function in two variables is a function $A(n,k)$ whereby

$$\frac{A(n+1,k)}{A(n,k)} \quad \text{and} \quad \frac{A(n,k+1)}{A(n,k)}$$

are rational functions. A WZ pair is a pair (F,G) of functions satisfying the following: Both F and G are two-variable hypergeometric functions that satisfy

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k), \tag{7}$$

and G is also such that $G(n,0) = 0$ and $\lim_{k\to\infty} G(n,k) = 0$ [23]. Given a hypergeometric function $F(n,k)$, the WZ method determines a companion function $G(n,k)$ whereby F and G form a WZ pair, if there actually is such a function G [23]. Summing both sides of the equality in (7) over k, we find that

$$\sum_k F(n+1,k) = \sum_k F(n,k), \tag{8}$$

i.e., so that the sum $\sum_k F(n,k)$ must be constant. So, if we want to prove a hypergeometric identity of the form

$$\sum_k \text{summand}(n,k) = \text{rhs}(n),$$

where the above sum vanishes for all indices k outside of some finite interval [23, §2.3], we set

$$F(n,k) := \frac{\text{summand}(n,k)}{\text{rhs}(n)},$$

and then apply the WZ method so as to form a WZ pair (F,G), so that we only need to check that the aforementioned constant, as in either side of (8), is equal to 1. The WZ method allows us to express the companion G so that $G(n,k) = R(n,k)F(n,k)$ for a rational function $R(n,k)$, so that simply evaluating this function R, which we refer to as a WZ *proof certificate*, is a proof in itself [25], since this function certifies that (8) holds, i.e., that either side of (8) is constant. So, the WZ method allows us to simply provide the proof certificate $R(n,k)$ as a one-line proof, for a given finite binomial sum identity to which we may apply the WZ method [23]. We now find ourselves in more of a position to go over Zeilberger's well-known proof of (1).

We let the Pochhammer symbol be defined and denoted as per usual, with $(a)_k = a(a+1)\cdots(a+k-1)$. We find it worthwhile to express this rising factorial function so that $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, letting the famous Γ-function be defined as per usual, with the following Euler integral, letting $\Re(x)$ be positive: $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du$. We are to later apply the famous Legendre duplication formula

$$\Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}\left(\frac{1}{4}\right)^k \binom{2k}{k}\Gamma(k+1),$$

noting that we may express the Pochhammer symbol $\left(\frac{1}{2}\right)_k$ as $\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}}$. A *generalized hypergeometric function* is a series of the form

$$\sum_{i=0}^{\infty} \frac{(a_1)_i\,(a_2)_i\cdots(a_p)_i}{(b_1)_i\,(b_2)_i\cdots(b_q)_i}\frac{x^i}{i!} = {}_pF_q\!\left[\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\;\middle|\; x\right],$$

and we are to later make use of Gauss' famous hypergeometric identity whereby

$${}_2F_1\!\left[\begin{matrix}a,b\\ c\end{matrix}\;\middle|\; 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

for $\Re(c-a-b)>0$.

The main idea behind Zeilberger's proof of (1) comes down to the terminating series identity whereby

$$\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(n+1)}=\sum_{k=0}^{n}(-1)^k(4k+1)\frac{\left(\frac{1}{2}\right)_k^2(-n)_k}{(k!)^2\left(n+\frac{3}{2}\right)_k}, \tag{9}$$

which, for $n\in\mathbb{N}_0$, we may rewrite as

$$\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(n+1)}=\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{\left(\frac{1}{2}\right)_k^2(-n)_k}{(k!)^2\left(n+\frac{3}{2}\right)_k}, \tag{10}$$

since Pochhammer expressions of the form $(-n)_k$ vanish for $k>n\in\mathbb{N}_0$. Following through with the aforementioned proof due to Zeilberger, let us employ the WZ method to prove (9). If we define $F(n,k)$ to be the hypergeometric expression given by dividing the summand in (9) by the left-hand side of this same equality, then we find that the WZ proof certificate, in this case, is $R(n,k)=\frac{2k^2}{(4k+1)(k-n-1)}$, giving us a WZ proof of (9) for $n\in\mathbb{N}_0$. However, the non-terminating version of the identity under consideration actually holds for suitably bounded real values n due to Carlson's theorem [1, p. 40] [12]. Setting $n=-\frac{1}{2}$, this gives us a full proof of Ramanujan's formula, as displayed in (1).

3. A WZ Proof for a Recalcitrant Lemniscate-like Series

By comparing the summand in Zeilberger's series in (10) to that from Ramanujan's series, as in (1), one might hope that it would be feasible, quite broadly, to mimic Zeilberger's proof so as to evaluate series involving centered binomial coefficients by replacing a summand factor as in

$$\left(\frac{1}{4}\right)^k\binom{2k}{k}\quad\text{with}\quad\frac{(-n)_k}{\left(n+\frac{3}{2}\right)_k}, \tag{11}$$

in the hope that the resultant sum would admit a closed-form evaluation as a single hypergeometric function. However, this seems to rarely turn out to be the case. For example, in order to evaluate (2) by mimicking Zeilberger's proof [12], one might hope to be able to evaluate

$$\sum_{k=0}^{n}\frac{\left(\frac{1}{2}\right)_k(-n)_k}{(4k+3)k!\left(n+\frac{3}{2}\right)_k}$$

and then apply the WZ method, but this is not possible, since the above finite sum can only be written as a sum of two two-variable hypergeometric functions that cannot be expressed as a single such function. The foregoing considerations emphasize the remarkable nature about our proof of Theorem 1.

It is quite relevant to note that the above outlined approach, as given by applying the substitution shown in (11) to an infinite series involving normalized central binomial coefficients in the hope that the WZ method could then be applied to the resultant finite sum, also cannot be applied to our previous work on the *hypergeometry of the parbelos* [9], which, as in with our present article, concerned the problem of evaluating an exotic ${}_3F_2(1)$-series of geometric interest or significance, namely, the following expression, which we had shown to be equal to a mathematical constant related to the parbelos figure introduced by Sondow [24]:

$$ {}_3F_2\left[\begin{matrix} -\frac{1}{2}, \frac{1}{4}, \frac{3}{4} \\ \frac{1}{2}, 1 \end{matrix} \,\middle|\, 1\right]. $$

By expressing the corresponding summand using central binomial coefficients, and then applying the substitution in (11), the resulting finite sum does not seem to admit any closed form. Again, this illustrates the interest in our WZ proof of Theorem 1 below.

The key idea behind our simplified proof of the known evaluation for the lemniscate-like constant in (2) is given by applying the WZ method to the series obtained from Zeilberger's series in (10) in the manner indicated as follows.

Lemma 1. *The identity whereby the sum obtained by removing the summand factor* $(-1)^k$ *from* (9) *evaluates as*

$$ \frac{2^{4n-1}(2n+1)\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(n+\frac{1}{4}\right)\Gamma^2(2n+1)}{\pi^2\Gamma^2(n+1)\Gamma(4n+1)} $$

holds true for $n \in \mathbb{N}_0$.

Proof. $R(n,k) = \frac{2k^2(4k-1)}{(4k+1)(4n+1)(k-n-1)}$. □

Again by Carlson's theorem [1, p. 40], we see that the identity given as the above lemma also holds for suitably bounded real n if we replace the upper parameter of the finite sum under consideration with ∞. This leads us toward the below proof. To begin with, we recall that the famous sequence of Catalan numbers, as indexed as `A000108` in the OEIS [22], may be defined so that $C_n = \binom{2n}{n}/(n+1)$; we are to below make use of the generating function for the sequence $(C_n^2 : n \in \mathbb{N}_0)$, as given as `A001246` in the OEIS [22]. The below proof also invokes Gauss' theorem, which, famously, admits a one-line WZ proof [23, §7.2].

Theorem 1. *[7] The explicit evaluation whereby*

$$ \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \binom{2k}{k}^2 \frac{1}{4k+3} = \frac{2}{\pi} - \frac{2\Gamma^2\left(\frac{3}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)} \tag{12} $$

holds true.

Proof. Setting $n = \frac{1}{4}$ in Lemma 1 and applying the Legendre duplication formula, this immediately gives us that

$$\sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^{k} \binom{2k}{k}^{2} \frac{4k+1}{(4k-1)(4k+3)} = -\frac{2\Gamma^{2}\left(\frac{3}{4}\right)}{\Gamma^{2}\left(\frac{1}{4}\right)}. \tag{13}$$

Applying partial fraction decomposition and a re-indexing argument, this gives us that:

$$-\frac{1}{8}\sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^{k} C_k^2 + \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^{k} \binom{2k}{k}^{2} \frac{1}{4k+3} = \frac{1}{2} - \frac{2\Gamma^{2}\left(\frac{3}{4}\right)}{\Gamma^{2}\left(\frac{1}{4}\right)}.$$

Since the generating function for the squares of the Catalan numbers may be written as

$$\frac{{}_2F_1\!\left[\begin{matrix} -\frac{1}{2}, -\frac{1}{2} \\ 1 \end{matrix} \,\middle|\, 16x\right] - 1}{4x},$$

we obtain the desired result, according to Gauss' theorem. □

We assert that our one-page, simplified proof of the above lemniscate-like constant evaluation stands in stark contrast to the relatively sophisticated methodologies involving the Caputo operator $D^{1/2}$ and the complementary semi-derivative operator $D_{\perp}^{1/2}$ employed in what appears to be the only previously known proof for this same evaluation [7, §2.2].

3.1. Further Results

The lemniscate-like constant obtained by replacing the summand factor $\frac{1}{4n+3}$ with $\frac{1}{4n+1}$ in (2) is easily evaluable using today's CAS software, so the evaluation of this constant is of less interest, for our purposes. However, we find it worthwhile to note that we may mimic Zeilberger's proof as given in Section 2 to formulate a brief and elegant proof of the exotic ${}_3F_2(1)$-series evaluation whereby

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^{2}}{16^{k}(4k+1)} = \frac{\Gamma^{4}\left(\frac{1}{4}\right)}{16\pi^{2}}. \tag{14}$$

The non-terminating version of the WZ-derived summation identity given below is valid for real variables, according to Carlson's theorem, and we are allowed to set $n = -\frac{1}{2}$, giving us the evaluation in (14).

Lemma 2. *The identity whereby*

$$\sum_{k=0}^{n} \left(\frac{1}{4}\right)^{k} \binom{2k}{k} \frac{(-n)_k}{\left(n+\frac{3}{2}\right)_k (4k+1)} = \frac{2^{-2n}\Gamma^{2}\left(\frac{5}{4}\right)\Gamma(2n+2)}{\Gamma^{2}\left(n+\frac{5}{4}\right)}$$

holds for $n \in \mathbb{N}_0$.

Proof. $R(n,k) = \frac{k(4k+1)}{8(n+1)(k-n-1)}$. □

By setting $n = -\frac{1}{4}$, we obtain a simplified WZ proof of the lemniscate-like constant evaluation whereby

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(4k+1)^2} = \frac{1}{16}\sqrt{\frac{\pi}{2}}\Gamma^2\left(\frac{1}{4}\right), \tag{15}$$

which played an important role in our recent work [6] on the closed-form evaluation of series involving

$$\left(\frac{1}{32}\right)^n \binom{2n}{n}^2 H_n \quad \text{for} \quad n \in \mathbb{N}_0, \tag{16}$$

letting $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denote the n^{th} entry in the sequence of harmonic numbers. Our WZ proof of (15) is dramatically simpler compared to our having previously [6] proved this result using a limiting property of ${}_4F_3(1)$-series of the following form:

$${}_4F_3\left[\begin{matrix} 1, m+\frac{5}{4}, m+\frac{5}{4}, m+\frac{3}{2} \\ m+2, m+\frac{9}{4}, m+\frac{9}{4} \end{matrix} \,\middle|\, 1\right].$$

In our recent work in which the concept of a lemniscate-like constant was introduced [8], it was noted that the evaluation in (15) is actually a special case of the classical ${}_3F_2(1)$-identity known as Dixon's identity, but our application of Carlson's theorem in our simplified proof of (15) is not equivalent to the known WZ proof of Dixon's theorem [23, §7.2].

4. A WZ Proof for a Watson-type Infinite Series Evaluation

We now turn our attention toward the problem of evaluating the lemniscate-like constant

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(4k+3)^2}, \tag{17}$$

which was also involved in our recent work on the evaluation of series involving (16) [6]. It is not obvious as to how to go about evaluating the above series, especially since the generating function for

$$\left(\frac{1}{(4k+3)^2} \ : \ k \in \mathbb{N}_0\right),$$

cannot be expressed in terms of elementary functions. It appears that there are only two known proofs for the known evaluation for (17) in the extant literature: a proof that we had previously provided [6] based on a limiting property of

$$ {}_4F_3\!\left[\begin{matrix} 1, m+\frac{3}{2}, m+\frac{7}{4}, m+\frac{7}{4} \\ m+2, m+\frac{11}{4}, m+\frac{11}{4} \end{matrix} \,\middle|\, 1\right], $$

and a proof [8] that requires the use of an evaluation for

$$ {}_3F_2\!\left[\begin{matrix} a,\ b,\ c \\ 2+a-b, 2+a-c \end{matrix} \,\middle|\, 1\right] $$

that had been proved by Chu in his work on analytical formulas for generalized Watson–Whipple–Dixon series [11]. These relatively sophisticated methods are in contrast, in a dramatic way, to a much simpler WZ proof that we have put together for the known symbolic form for (17), as below. However, it should be made clear that it is not obvious as to how the WZ method may be used to evaluate (17), as we consider below.

In view of the substitution shown in (11), one might naively hope that it would be feasible to evaluate

$$ \sum_{k=0}^{n} \frac{(-n)_k}{\left(n+\frac{3}{2}\right)_k (4k+1)^3}, \tag{18} $$

i.e., in the hope of mimicking Zeilberger's proof of (1). However, current versions of Maple and Mathematica are only able to express (18) as a ${}_4F_3$-series that seems to be inevaluable. This motivates the experimental approach used below to formulate a WZ proof for evaluating the lemniscate-like constant under consideration. The idea of a 'dialectic' or 'dialog' inherent in computer-human interactions within mathematical research is considered to be at the core of what is meant by the expression *experimental mathematics* [4, pp. vii–viii], and the following discussion seems to nicely reflect this idea.

If we want to construct a WZ proof for evaluating (17), based on or otherwise inspired by the famous 'automatic' proof of (1) due to Zeilberger [12], as a natural place to start, we look toward our proof of Theorem 1 for inspiration. Intuitively, since we had obtained the series in (12) by taking Zeilberger's summand in (9) and removing the factor given by the sequence $\left((-1)^k : k \in \mathbb{N}_0\right)$, in order to obtain the summand in (17), we want to 'remove' a normalized central binomial coefficient as a summand factor in (9), and contribute a factor of the form $\frac{1}{4k+3}$. However, this leads us to a problem: In particular, the resultant sum

$$ \sum_{k=0}^{n} \frac{(-n)_k \left(\frac{1}{2}\right)_k}{\left(n+\frac{3}{2}\right)_k k!(4k+3)} \tag{19} $$

cannot be expressed as a single hypergeometric expression, so that the WZ method cannot be applied, in this case; we also note that it seems somewhat remarkable that Mathematica is not able to evaluate (19).

The sum in (19) can be expressed as a sum of two hypergeometric functions, so this motivates the idea of trying to find, experimentally, a rational function $\rho(n,k)$ in the variables n and k such that (19) plus

$$\sum_{k=0}^{n} \frac{(-n)_k \left(\frac{1}{2}\right)_k}{\left(n+\frac{3}{2}\right)_k k!(4k+3)} \rho(n,k) \tag{20}$$

admits a closed form as a single hypergeometric mapping, and in such a way so that we may apply Carlson's theorem to obtain a copy of the lemniscate-like sum in (17). How can such a function $\rho(n,k)$ be determined?

With regard to the above question, as a natural place to start, we experimented with "natural-looking" variants of the summand in (19), and by comparing the Maple output for the sums in (19) and (20), using this experimental "trial-and-error" approach for various rational functions $\rho(n,k)$, this eventually led us to discover the following result, which we successfully apply, as below, to offer a new and greatly simplified proof for the known symbolic form for (17) [6, 8].

Lemma 3. *The identity whereby*

$$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)_k (-n)_k}{k! \left(n+\frac{3}{2}\right)_k} \frac{8k(2n+1)+8n+3}{4k+3} = \frac{\Gamma^2\left(\frac{3}{4}\right)\Gamma(2n+2)}{4^n \Gamma^2\left(n+\frac{3}{4}\right)}$$

holds true for each member n of $\mathbb{N}_0$.

Proof. $R(n,k) = \frac{k(4k+3)(16kn+16k-8n-7)}{8(n+1)(k-n-1)(16kn+8k+8n+3)}$. □

Again, in view of Carlson's theorem, by replacing the upper parameter of the sum in the above Lemma with ∞, we have that the same identity holds for suitably bounded real values n. This leads us to the following proof.

Theorem 2. *[6, 8] The explicit evaluation*

$$\frac{\sqrt{2}\pi^{3/2}}{\Gamma^2\left(\frac{1}{4}\right)} - \frac{1}{4}\sqrt{\frac{\pi}{2}}\Gamma^2\left(\frac{3}{4}\right) = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k(4k+3)^2}$$

holds true.

Proof. Setting $n = \frac{1}{4}$, the non-terminating version of the summation identity in Lemma 3 immediately gives us that

$$\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \binom{2k}{k} \frac{12k+5}{(4k-1)(4k+3)^2} = -\frac{1}{4}\sqrt{\frac{\pi}{2}}\,\Gamma^2\left(\frac{3}{4}\right). \tag{21}$$

Applying partial fraction decomposition to the rational function factor displayed in the above summand, i.e., so that

$$\frac{12k+5}{(4k-1)(4k+3)^2} = -\frac{1}{2(4k+3)} + \frac{1}{(4k+3)^2} + \frac{1}{2(4k-1)},$$

and by expanding the summand in (21) accordingly, and by then applying an index shift to the sum corresponding to the last partial fraction displayed above, this gives us the following, according to the classically known value for the classical lemniscate constant given as A076390 in the OEIS [22]:

$$\frac{1}{2} + \frac{\sqrt{2}\pi^{3/2}}{\Gamma^2\left(\frac{1}{4}\right)} - \frac{1}{4}\sqrt{\frac{\pi}{2}}\Gamma^2\left(\frac{3}{4}\right) = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k C_k + \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k(4k+3)^2}.$$

So, we see that the desired result now follows from the famous generating function for the Catalan sequence A000108, which nicely recalls our proof of Theorem 1. □

4.1. WZ Proofs for the Classical Lemniscate Constants

While the evaluations in (3) and (4) are classically known and follow immediately from the Γ-function evaluation for the beta integral, the WZ proofs introduced in this article motivate the consideration as to how the WZ method may be used to prove the series evaluations in (3) and (4). This is more straightforward compared to our preceding proofs, so we only briefly consider this.

Using the WZ method, this gives us a computer-generated proof of the identity

$$\sum_{k=0}^{n}\frac{\left(\frac{1}{2}\right)_k(-n)_k}{k!\left(n+\frac{3}{2}\right)_k} = \frac{4^n(2n+1)\Gamma^3(2n+1)}{(4n+1)\Gamma^2(n+1)\Gamma(4n+1)}$$

for $n \in \mathbb{N}_0$, with Maple providing the following proof certificate:

$$R(n,k) = \frac{k(4k-1)}{4(2n+1)(k-n-1)}.$$

Again, by using Carlson's theorem and setting $n = \frac{1}{4}$, we may readily obtain the famous constant evaluation in (4). A similar argument may be used to obtain the explicit form for the constant in (3), using the identity whereby

$$\sum_{k=0}^{n}\frac{\left(\frac{1}{2}\right)_k(-n)_k}{(4k+1)k!\left(n+\frac{3}{2}\right)_k} = \frac{\pi^2 2^{-2n-3}(2n+1)\Gamma(2n+1)}{\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(n+\frac{5}{4}\right)},$$

with the WZ method giving us the following proof certificate, in this case:

$$R(n,k) = \frac{k(4k+1)}{8(n+1)(k-n-1)}.$$

5. Conclusion

We encourage further explorations based on the application of the WZ method to sums involving summand factors as in

$$\frac{\left(\frac{1}{2}\right)_k (-n)_k}{k!\left(n+\frac{3}{2}\right)_k} \tag{22}$$

for $k = 0, 1, \dots, n$, inspired by our above results. We have come to find that: using WZ pairs associated with sums involving expressions as in (22), by "tweaking" WZ identities as in

$$\sum_{n=0}^{\infty} G(n,k) - \sum_{n=0}^{\infty} G(n,k+1) = F(0,k) - \lim_{n\to\infty} F(n,k), \tag{23}$$

this often leads to elegant series for $\frac{\sqrt{2}}{\pi}$, with reference to Examples 1 and 2 below, noting that Mathematica is not able to evaluate either of these series, even with the use of commands such as `FunctionExpand`. We conclude by offering our WZ proofs for these formulas for $\frac{\sqrt{2}}{\pi}$. With regard to the following Lemma and to Theorem 3 below, we are making use of the notational convention whereby

$$\Gamma\begin{bmatrix} x_1, x_2, \dots, x_{n_1} \\ y_1, y_2, \dots, y_{n_2} \end{bmatrix} = \frac{\Gamma(x_1)\,\Gamma(x_2)\cdots\Gamma(x_{n_1})}{\Gamma(y_1)\,\Gamma(y_2)\cdots\Gamma(y_{n_2})}.$$

Lemma 4. *The identity whereby*

$$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)_k (-n)_k}{k!\left(n+\frac{3}{2}\right)_k} = \frac{4^n(2n+1)}{4n+1}\Gamma\begin{bmatrix} 2n+1, 2n+1, 2n+1 \\ n+1, n+1, 4n+1 \end{bmatrix}$$

holds true for $n \in \mathbb{N}_0$.

Proof. $R(n,k) = \frac{k(4k-1)}{4(2n+1)(k-n-1)}$. □

This leads us to the nontrivial ${}_3F_2(1)$-identity given below, noting that neither Maple nor Mathematica is able to evaluate the infinite series given below.

Theorem 3. *The identity whereby*

$$\sum_{n=0}^{\infty} \frac{16k(2k+1)(n+1)+6n+9}{4^n}\,\Gamma\begin{bmatrix} 2n+\frac{5}{2}, k-n-\frac{3}{2} \\ -n-\frac{1}{2}, n+2, k+n+3 \end{bmatrix}$$

evaluates as

$$\frac{128\sqrt{\frac{2}{\pi}}(-1)^{k+1}}{(-1)^{2k}+1} - 12\,\Gamma\begin{bmatrix} k-\frac{1}{2} \\ k+2 \end{bmatrix}$$

holds true, if k is suitably bounded.

Proof. Using the WZ pair associated with our above WZ proof, we rewrite the difference equation in (7) so that

$$F\left(n+\frac{3}{2},k\right)-F\left(n+\frac{1}{2},k\right)=G\left(n+\frac{1}{2},k+1\right)-G\left(n+\frac{1}{2},k\right).$$

By summing both sides of this equality over $n \in \mathbb{N}_0$, and then simplifying the resultant equality, this can be shown to give us the desired result, for suitably bounded k. □

Example 1. Setting $k=-1$ in the above theorem, this gives us a WZ proof of the formula

$$\frac{4\sqrt{2}}{\pi}=\sum_{n=0}^{\infty}\left(\frac{1}{64}\right)^{n}\binom{2n}{n}\binom{4n}{2n}\frac{22n+3}{(2n+1)(2n+3)},$$

noting that Maple is able to evaluate the above series.

Example 2. Setting $k=-2$ in Theorem 3, we obtain a WZ proof of the identity whereby

$$\frac{2\sqrt{2}}{3\pi}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}\binom{4n}{2n}n(34n+1)}{64^{n}(2n+1)(2n+3)(2n+5)},$$

which Maple is also able to confirm directly.

Guillera, in 2013 [15], applied the WZ method to generalize (1), but our methods and results are fundamentally different compared to how Guillera has obtained identities for evaluating series as in (24) below [15]:

$$\sum_{n=0}^{\infty}(-1)^{n}\frac{\left(\frac{1}{2}+x\right)^{3}_{n}}{(1+x)^{3}_{n}}(1+4x+4n). \tag{24}$$

We conclude by suggesting that our WZ proofs, as introduced in this article, motivate the consideration as to how such proofs may be mimicked and applied using Guillera's many WZ proofs for Ramanujan-like series for $\frac{1}{\pi}$ and $\frac{1}{\pi^2}$ [14, 15, 16, 17, 18, 19, 20, 21]. Recalling (1), (5), and (13), the evaluation

$$\sum_{n=0}^{\infty}\left(\frac{1}{4^{n}}\binom{2n}{n}\right)^{3}\frac{(-1)^{n+1}(4n+1)^{2}}{(4n-1)(4n+3)}=\frac{32\left(2+\sqrt{2}\right)\Gamma^{2}\left(\frac{1}{4}\right)}{\Gamma^{4}\left(\frac{1}{8}\right)}$$

was proved, along with many other remarkable results, quite recently in [10], using the Clebsch–Gordan integral; we encourage the exploration as to how our WZ-based methods may be applied through the use of the material from [10].

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