ON PILLAI'S PROBLEM WITH LUCAS NUMBERS AND POWERS OF 3

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Abstract

In this paper, we find all integers m having at least two representations as a difference between a Lucas number and a power of 3. The sequence of Lucas numbers, $(L_k)_{k\geq 0}$, is given by $L_0 = 2, L_1 = 1$ and $L_{k+2} = L_{k+1} + L_k$, for $k \geqslant 0$. The tools used to solve our main theorem are linear forms in logarithms, properties of continued fractions and a version of the Baker-Davenport reduction method in Diophantine approximation.

1. Introduction

The Lucas sequence $(L_k)_{k\geq 0}$ is a linear recurrence given by $L_0 = 2, L_1 = 1$ and

$$L_{k+2} = L_{k+1} + L_k, \quad \text{for } k \geqslant 0.$$

It follows the same recursive definition as the Fibonacci sequence $(F_k)_{k\geq 0}$ given by $F_0 = 0, F_1 = 1$ and

$$F_{k+2} = F_{k+1} + F_k, \quad \text{for } k \geqslant 2.$$

The Diophantine equation

$$L_k + L_l = 2^t$$

is studied in [3] in positive integers k, l and t. Similar equations involving Fibonacci and Padovan sequences are solved in [6, 8].

In this paper, we study the Diophantine equation

$$L_k - 3^l = m,$$

where m is a fixed integer and k, l are positive variable integers. We are interested in finding those integers m admitting at least two representations as a difference between a Lucas number and a power of 3.

All integers c which have at least two representations as a difference between a Fibonacci number and a power of three are found in [4]. Specifically, the author proved the following theorem.

Theorem 1. *The only integers having at least two representations of the form $F_n - 3^m$ are -26, -6, -1, 0, 2, 4, 7, and 12. Furthermore, all the representations of the above integers as $F_n - 3^m$ with $n \geq 2$ and $m \geq 0$ are given by*

$$\begin{aligned} -26 &= F_{10} - 3^4 = F_2 - 3^3, \\ -6 &= F_8 - 3^3 = F_4 - 3^2, \\ -1 &= F_6 - 3^2 = F_3 - 3^1, \\ 0 &= F_4 - 3^1 = F_2 - 3^0, \\ 2 &= F_5 - 3^1 = F_4 - 3^0, \\ 4 &= F_7 - 3^2 = F_5 - 3^0, \\ 7 &= F_9 - 3^3 = F_6 - 3^0, \\ 12 &= F_8 - 3^2 = F_7 - 3^0. \end{aligned} \tag{1.1}$$

In this paper, we prove an extension of Theorem 1 when the Fiboncci numbers are replaced by Lucas numbers and determine all integers m having at least two representations of the form $L_k - 3^l$. We prove the following result.

Theorem 2. *The only integers having at least two representations of the form $L_k - 3^l$ are -5, -2, 0, 2, 20, and 114. Furthermore, all the representations of the above integers as $L_k - 3^l$ with $k \geq 1$ and $l \geq 0$ are given by*

$$\begin{aligned} -5 &= L_3 - 3^2 = L_9 - 3^4, \\ -2 &= L_1 - 3^1 = L_4 - 3^2, \\ 0 &= L_1 - 3^0 = L_2 - 3^1, \\ 2 &= L_2 - 3^0 = L_5 - 3^2 = L_7 - 3^3, \\ 20 &= L_7 - 3^2 = L_8 - 3^3 = L_{16} - 3^7, \\ 114 &= L_{10} - 3^2 = L_{14} - 3^6. \end{aligned} \tag{1.2}$$

2. Preliminaries

Before proceeding further, we recall the Binet formula for the Lucas numbers $(L_k)_{k\geq 0}$:

$$L_k = \alpha^k + \beta^k,$$

for $k \geq 0$, where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the characteristic equation $x^2 - x - 1 = 0$. In particular, the inequality

$$\alpha^{k-1} \leqslant L_k \leqslant 2\alpha^k \tag{2.1}$$

holds for all $k \geq 0$.

To prove Theorem 2, using a result on linear forms in two logarithms, we require some notation. Let δ be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \delta^{(i)})$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\delta^{(i)}$ denotes the conjugates of δ. Then,

$$h(\delta) = \frac{1}{d}(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\delta^{(i)}|, 1\}))$$

is called the logarithmic height of δ. In particular, if $\delta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then,

$$h(\delta) = \log \max\{|p|, q\}.$$

The following properties of the logarithmic height, will be used in the next section. Let δ, ν be algebraic numbers and $r \in \mathbb{Z}$. Then,

- $h(\delta \pm \nu) \leq h(\delta) + h(\nu) + \log 2$,
- $h(\delta\nu^{\pm 1}) \leq h(\delta) + h(\nu)$,
- $h(\delta^r) = |r| h(\delta)$.

Using the above notation,we restate the result in [7, Cor. 1].

Theorem 3. *Let δ_1, δ_2 be two non-zero algebraic numbers, and let $\log \delta_1$ and $\log \delta_2$ be any determinations of their logarithms. Set*

$$D = [\mathbb{Q}(\delta_1, \delta_2) : \mathbb{Q}]/[\mathbb{R}(\delta_1, \delta_2) : \mathbb{R}]$$

and

$$\Gamma := b_2 \log \delta_2 - b_1 \log \delta_1,$$

where b_1 and b_2 are positive integers. Further, let $A_1, A_2 > 1$ be real numbers such that

$$\log A_i \geq \max\{h(\delta_i), \frac{|h(\delta_i)|}{D}, \frac{1}{D}\}, \quad i = 1, 2.$$

Then, assuming that δ_1 and δ_2 are multiplicatively independent, we have

$$\log |\Gamma| > -30.9 \cdot D^4 (\max\{\log b', \frac{21}{D}, \frac{1}{2}\})^2 \log A_1 \log A_2,$$

where

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

We also need the following general lower bound for linear forms in logarithms due to Matveev [9].

Theorem 4. *Assume that $\delta_1, \cdots, \delta_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D. Let $b_1, \cdots, b_n$ be rational integers, and*

$$\Lambda := \delta_1^{b_1} \cdots \delta_t^{b_t} - 1$$

be not zero. Then,

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\right),$$

where

$$B \geqslant \max\{|b_1|, \cdots, |b_t|\},$$

and

$$A_i \geqslant \max\{D h(\delta_i), |\log \delta_i|, 0.16\}, \quad \text{for all} \quad i = 1, \cdots, t.$$

Finally, we present a version of the reduction method in [1] and [5]. This will be one of the key tools used to reduce the upper bounds on the variables.

Lemma 1. *Let N be a positive integer, let p/q be a convergent of the irrational number γ such that $q > 6N$ and let A, B, μ be real numbers with $A > 0$ and $B > 1$. Define*

$$\xi := \|\mu q\| - N \|\gamma q\|,$$

where $\|\cdot\|$ denotes the distance to the nearest integer. If $\xi > 0$, then there is no solution to the inequality

$$0 < u\gamma - v + \mu < AB^{-w}$$

in positive integers u, v, and w with $u \leqslant N$ and $w \geqslant \frac{\log(Aq/\xi)}{\log B}$.

3. The Proof of Theorem 2

Let us establish here a relation between k and l.

First of all, observe that if $L_k - 3^l = L_{k_1} - 3^{l_1}$, with $\min\{k, k_1\} > 1$, $\min\{l, l_1\} \geq 0$ and $(k, k_1) \neq (l, l_1)$.

Then, assume that $l = l_1$;

$l = l_1$ implies that $L_k = L_{k_1}$. This is a contradiction of our assumption. Thus , $l > l_1$. We have

$$L_k - L_{k_1} = 3^l - 3^{l_1} > 0. \tag{3.1}$$

Combining (2.1) with (3.1), one gets that

$$\alpha^{k-3} \leq L_{k-2} \leq L_k - L_{k_1} = 3^l - 3^{l_1} < 3^l,$$

which implies that

$$k < l\left(\frac{\log 3}{\log \alpha}\right) + 3. \tag{3.2}$$

And

$$L_k - L_{k_1} = 3^l - 3^{l_1} = 3^{l-1}(3 - 3^{l_1 - l + 1}) < 2\alpha^k$$

implies $3^{l-1} < 2\alpha^k$, so that

$$(l-1)\frac{\log 3}{\log \alpha} - \frac{\log 2}{\log \alpha} < k$$

and we get:

$$(l-2)\frac{\log 3}{\log \alpha} < k. \tag{3.3}$$

Combining (3.2) with (3.3), we obtain

$$(l-2)\frac{\log 3}{\log \alpha} < k < l\left(\frac{\log 3}{\log \alpha}\right) + 3. \tag{3.4}$$

When $k \leq 250$, we have $l \leq 112$. Then, a brute force search using Sagemath in the range $2 \leq k_1 < k \leq 250$ and $0 \leq l_1 < l \leq 112$ produces the solutions $m \in \{-5, -2, 0, 2, 20, 114\}$, whose representation is (1.2).

Thus, for the remainder of the paper, we assume that $k > 250$ and $l > 112$.

3.1. Bounding k

We rewrite (3.1) as
$$\alpha^k - 3^l = L_{k_1} - 3^{l_1} - \beta^k.$$
Now, taking absolute values, we obtain:

$$|\alpha^k - 3^l| < L_{k_1} + 3^{l_1} + |\beta|^k < 2\alpha^{k_1} + 3^{l_1} + \frac{1}{2} < 2\max\{\alpha^{k_1+2}, 3^{l_1+1}\}.$$

Dividing both sides of the above expression by α^k and taking into account that $2 < \alpha^2$ and $\alpha < 3$, we get:
$$|1 - 3^l\alpha^{-k}| < 2\alpha^{-k}\max\{\alpha^{k_1+2}, 3^{l_1+1}\} < \max\{\alpha^{k_1-k+4}, 3^{l_1-k+3}\}. \tag{3.5}$$
We apply Theorem 3 to
$$\Gamma := l\log 3 - k\log\alpha.$$
Therefore, estimation (3.5) can be rewritten as

$$|1 - e^{\Gamma}| < \max\{\alpha^{k_1-k+4}, 3^{l_1-k+3}\}. \tag{3.6}$$

The algebraic number field containing 3 and α is $\mathbb{Q}(\sqrt{5})$, so we can take $D := 2$. Hence,
$$\log|\Gamma| < \max\{(k_1 - k + 4)\log\alpha, (l_1 - k + 3)\log 3\}. \tag{3.7}$$
Note further that $h(\alpha) = \log\alpha/2$ and $h(3) = \log 3$. Thus, we can choose
$$\log A_1 := \log\alpha \quad \text{and} \quad \log A_2 := \log 3.$$
Finally, recall that $l \le k$ and so
$$b' = \frac{k}{2\log 3} + \frac{l}{2\log\alpha} < 3k.$$
Since α and 3 are multiplicatively independent, we have, by Theorem 3, that
$$\log\Gamma \ge -30.9 \cdot 2^4 \cdot (\max\{\log(3k), 21/2, 1/2\})^2 \cdot \log\alpha \cdot \log 3.$$
Thus,
$$\log|\Gamma| > -174 \cdot (\max\{\log(3k), 21/2, 1/2\})^2. \tag{3.8}$$
Combining (3.7) and (3.8), we obtain:
$$\min\{(-k_1+k-4)\log\alpha, (-l_1+k-3)\log 3\} < 174\cdot(\max\{\log(3k), 21/2, 1/2\})^2. \tag{3.9}$$
Therefore, either
$$\begin{aligned} &\min\{(-k_1 + k)\log\alpha, (-l_1 + k)\log 3\} < 174 \cdot (\log(3k))^2, \quad \text{or} \\ &\min\{(-k_1 + k)\log\alpha, (-l_1 + k)\log 3\} < 19184. \end{aligned} \tag{3.10}$$

Finally, we consider a third linear form in logarithms. We now rewrite equation (3.1) as follows:

$$\alpha^k - \alpha^{k_1} - 3^l + 3^{l_1} = \beta^{k_1} - \beta^k.$$

Taking absolute values in the above relation and using the fact that $\beta = (1 - \sqrt{5})/2$, we get:

$$|\alpha^{k_1}(\alpha^{k-k_1} - 1) - 3^{l_1}(3^{l-l_1} - 1)| = |\beta^{k_1} - \beta^k| < |\beta^{k_1}| + |\beta^k| < 2.$$

Dividing both sides of the above inequality by the second term of the left-hand side, we obtain:

$$|\alpha^{k_1}(\alpha^{k-k_1} - 1)3^{-l_1}(3^{l-l_1} - 1)^{-1} - 1| < \frac{2}{3^l(3^{-l+l_1} - 1)} < \frac{4}{3^l} < \frac{4}{\alpha^{k-3}}. \tag{3.11}$$

We apply Theorem 4 to

$$\Lambda = \alpha^{k_1}(\alpha^{k-k_1} - 1)3^{-l_1}(3^{l-l_1} - 1)^{-1} - 1$$

with the parameters $n := 3$, $\delta_1 := 3$, $\delta_2 := \alpha$, $\delta_3 := (\alpha^{k-k_1} - 1)(3^{l-l_1} - 1)^{-1}$, $b_1 := -l_1$, $b_2 := k_1$ and $b_3 := 1$, $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ contains $\delta_1, \delta_2, \delta_3$, $D := [\mathbb{K} : \mathbb{Q}] = 2$. To see why the left-hand side of (3.11) is not zero, note that otherwise, we would get the relation

$$\alpha^k - \alpha^{k_1} = 3^l - 3^{l_1}.$$

This is impossible because $\alpha^k - \alpha^{k_1} \notin \mathbb{Q}$. Thus,

$$\alpha^{k_1}(\alpha^{k-k_1} - 1)3^{-l_1}(3^{l-l_1} - 1)^{-1} - 1$$

is not zero.

We now apply Theorem 4 with $A_1 := 2\log 3$, $A_2 := \log\alpha$. Since $l \le k$; it follows that we can take $B := k$. Let us now estimate $h(\delta_3)$. We begin by observing that

$$h(\delta_3) \le h(\alpha^{k-k_1} - 1) + h(3^{l-l_1} - 1).$$

Next, notice that

$$h(\delta_3) \le (k - k_1)\log\alpha + (l - l_1)\log 3 + 2\log 2 \le 2 \cdot 174 \cdot (\max\{\log(3k), 21/2, 1/2\})^2.$$

Hence, we can take

$$A_3 := 2 \cdot 174 \cdot (\max\{\log(3k), 21/2, 1/2\})^2 > \max\{2h(\delta_3), |\log\delta_3|, 0.16\}.$$

Now, from Theorem 4 we have that

$$\log|\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log(k)) \cdot 2\log 3 \cdot 2\log\alpha$$
$$\times 2 \cdot 174 \cdot (\max\{\log(3k), 21/2, 1/2\})^2.$$

So, Inequality (3.11) gives

$$k < 2 \cdot 10^{15}\log(k) \cdot (\max\{\log(3k), 21/2, 1/2\})^2. \tag{3.12}$$

If $\max\{\log(3k), 21/2\} = 21/2$, it then follows from (3.12) that

$$k < 221 \cdot 10^{15}(\log(k)),$$

giving

$$k < 10^{19}.$$

If on the other hand we have that $\max\{\log(3k), 21/2\} = \log(3k)$, then Inequality (3.12) gives that

$$k < 2 \cdot 10^{15}(\log(k)) \cdot (\log(3k))^2,$$

and so $k < 8 \cdot 10^{16}$. In any case, we have that

$$k < 10^{19}$$

always holds. We summarize what we have so far in the following lemma.

Lemma 2. *If an integer m has at least two representations of the form $L_k - 3^l$ with $k > 250$, then inequalities $l < k$ and $k < 10^{19}$ hold.*

4. The Final Computations

In this section, we will reduce the upper bound on k. Firstly, we determine a suitable upper bound on $k - k_1, l - l_1$, and later we use Lemma 1 to conclude that k must be smaller than 250.

Turning back to Inequality (3.5),

$$|1 - e^\Gamma| < \max\{\alpha^{k_1-k+4}, 3^{l_1-k+3}\}.$$

If

$$\Gamma := l\log 3 - k\log\alpha > 0,$$

we obtain:

$$0 < \Gamma < e^\Gamma - 1 < \max\{\alpha^{k_1-k+4}, 3^{l_1-k+3}\}.$$

Thus,

$$0 < l\left(\frac{\log 3}{\log\alpha}\right) - k < \frac{1}{\log\alpha}\max\{\alpha^{k_1-k+4}, 3^{l_1-k+3}\} < \max\{\alpha^{k_1-k+6}, 3^{l_1-l+4}\}.$$

Dividing across by $\log\alpha$, we get:

$$0 < l\gamma - k < \max\{\alpha^{k_1-k+6}, 3^{l_1-l+4}\}, \tag{4.1}$$

where

$$\gamma := \frac{\log 3}{\log\alpha}.$$

Let $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] = [2; 3, 1, 1, 6, 1, 49, 1, 2, \cdots]$ be the continued fraction expansion of γ, and let denote p_n/q_n its nth convergent. Recall also that $l < 10^{19}$ by Lemma 2. A quick inspection using Sagemath reveals that

$$4977896525362041575 = q_{41} < 6 \cdot 10^{19} < q_{42} = 805929983250536127817.$$

Furthermore, $a_N := \max\{a_i; i = 0, 1, \ldots, 42\} = a_6 = 49$. So, from the known properties of continued fractions, we obtain that

$$|l\gamma - k| > \frac{1}{(a_N + 2)l}. \tag{4.2}$$

Comparing estimates (4.1) and (4.2), we get right away that

$$1 < 51l \cdot \max\{\alpha^{k_1-k+6}, 3^{l_1-l+4}\} < 51 \cdot 10^{19} \cdot \max\{\alpha^{k_1-k+6}, 3^{l_1-l+4}\},$$

leading to the following two cases.

`Case 1.` If $\max\{\alpha^{k_1-k+6}, 3^{l_1-l+4}\} = \alpha^{k_1-k+6}$, then

$$k - k_1 < \frac{\log(51 \cdot 10^{19} \cdot \alpha^6)}{\log\alpha} < 106.$$

`Case 2.` If $\max\{\alpha^{k_1-k+6}, 3^{l_1-l+4}\} = 3^{l_1-l+4}$, then

$$l - l_1 < \frac{\log(51 \cdot 10^{19} \cdot 3^4)}{\log 3} < 50.$$

Suppose now that $\Gamma < 0$. First, note that $\max\{\alpha^{k_1-k+4}, 3^{l_1-l+3}\} < 9$ since $k > k_1$ and $l > l_1$. Then, from (3.6), we have that $|1 - e^{\Gamma}| < 9$. Thus, $0 < e^{\Gamma} < 10$. Therefore,

$$e^{|\Gamma|} < 10.$$

Since $\Gamma < 0$, we have

$$0 < |\Gamma| \leqslant e^{|\Gamma|} - 1 = e^{|\Gamma|}|e^{-|\Gamma|} - 1| = e^{|\Gamma|}|e^{\Gamma} - 1| < 10 \cdot \max\{\alpha^{k_1-k+4}, 3^{l_1-l+3}\}.$$

Then, we obtain:

$$0 < -l\log 3 + k\log\alpha < 10 \cdot \max\{\alpha^{k_1-k+4}, 3^{l_1-l+3}\}.$$

By the same arguments used in the case $\Gamma > 0$, we obtain:

$$0 < k\left(\frac{\log \alpha}{\log 3}\right) - l < \frac{10}{\log 3} \cdot \max\{\alpha^{k_1-k+4}, 3^{l_1-l+3}\} < \max\{\alpha^{k_1-k+9}, 3^{l_1-l+6}\}. \quad (4.3)$$

We put

$$\gamma := \frac{\log \alpha}{\log 3}.$$

Let $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] = [0; 2, 3, 1, 1, 6, 1, 49, 1, 2, 2, 1, 1, 2, \cdots]$ be the continued fraction expansion of γ, and let p_n/q_n denote its nth convergent. Recall also that $k < 10^{19}$ by Lemma 2. A quick inspection using Sagemath reveals that

$$11364596648895014778 = q_{42} < 6 \cdot 10^{19} < q_{43} = 18399476847757338044 95.$$

Furthermore, $a_N := \max\{a_i; i = 0, 1, \ldots, 43\} = a_7 = 49$. So, from the known properties of continued fractions, we obtain that

$$|k\gamma - l| > \frac{1}{(a_N + 2)k}. \quad (4.4)$$

Comparing estimates (4.3) and (4.4), we get right away that

$$1 < 51k \cdot \max\{\alpha^{k_1-k+9}, 3^{l_1-l+6}\} < 51 \cdot 10^{19} \cdot \max\{\alpha^{k_1-k+9}, 3^{l_1-l+6}\},$$

leading to the following two cases.

Case 1. If $\max\{\alpha^{k_1-k+9}, 3^{l_1-l+6}\} = \alpha^{k_1-k+9}$, then

$$k - k_1 < \frac{\log(51 \cdot 10^{19} \cdot \alpha^9)}{\log \alpha} < 109.$$

Case 2. If $\max\{\alpha^{k_1-k+9}, 3^{l_1-l+6}\} = 3^{l_1-l+6}$, then

$$l - l_1 < \frac{\log(51 \cdot 10^{19} \cdot 3^6)}{\log 3} < 50.$$

As a conclusion, we have that either $k - k_1 \leq 108$ or $l - l_1 \leq 49$ whenever $\Gamma \neq 0$.

Finally, we shall use (3.11) to reduce the upper bound on k. we distinguish between the case $k - k_1 \leq 108$ and $l - l_1 \leq 49$.

Put

$$\Gamma = k_1 \log \alpha - l \log 3 + \log\left(\frac{\alpha^u - 1}{3^v - 1}\right). \quad (4.5)$$

where $u = k - k_1, v = l - l_1$.

Note that $\Gamma \neq 0$. Thus, we distinguish the following cases. If $\Gamma > 0$ then, from (3.11), we obtain:

$$0 < \Gamma \leqslant e^{\Gamma} - 1 < \frac{4}{\alpha^{k-3}}. \quad (4.6)$$

In Inequality (4.6) replace Γ by its formula (4.5) and dividing both sides of the resulting inequality by $\log 3$, we get:

$$0 < k_1 \left(\frac{\log \alpha}{\log 3} \right) - l + \frac{\log \left(\frac{\alpha^u - 1}{3^v - 1} \right)}{\log 3} < \frac{4}{\alpha^{k-3}}. \tag{4.7}$$

We now put:

$$\gamma := \frac{\log \alpha}{\log 3}, \quad \mu := \frac{\log \left(\frac{\alpha^u - 1}{3^v - 1} \right)}{\log 3}, \quad A := 4 \quad \text{and} \quad B := \alpha.$$

Clearly γ is an irrational number. We also put $N := 10^{19}$, which is an upper bound on k_1 by Lemma 2. We therefore apply Lemma 1 to Inequality (4.7) for all choices of $u \in \{1, \ldots, 108\}, v \in \{1, \ldots, 49\}$ and get that

$$k - 3 < \frac{\log(Aq/\xi)}{\log B},$$

where $q > 6N$ is a denominator of a convergent of the continued fraction of γ such that $\xi = \|\mu q\| - N\|\gamma q\| > 0$. Indeed, using Sagemath, we have that

$$q = q_{43} = 1839947684775733804495.$$

Futher this yields $\xi \geq 0.0000102818012237548828125 > 0.00001$. Therefore, we find that if $\Gamma > 0$, then $k < 127$. This is false because our assumption is that $k > 250$.

Suppose now that $\Gamma < 0$. First, note that $\frac{4}{\alpha^{k-3}} < \frac{1}{2}$ since $k > 250$. Then, from (3.11), we have that $|1 - e^{\Gamma}| < \frac{1}{2}$; thus $\frac{1}{2} < e^{\Gamma} < \frac{3}{2}$. Therefore,

$$e^{|\Gamma|} < 2.$$

Since $\Gamma < 0$, we have:

$$0 < |\Gamma| \leqslant e^{|\Gamma|} - 1 = e^{|\Gamma|}|e^{-|\Gamma|} - 1| = e^{|\Gamma|}|e^{\Gamma} - 1| < \frac{4}{\alpha^{k-3}}.$$

Then we obtain:

$$0 < l \log 3 - k_1 \log \alpha - \log \left(\frac{\alpha^u - 1}{3^v - 1} \right) < \frac{4}{\alpha^{k-3}}.$$

By the same arguments used for proving (3.11), we obtain:

$$0 < l \left(\frac{\log 3}{\log \alpha} \right) - k_1 - \frac{\log \left(\frac{\alpha^u - 1}{3^v - 1} \right)}{\log \alpha} < \frac{9}{\alpha^{k-3}}. \tag{4.8}$$

We now put:

$$\gamma := \frac{\log 3}{\log \alpha}, \quad \mu := -\frac{\log\left(\frac{\alpha^u - 1}{3^v - 1}\right)}{\log \alpha}, \quad A := 9 \quad \text{and} \quad B := \alpha.$$

Clearly, γ is an irrational number. We also put $N := 10^{19}$ which is an upper bound on l by Lemma 2. We therefore apply Lemma 1 to Inequality (4.8) for all choices of $u \in \{1, \ldots, 108\}, v \in \{1, \ldots, 49\}$ and get that

$$k < \frac{\log(Aq/\xi)}{\log B},$$

where $q > 6N$ is a denominator of a convergent of the continued fraction of γ such that $\xi = \|\mu q\| - N\|\gamma q\| > 0$. Indeed, using Sagemath, we have hat

$$q = q_{42} = 805929983250536127817.$$

Therefore, we find that if $\Gamma < 0$, then $k < 127$. This is false because our assumption is that $k > 250$. Thus, Theorem 2 is proven.

References

[1] A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J. Math. Oxford Ser.* **20** (1969), 129–137.

[2] E. F. Bravo and J. J. Bravo, On the Diophantine equation $F_n + F_m + F_l = 2^d$, *Lith. Math. J.* **55** (2015), 301–311.

[3] J. J. Bravo and F. Luca, On the Diophantine equation $L_n + L_m = 2^a$, *J. Integer Seq.* **17** (2014), Article 14.8.3.

[4] M. Ddamulira, On the Problem of Pillai with Fibonacci numbers and powers of three, *Bol. Soc. Mat. Mex.* **26**(3) (2020), 263–277. 10.1007/s40590–019–00263–1. hal-02013056v2

[5] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser.* **49** (1998), 291–306.

[6] A. C. G. Lomelí and S. H. Hernández, Powers of two as sums of two Padovan numbers, *Integers* **18** (2018), A84.

[7] M. Laurent, M. Mignotte, and Y. Nesterenko, Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory* **55** (1995), 285–321.

[8] F. Luca and S. Siksek, Factorials expressible as sums of two and three Fibonacci numbers, *Proc. Edinb. Math. Soc.* **53** (2010), 747–763.

[9] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II, *Izv. Ran.Ser.Mat.* **64** (2000), 125–180; *Izv. Math.* **64** (2000), 1217–1269.