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THE FACTORIAL MOMENTS OF THE GENERALIZED BERNOULLI-FIBONACCI DISTRIBUTION

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Abstract

The generalized Bernoulli-Fibonacci distribution describes the waiting time until k successive successes occur in a Bernoulli process. In the case of a symmetric Bernoulli process, when the counting starts at time 1, we prove that the factorial moments of this distribution are multiples of the terms of a subsequence of the generalized k-step Fibonacci numbers. As a result, when the counting starts at time k, the factorial moments of the Bernoulli-Fibonacci distribution are linear combinations of the aforementioned subsequence of the k-step Fibonacci numbers. Up to now, most authors used the second version of the Bernoulli-Fibonacci distribution, and thus have been unable to provide formulas for all the factorial moments of the distribution for a general k. To establish the main result in our paper, we first prove a number of identities involving the roots of unity of order $k+1$ and the inverses of the roots of the characteristic polynomial of the k-step Fibonacci numbers.

1. Introduction

For each $k \in \mathbb{Z}_{>0}$, let $(F_n^{(k)} : n \in \mathbb{Z})$ be the sequence of *k-step generalized Fibonacci numbers* defined by

$$F_n^{(k)} = 0 \quad \text{for all } n \leq 0, \quad F_1^{(k)} = 1, \quad \text{and} \quad F_n^{(k)} = \sum_{i=1}^{k} F_{n-i}^{(k)} \quad \text{for all } n \geq 2. \tag{1}$$

This sequence has been studied by many authors; see, for example, Capocelli and Cull [4], Christensen [5], Koutras [9], Medhi [11], Philippou et al. [14], Wolfram [19], and Zhang and Hadjicostas [20].

It is well-known that

$$F_n^{(k)} = 2^{n-2} \quad \text{for } n = 2, \ldots, k+1 \quad \text{and} \quad F_{k+2}^{(k)} = 2^k - 1. \tag{2}$$

The sequence $(F_n^{(k)} : n \in \mathbb{Z}_{>0})$ may be calculated either through recurrence (1) or through the more explicit formula

$$F_n^{(k)} = S(k, n-2) - S(k, n-k-2) \quad \text{for } n \geq 2, \tag{3}$$

where

$$S(k,n) = \sum_{j=0}^{\lfloor n/(k+1) \rfloor} (-1)^j \binom{n-kj}{j} 2^{n-(k+1)j}. \tag{4}$$

(Empty sums are assumed to be zero; e.g., $S(k, n-k-2) = 0$ for $2 \leq n \leq k+1$.) Equation (3) can proved using results from Dunkel [6].

For an equation similar to Equation (3), but apparently different, see also Theorem 2.4 in Howard and Cooper [8], who worked with the shifted sequence $(G_n^{(k)} : n \in \mathbb{Z}) = (F_{n-k+2}^{(k)} : n \in \mathbb{Z})$. In our notation, for $k \geq 2$ and $n \geq k+2$, their formula becomes

$$F_n^{(k)} = \sum_{j=0}^{\lfloor n/(k+1) \rfloor} (-1)^j \left(\binom{n-kj}{j} - \binom{n-kj-2}{j-2} \right) 2^{n-(k+1)j-2}, \tag{5}$$

where we assume that $\binom{a}{b} = 0$ if $b < 0$. (Since Howard and Cooper [8] assume $\binom{a}{b} = 0$ when $a < b$, we have modified the upper limit of summation in their formula.)

It is well-known (see most of the references mentioned above) that

$$\sum_{n=0}^{\infty} F_n^{(k)} x^n = \frac{x}{1 - (x + x^2 + \cdots + x^k)}. \tag{6}$$

Let R_k^* be the radius of convergence of power series (6). For $k = 1$, we clearly have $R_1^* = 1$. For $k \geq 2$, using results from Capocelli and Cull [4] and Wolfram [19], we prove in Appendix A that

$$R_k^* > \frac{1}{2 - \frac{1}{2^k}}. \tag{7}$$

Of course, inequality (7) is valid even for $k = 1$.

Generalized k-step Fibonacci numbers (as well related sequences) play a role in describing the probabilistic properties of the run length or waiting time of a process. We model the status of a process, whether it be "in control" or "out of control", with an infinite sequence of events, and define the *run length* (or *waiting time*) of this process to be the first time the process is "out of control". See Equations (29) and (38) in Section 3.

We consider the control statistic of a *moving average (MA) control chart* as our prototype example in deciding whether the sequence of events is in or out of control at a certain time; see, for example, Böhm and Hackl [3], Lai [10], and Zhang et al. [21]. Essentially, an MA control chart is a linear combination (of k

at a time) independent and identically distributed random variables $(X_i)_{i=1}^{\infty}$ with positive coefficients. See Equations (34) and (37) in Section 3.

When the random variables $(X_i)_{i=1}^{\infty}$ form a Bernoulli process with

$$\mathbb{P}(X_i = 1) = p \quad \text{and} \quad \mathbb{P}(X_i = 0) = q \quad \text{for each } i \in \mathbb{Z}_{>0},$$

where $0 < p, q < 1$ and $p+q = 1$, and the MA control chart is the sum of k successive X_i's (e.g., see Equations (53) and (54) in Section 4), we may declare the process to be "out of control" when the MA control chart is greater than a fixed constant c that depends only on k. When $c = k-1$, then the process is "out of control" if and only if k successive 1's are observed; e.g., see Feller [7, Section XIII.7]. See also Equation (43) in Section 4 of our paper.

When $p = q = \frac{1}{2}$, we are dealing with a *symmetric Bernoulli process*; see Equations (42) in Section 4. The probability distribution of the run length of a symmetric Bernoulli process that is "out of control" when k successive 1's occur (i.e., when $c = k-1$) is usually known as the *generalized k-step Bernoulli-Fibonacci distribution.* Of course, one may reduce the value of c and get other kinds of "generalized k-step Bernoulli-Fibonacci" distributions, but in this paper we only concentrate with the case $c = k-1$.

We provide explicit formulas for all the factorial moments of the generalized k-step Bernoulli-Fibonacci distribution. See Theorem 2 in Section 4, which is essentially the main result of the paper. Our explicit formulas involve the subsequence $(F^{(k)}_{(k+1)r+k} : r \in \mathbb{Z}_{\geq 0})$ of the generalized k-step Fibonacci numbers.

The proof of Theorem 2 relies on Theorem 1, which is the main result of Section 2. Theorem 1 gives the generating function of $(F^{(k)}_{(k+1)r+k} : r \in \mathbb{Z}_{\geq 0})$. (For the generating functions of other subsequences of the generalized k-step Fibonacci numbers, see Equation (60) in Section 5.)

The proof of Theorem 1 relies on a number of results about the generalized k-step Fibonacci numbers that involve the roots of unity of order $k+1$ and the inverses of the roots of the characteristic polynomial of the k-step Fibonacci numbers.

In Theorem 2, we also give the *probability generating function (p.g.f.)* of the above k-step Bernoulli-Fibonacci distribution. This p.g.f. was essentially obtained by Medhi [11] and Shane [18] as well. The authors, however, work with a slightly different random length or waiting time: the number of symmetric Bernoulli trials needed until k successive 1's are observed for the first time starting the counting at time k. We start the counting at time 1; in our paper, compare Equations (29) and (34) with Equations (37) and (38).

Even though the p.g.f. obtained by Medhi [11] and Shane [18] is t^{k-1} times our p.g.f. in Theorem 2, the factorial moments obtained using the p.g.f. of those two authors are more complicated than the factorial moments obtained using our p.g.f. For more details, see the discussion after Equations (37) and (38) (at the end of Section 3) and Remark 1 in Section 4 in our paper.

We also note that Medhi [11] did not obtain any higher factorial moments for his/her k-step Bernoulli-Fibonacci distribution. Shane [18] did obtain (in some form) all the factorial moments for his/her k-step Bernoulli-Fibonacci distribution when $k = 2$ (the usual Fibonacci case), but he/she stated that the factorial moments for a general k are not known in closed form. Shane's [18] formula for the case $k = 2$ is essentially a special case of our formula (55) in Section 4.

The organization of our paper is as follows. In Section 2, we derive the generating function of the subsequence $(F^{(k)}_{(k+1)r+k} : r \in \mathbb{Z}_{\geq 0})$ of the generalized k-step Fibonacci numbers. In Section 3, we review the theory of the run length of an MA process. In Section 4, we derive all the factorial moments of both versions of the generalized k-step Bernoulli-Fibonacci distribution (in the symmetric case). Finally, in Section 5, we give some concluding remarks, including an extension of Theorem 1.

Note that, in Section 3, we use Abel's partial summation (or summation by parts): If $A_j = \sum_{i=0}^{j} a_i$ and $B_j = \sum_{i=0}^{j} b_i$, then

$$\sum_{j=0}^{n} a_j B_j = A_n B_n - \sum_{j=1}^{n} A_{j-1} b_j. \tag{8}$$

See, for example, Problem 18.19 in Billingsley [2, p. 244] (where we have corrected a sign error). It is trivial, however, to prove Equation (8) by mathematical induction.

Also in Section 3, we use the *falling factorial*

$$[a]_r := a(a-1)(a-2)\cdots(a-r+1) \quad \text{for } r \in \mathbb{Z}_{>0}. \tag{9}$$

We also let $[a]_0 := 1$. (Note that $[0]_r = 0$ for $r \in \mathbb{Z}_{>0}$ and $[0]_0 = 1$.) It is easy to prove that

$$r \sum_{i=1}^{j} [i-1]_{r-1} = [j]_r \quad \text{for } r \in \mathbb{Z}_{>0} \text{ and } j \in \mathbb{Z}_{\geq 0}. \tag{10}$$

2. Some Results About the Generalized k-step Fibonacci Numbers

The purpose of this section is to prove a result about the generating function of a subsequence of the sequence $(F^{(k)}_n : n \in \mathbb{Z}_{\geq 0})$. In the process of doing so, we prove a number of auxiliary results that are interesting in their own right.

Theorem 1. *For $k \geq 2$, we have*

$$\sum_{r=0}^{\infty} F^{(k)}_{(k+1)r+k} x^r = \frac{2^{k-2}(x+1)}{2^k - \sum_{\ell=1}^{k} 2^{k-\ell}(x+1)^{\ell}} \quad \textit{for } |x| < \left(\frac{1}{2 - \frac{1}{2^k}}\right)^{k+1}. \tag{11}$$

We first prove an auxiliary result about the factorization of a polynomial of degree $k(k+1)$ that is a function of the polynomial that appears in the denominator of the generating function in Equation (11). Denote by $\rho_1, \ldots, \rho_n$ the roots of the polynomial

$$q_k(x) := 1 - x - x^2 - \cdots - x^k = x^k r_k\left(\frac{1}{x}\right), \tag{12}$$

where

$$r_k(y) := y^k - y^{k-1} - \cdots - y - 1. \tag{13}$$

The polynomial $q_k(x)$ is the denominator of the generating function (6), while $r_k(y)$ is the characteristic polynomial of the sequence $(F_n^{(k)} : n \in \mathbb{Z}_{\geq 0})$.

Lemma 1. *For each $k \in \mathbb{Z}_{>0}$, let $d_k(t) = 2^k - 2^{k-1}t - 2^{k-2}t^2 - \cdots - t^k$ and let $\omega_{0,k} = 1$, $\omega_{1,k}$, $\ldots$, $\omega_{k,k}$ be all the roots of unity of order $k+1$. Then*

$$d_k(x^{k+1} + 1) = \prod_{i=0}^{k}(1 - \omega_{i,k}x - \omega_{i,k}^2 x^2 - \cdots - \omega_{i,k}^k x^k). \tag{14}$$

Proof. We treat the case $k = 1$ by itself. In this case, we have $d_1(t) = 2-t$, $\omega_{0,1} = 1$, $\omega_{1,1} = -1$, and

$$d_1(x^2 + 1) = 2 - (x^2 + 1) = 1 - x^2 = \prod_{i=0}^{1}(1 - \omega_{i,1}x).$$

Assume now $k \geq 2$ and define $\phi_k(x)$ to be the polynomial on the right-hand side of Equation (14). We shall prove that the polynomial $d_k(x^{k+1} + 1) - \phi_k(x)$, which is of degree at most $k(k+1)$, vanishes for (at least) $k(k+1)+1$ different values of x. This would prove that $d_k(x^{k+1}+1) - \phi_k(x)$ is identically zero, thus establishing Equation (14) for all x.

None of the roots $\rho_1, \ldots, \rho_k$ of the polynomial $q_k(x)$ in Equation (12) is equal to 0 or 1. We claim that the set

$$A_k := \{\rho_j/\omega_{i,k} : j \in \{1, \ldots, k\},\, i \in \{0, \ldots, k\}\} \cup \{0\} \tag{15}$$

has exactly $k(k+1)+1$ elements. To prove this claim, first note that Equations (12) and (13) imply that $1/\rho_1, \ldots, 1/\rho_k$ are the roots of the polynomial $r_k(y)$, which is the characteristic polynomial of the sequence $(F_n^{(k)} : n \in \mathbb{Z}_{\geq 0})$.

Capocelli and Cull [4] and Wolfram [19] proved that the roots of $r_k(y)$ are simple, so $\rho_1, \ldots, \rho_k$ are distinct. Assume $\rho_j/\omega_{i,k} = \rho_\ell/\omega_{m,k}$ for $1 \leq j, \ell \leq k$ and $0 \leq i, m \leq k$. It follows that

$$\rho_j^{k+1} = (\rho_j/\omega_{i,k})^{k+1} = (\rho_\ell/\omega_{m,k})^{k+1} = \rho_\ell^{k+1}. \tag{16}$$

Since the roots $\rho_1, \ldots, \rho_k$ of $q_k(x)$ are also roots of $(1-x)q_k(x) = 1 - 2x + x^{k+1}$, we have

$$1 - 2\rho_j + \rho_j^{k+1} = 0 = 1 - 2\rho_\ell + \rho_\ell^{k+1}. \tag{17}$$

From Equations (16) and (17), we get $\rho_j = \rho_\ell$, and so $j = \ell$. Since $\rho_j/\omega_{i,k} = \rho_\ell/\omega_{m,k}$, we conclude that $\omega_{i,k} = \omega_{m,k}$, and so $i = m$. Thus, $(j, i) = (\ell, m)$.

Clearly, $\rho_j/\omega_{i,k} \neq 0$ for $1 \le j \le k$ and $0 \le i \le k$. This completes the proof of the claim that $\#A_k = k(k+1)+1$. We shall now prove that $d_k(x^{k+1}+1) - \phi_k(x) = 0$ for all $x \in A_k$, which would establish Equation (14).

Note that $(\rho_j/\omega_{i,k})^{k+1} + 1 = \rho_j^{k+1} + 1 = 2\rho_j$, and thus

$$\begin{aligned} d_k((\rho_j/\omega_{i,k})^{k+1}+1) &= 2^k - 2^{k-1}(2\rho_j) - 2^{k-2}(2\rho_j)^2 - \cdots - (2\rho_j)^k \\ &= 2^k(1 - \rho_j - \rho_j^2 - \cdots - \rho_j^k) = 0. \end{aligned}$$

Also, $\phi_k(\rho_j/\omega_{i,k}) = 0$ because

$$1 - \omega_{i,k}\left(\frac{\rho_j}{\omega_{i,k}}\right) - \omega_{i,k}^2\left(\frac{\rho_j}{\omega_{i,k}}\right)^2 - \cdots - \omega_{i,k}^k\left(\frac{\rho_j}{\omega_{i,k}}\right)^k = 1 - \rho_j - \rho_j^2 - \cdots - \rho_j^k = 0.$$

Finally,

$$d_k(0^{k+1}+1) = 2^k - 2^{k-1} - 2^{k-2} - \cdots - 1 = 1 = \phi_k(0),$$

and this establishes that $d_k(x^{k+1}+1) = \phi_k(x)$ for all $x \in A_k$, which finishes the proof the lemma. $\square$

Lemma 2. *For each $k \in \mathbb{Z}_{>0}$, let $\omega_{0,k} = 1$, $\omega_{1,k}$, $\ldots$, $\omega_{k,k}$ be all the roots of unity of order $k+1$, and let $\rho_1, \rho_2, \ldots, \rho_k$ be the roots of the polynomial $q_k(x)$ defined by Equation (12). For each $m \in \{0, 1, \ldots, k\}$ and each root ρ_n of $q_k(x)$, we have*

$$\prod_{\substack{j=0\\ j\neq m}}^{k} (\omega_{m,k} - \omega_{j,k}) = \frac{k+1}{\omega_{m,k}} \quad \textit{and} \quad \prod_{\substack{j=0\\ j\neq m}}^{k} (\omega_{m,k} - \rho_n\omega_{j,k}) = \frac{2}{\omega_{m,k}}. \tag{18}$$

Proof. We treat the case $k = 1$ by itself. In this case, we have $\omega_{0,1} = 1$, $\omega_{1,1} = -1$, and $\rho_1 = 1$. It is then easy to prove both Equations (18) when $k = 1$.

Assume now $k \ge 2$. We then have

$$\begin{aligned} \prod_{j=0}^{k}(y - \omega_{j,k}) = y^{k+1} - 1 &\Longrightarrow \prod_{\substack{j=0\\ j\neq m}}^{k} (y - \omega_{j,k}) = \frac{y^{k+1}-1}{y - \omega_{m,k}} \\ &\Longrightarrow \prod_{\substack{j=0\\ j\neq m}}^{k} (x\omega_{m,k} - \omega_{j,k}) = \frac{(x\omega_{m,k})^{k+1}-1}{x\omega_{m,k} - \omega_{m,k}} \end{aligned}$$

$$\implies \prod_{\substack{j=0\\ j\neq m}}^{k}(x\omega_{m,k}-\omega_{j,k}) = \frac{1}{\omega_{m,k}}(1+x+x^2+\cdots+x^k).$$

Letting $x = 1$ in the last equation above, we get the first equation in (18). On the other hand,

$$\prod_{\substack{j=0\\ j\neq m}}^{k}(y-\omega_{j,k}) = \frac{y^{k+1}-1}{y-\omega_{m,k}} \implies \prod_{\substack{j=0\\ j\neq m}}^{k}\left(\frac{x}{\rho_n}-\omega_{j,k}\right) = \frac{\left(\frac{x}{\rho_n}\right)^{k+1}-1}{\frac{x}{\rho_n}-\omega_{m,k}}$$
$$\implies \prod_{\substack{j=0\\ j\neq m}}^{k}(x-\rho_n\omega_{j,k}) = \frac{x^{k+1}-\rho_n^{k+1}}{x-\rho_n\omega_{m,k}}.$$

Letting $x = \omega_{m,k}$ in the last equality, we get

$$\prod_{\substack{j=0\\ j\neq m}}^{k}(\omega_{m,k}-\rho_n\omega_{j,k}) = \frac{\omega_{m,k}^{k+1}-\rho_n^{k+1}}{\omega_{m,k}-\rho_n\omega_{m,k}} = \frac{1-\rho_n^{k+1}}{\omega_{m,k}(1-\rho_n)}$$
$$= \frac{1}{\omega_{m,k}}(1+\rho_n+\rho_n^2+\cdots+\rho_n^k) = \frac{2}{\omega_{m,k}}$$

because $1-\rho_n-\rho_n^2-\cdots-\rho_n^k = 0$. This proves the second equation in (18) and completes the proof of the lemma. □

Lemma 3. *For each $k \in \mathbb{Z}_{>0}$, let $\omega_{0,k} = 1$, $\omega_{1,k}$, …, $\omega_{k,k}$ be all the roots of unity of order $k+1$. Then*

$$\sum_{i=0}^{k}\omega_{i,k}^{m+1} = \begin{cases} 0, & \textit{if } m = 0,1,\ldots,k-1;\\ k+1, & \textit{if } m = k. \end{cases}$$

Proof. Clearly, $\sum_{i=0}^{k}\omega_{i,k}^{k+1} = \sum_{i=0}^{k} 1 = k+1$. Thus, we only prove the lemma for $m \in \{0,1,\ldots,k-1\}$.

Following Aigner [1, pp. 161–163], for $r, n \in \mathbb{Z}_{>0}$ with $1 \le n \le r$, we define the elementary symmetric function in r variables

$$a_n(x_1,\ldots,x_r) := \sum_{i_1<\cdots<i_n} x_{i_1}\cdots x_{i_n},$$

where the sum extends over all $\binom{r}{n}$ possible products of n of the variables $x_1,\ldots,x_r$. We also define the power function

$$s_n(x_1,\ldots,x_r) := \sum_{i=1}^{r} x_i^n.$$

Aigner [1, Proposition 4.25] proved the following result that he attributes to Waring:

$$a_n(x_1,\ldots,x_r)=\sum_{\substack{b_1,\ldots,b_n\geq 0\\ \sum_{i=1}^n ib_i=n}}\frac{(-1)^{n-\sum_{i=1}^n b_i}}{\prod_{i=1}^n(b_i!)\prod_{i=1}^n i^{b_i}}s_1^{b_1}\cdots s_n^{b_n}, \tag{19}$$

where $s_j:=s_j(x_1,\ldots,x_r)$ for all $j\in\{1,\ldots,r\}$. In Equation (19), we let $r=k+1$, $x_1=\omega_{0,k}=1$, $x_2=\omega_{1,k}$, …, $x_{k+1}=\omega_{k,k}$, and $n=1,\ldots,k$. It is well-known that

$$a_n(\omega_{0,k},\omega_{1,k},\ldots,\omega_{k,k})=0 \quad \text{for } n=1,\ldots,k.$$

Thus,

$$\sum_{\substack{b_1,\ldots,b_n\geq 0\\ \sum_{i=1}^n ib_i=n}}\frac{(-1)^{n-\sum_{i=1}^n b_i}}{\prod_{i=1}^n(b_i!)\prod_{i=1}^n i^{b_i}}s_1^{b_1}\cdots s_n^{b_n}=0 \quad \text{for } n=1,\ldots,k. \tag{20}$$

To finish the proof of the lemma, we prove by finite induction that

$$s_n(\omega_{0,k},\omega_{1,k},\ldots,\omega_{k,k})=0 \quad \text{for } n=1,\ldots,k.$$

We have $s_1(\omega_{0,k},\omega_{1,k},\ldots,\omega_{k,k})=a_1(\omega_{0,k},\omega_{1,k},\ldots,\omega_{k,k})=0$. Let m be a positive integer less than k and assume $s_n(\omega_{0,k},\omega_{1,k},\ldots,\omega_{k,k})=0$ for $n=1,\ldots,m$. For $n=m+1$ in Equation (20), the only term that does *not* involve $s_j(\omega_{0,k},\omega_{1,k},\ldots,\omega_{k,k})$ for at least one $j\in\{1,\ldots,k\}$ is the one corresponding to $(b_1,b_2,\ldots,b_m,b_{m+1})=(0,0,\ldots,0,1)$. All the other terms are 0 (by the inductive hypothesis). We then get

$$\frac{(-1)^{m+1-1}}{(1!)(m+1)^1}s_{m+1}(\omega_{0,k},\omega_{1,k},\ldots,\omega_{k,k})=0,$$

which implies $s_{m+1}(\omega_{0,k},\omega_{1,k},\ldots,\omega_{k,k})=0$. This completes the finite induction and the proof of the lemma. □

In the next lemma, we evaluate a power series involving the subsequence $(F^{(k)}_{(k+1)r+k}: r\in\mathbb{Z}_{\geq 0})$ of the k-step generalized Fibonacci numbers.

Lemma 4. *For each $k\in\mathbb{Z}_{>0}$, let $\omega_{0,k}=1$, $\omega_{1,k}$, …, $\omega_{k,k}$ be all the roots of unity of order $k+1$. Then*

$$\sum_{r=0}^{\infty}F^{(k)}_{(k+1)r+k}x^{(k+1)r}=\frac{1}{k+1}\sum_{i=0}^{k}\frac{\omega_{i,k}^2x^{1-k}}{1-\omega_{i,k}x-\omega_{i,k}^2x^2-\cdots\omega_{i,k}^kx^k} \tag{21}$$

for $|x|<\frac{1}{2-\frac{1}{2^k}}$.

Proof. It follows from Equation (6) in Section 1 that

$$\sigma_k(x):=\frac{x}{1-(x+x^2+\cdots+x^k)}=\sum_{n=0}^{\infty}F_n^{(k)}\,x^n \quad \text{for } |x|<\frac{1}{2-\frac{1}{2^k}}.$$

Re-arranging the terms of the above (absolutely convergent) power series, we get

$$\sum_{s=0}^{k}\sum_{r=0}^{\infty} F^{(k)}_{(k+1)r+s}x^{(k+1)r+s} = \sigma_k(x) \quad \text{for } |x| < \frac{1}{2-\frac{1}{2^k}}.$$

Thus,

$$\sum_{i=0}^{k}\sum_{s=0}^{k}\sum_{r=0}^{\infty} F^{(k)}_{(k+1)r+s}\omega_{i,k}(\omega_{i,k}x)^{(k+1)r+s} = \sum_{i=0}^{k}\omega_{i,k}\sigma_k(\omega_{i,k}x).$$

It follows that

$$\sum_{s=0}^{k}\sum_{r=0}^{\infty}\left(\sum_{i=0}^{k}\omega_{i,k}^{s+1}\right) F^{(k)}_{(k+1)r+s}x^{(k+1)r+s} = \sum_{i=0}^{k}\omega_{i,k}\sigma_k(\omega_{i,k}x).$$

By Lemma 3, we have $\sum_{i=0}^{k}\omega_{i,k}^{s+1} = 0$ for $s = 0, 1, \ldots, k-1$ and $\sum_{i=0}^{k}\omega_{i,k}^{s+1} = k+1$ for $s = k$. Thus,

$$(k+1)\sum_{r=0}^{\infty} F^{(k)}_{(k+1)r+k}x^{(k+1)r+k} = \sum_{i=0}^{k}\omega_{i,k}\sigma_k(\omega_{i,k}x),$$

from which we can easily derive Equation (21). □

We are now ready to prove the main result of the section, that is, Theorem 1.

Proof of Theorem 1. Assume $k \geq 2$. We first establish that

$$\sum_{i=0}^{k}\frac{\omega_{i,k}^2 x^{1-k}}{1-\omega_{i,k}x-\omega_{i,k}^2x^2-\cdots-\omega_{i,k}^k x^k} = \frac{2^{k-2}(k+1)(x^{k+1}+1)}{2^k-\sum_{\ell=1}^{k}2^{k-\ell}(x^{k+1}+1)^{\ell}}. \tag{22}$$

In view of Lemma 1, this is equivalent to

$$\begin{aligned}\beta_k(x) &:= \sum_{i=0}^{k}\omega_{i,k}^2\prod_{\substack{j=0\\ j\neq i}}^{k}(1-\omega_{j,k}x-\omega_{j,k}^2x^2-\cdots-\omega_{j,k}^k x^k)\\ &= 2^{k-2}(k+1)x^{k-1}(x^{k+1}+1).\end{aligned} \tag{23}$$

To prove Equation (23), we use the same technique we used in the proof of Lemma 1: we show that $\beta_k(x) = 2^{k-2}(k+1)x^{k-1}(x^{k+1}+1)$ for all $x \in A_k$, where the set A_k is defined in Equation (15). Since the degree of the polynomial $\beta_k(x)$ is at most k^2 and $\#A_k = k(k+1)+1$, proving the latter claim is more than enough in establishing Equation (23).

Putting $x = 0$ on both sides of Equation (23), we get $\sum_{i=0}^{k} \omega_{i,k}^2 = 0$, which is true because $k \geq 2$. Setting $x = \rho_n/\omega_{m,k}$ on both sides of Equation (23), we get

$$\omega_{m,k}^2 \prod_{\substack{j=0 \\ j\neq m}}^{k} \left(1 - \omega_{j,k}\left(\frac{\rho_n}{\omega_{m,k}}\right) - \omega_{j,k}^2\left(\frac{\rho_n}{\omega_{m,k}}\right)^2 - \cdots \omega_{j,k}^k\left(\frac{\rho_n}{\omega_{m,k}}\right)^k\right)$$

$$= 2^{k-2}(k+1)\left(\frac{\rho_n}{\omega_{m,k}}\right)^{k-1}\left(\left(\frac{\rho_n}{\omega_{m,k}}\right)^{k+1} + 1\right).$$

The above equation is equivalent to

$$\prod_{\substack{j=0 \\ j\neq m}}^{k} \frac{1 - 2\left(\frac{\rho_n \omega_{j,k}}{\omega_{m,k}}\right) + \left(\frac{\rho_n \omega_{j,k}}{\omega_{m,k}}\right)^{k+1}}{1 - \left(\frac{\rho_n \omega_{j,k}}{\omega_{m,k}}\right)} = 2^{k-2}\rho_n^{k-1}(k+1)(\rho_n^{k+1}+1) \tag{24}$$

$$= 2^{k-1}\rho_n^k(k+1), \tag{25}$$

where we have used the equality $1 - 2\rho_n + \rho_n^{k+1} = 0$. After some simple algebra, we see that Equations (24) and (25) are equivalent to

$$\prod_{\substack{j=0 \\ j\neq m}}^{k} \frac{2\rho_n\left(1 - \frac{\omega_{j,k}}{\omega_{m,k}}\right)}{1 - \left(\frac{\rho_n \omega_{j,k}}{\omega_{m,k}}\right)} = 2^{k-1}\rho_n^k(k+1) \Longleftrightarrow \prod_{\substack{j=0 \\ j\neq m}}^{k} \frac{\omega_{m,k} - \omega_{j,k}}{\omega_{m,k} - \rho_n\omega_{j,k}} = \frac{k+1}{2}.$$

But the last equation follows immediately from Equations (18) in Lemma 2. This means that Equation (23) is true for $x = \rho_n/\omega_{m,k}$. Thus, $\beta_k(x) = 2^{k-2}(k+1)x^{k-1}(x^{k+1}+1)$ for all $x \in A_k$, and this establishes Equation (23) for all x, and thus Equation (22) for all x for which the denominators are not zero.

Lemma 4 and Equation (22) imply that

$$\sum_{r=0}^{\infty} F_{(k+1)r+k}^{(k)} x^{r(k+1)} = \frac{2^{k-2}(x^{k+1}+1)}{2^k - \sum_{\ell=1}^{k} 2^{k-\ell}(x^{k+1}+1)^\ell} \quad \text{for} \quad |x| < \frac{1}{2 - \frac{1}{2^k}}.$$

Given real $y \in [0, 1/(2-1/2^k)^{k+1})$, we let $x = y^{1/(k+1)}$ in the above equation, and we obtain

$$\sum_{r=0}^{\infty} F_{(k+1)r+k}^{(k)} y^r = \frac{2^{k-2}(y+1)}{2^k - \sum_{\ell=1}^{k} 2^{k-\ell}(y+1)^\ell}. \tag{26}$$

Using Equation (26) and the Uniqueness Theorem from Complex Analysis, we may prove that Equation (11) holds for all complex x such that $|x| < \left(\frac{1}{2-\frac{1}{2^k}}\right)^{k+1}$. $\square$

3. Review of the Theory of the Run Length of a Moving Average Process

All random variables are assumed to be defined on a common probability measure space $(\Omega, \mathscr{F}, \mathbb{P})$ (where $\mathscr{F}$ is a σ-algebra of subsets of the sample space Ω). For simplicity, the intersection of events A and B in $\mathscr{F}$ is denoted by AB (and the definition can be extended to the intersection of a finite number of events in $\mathscr{F}$). Consider a sequence of events $(E_i)_{i=1}^{\infty}$ in $\mathscr{F}$ such that

$$\mathbb{P}(E_{i+1}E_{i+2}\cdots E_{i+j}) = \mathbb{P}(E_1E_2\cdots E_j) > 0 \quad \text{for all } i, j \in \mathbb{Z}_{>0}. \tag{27}$$

We also assume

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} E_i\right) = 0. \tag{28}$$

When E_i occurs, we say the "sequence is in control at time i," while when E_i^c occurs, we say "the sequence is out of control at time i." We define the *run length* of the sequence $(E_i)_{i=1}^{\infty}$ by

$$\mathrm{RL} = \inf\{i \in \mathbb{Z}_{>0} : E_i^c\} \quad (\text{where } \inf\emptyset := +\infty). \tag{29}$$

Consider the probabilities

$$p_0 = 1, \quad p_i = \mathbb{P}(E_1E_2\cdots E_i) \quad \text{where } i \in \mathbb{Z}_{>0}.$$

By Equation (27), we have $p_i > 0$ for all $i \in \mathbb{Z}_{>0}$, even though Equation (28) is equivalent to $\lim_{n\to\infty} p_n = 0$. Since

$$\mathbb{P}(\mathrm{RL} = i) = \mathbb{P}(E_1E_2\cdots E_{i-1}E_i^c) = p_{i-1} - p_i \quad \text{for } i \in \mathbb{Z}_{>0},$$

it is also trivial to show that $\mathrm{RL} : \Omega \to \mathbb{R} \cup \{\infty\}$ is an extended random variable with $\mathbb{P}(\mathrm{RL} = +\infty) = 0$ because of Equation (28).

Many authors have studied the *average run length* (ARL) of the sequence of the events $(E_i)_{i=1}^{\infty}$, defined as

$$\mathrm{ARL} = \mathbb{E}(\mathrm{RL}) = \sum_{\ell=1}^{\infty} \ell(p_{\ell-1} - p_\ell).$$

See, for example, Böhm and Hackl [3], Lai [10], and Zhang et al. [21].

Note that, if

$$\lim_{n\to\infty} np_n = 0, \tag{30}$$

we may easily show that

$$\mathrm{ARL} = \sum_{i=0}^{\infty} p_i.$$

Of course, Equation (30) implies Equation (28). In addition, if Equation (30) holds, either both of the series

$$\sum_{\ell=1}^{\infty} \ell(p_{\ell-1} - p_\ell) \quad \text{and} \quad \sum_{\ell=0}^{\infty} p_\ell$$

converge or both diverge to ∞.

We define the r^{th} *factorial moment* of the run length by

$$\mathrm{ARL}_r := \mathbb{E}([\mathrm{RL}]_r) = \sum_{\ell=r}^{\infty} [\ell]_r (p_{\ell-1} - p_\ell) \quad \text{for } r \in \mathbb{Z}_{>0},$$

where the falling factorial $[a]_r$ is defined in Equation (9) in Section 1. Some of these factorial moments, of course, may be infinite. (Note that $\mathrm{ARL} = \mathrm{ARL}_1$.)

Assuming that, for a given $r \in \mathbb{Z}_{>0}$,

$$\lim_{n\to\infty} [n]_r p_n = 0, \tag{31}$$

one can use Equation (10) in Section 1 and summation by parts (Abel partial summation) to show that ARL_r equals

$$\mathrm{ARL}_r = r \sum_{\ell=r-1}^{\infty} [\ell]_{r-1} p_\ell. \tag{32}$$

In more detail, to prove Equation (32), first apply Equation (8) in Section 1 with

$$a_0 = A_0 = -p_0,\ a_j = p_{j-1} - p_j \text{ and } A_j = -p_j \text{ for } j \geq 1,$$

$$b_0 = B_0 = 0,\ b_j = r[j-1]_{r-1} \text{ and } B_j = [j]_r \text{ for } j \geq 1.$$

To prove that $\sum_{i=0}^{j} b_i = B_j$, use Equation (10). Afterwards, let $n \to \infty$ and use Equation (31) and the fact that $[\ell]_r = 0$ for $0 \leq \ell < r$. (Clearly, if Equation (31) holds for $r = r^* \in \mathbb{Z}_{>0}$, then it also holds for all $r \in \mathbb{Z}_{>0}$ with $r \leq r^*$.)

The *probability generating function* (*p.g.f.*) of the run length, RL, of the sequence of events $(E_i)_{i=1}^{\infty}$ is defined by

$$G(t) = E(t^{\mathrm{RL}}) = \sum_{\ell=1}^{\infty} t^\ell (p_{\ell-1} - p_\ell).$$

We have the following result whose proof we omit.

Proposition 1. *Let $(E_i)_{i=1}^{\infty}$ be a sequence of events satisfying conditions (27) and (28) with p.g.f. $G(t)$ for its run length. If R is the radius of convergence of the p.g.f., then $R \geq 1$ and*

$$G(t) = 1 + (t-1) \sum_{\ell=0}^{\infty} p_\ell t^\ell \quad \textit{for } |t| < R. \tag{33}$$

If $p_1 = \mathbb{P}(E_1) < 1$, then $R \geq 1/p_1^{1/k} > 1$, and Equation (33) holds for all $t \in [1, 1/p_1^{1/k})$.

In case $R > 1$, then for all $r \in \mathbb{Z}_{>0}$, the derivative $\frac{d^r G(t)}{dt^r}$ exists at $t = 1$, ARL_r is finite, and

$$\mathrm{ARL}_r = \left.\frac{d^r G}{dt^r}\right|_{t=1}.$$

Next we explain the main statistical model used in this paper. Let k be a fixed positive integer, $(X_n)_{n=1}^{\infty}$ be a sequence of independent random variables with common cumulative distribution function $F : \mathbb{R} \to [0, 1]$, with respect to $(\Omega, \mathscr{F}, \mathbb{P})$, and $(a_i)_{i=1}^{k}$ be a finite sequence of known positive constants. The *moving average control chart* (*MA chart*) has the *control statistic*

$$Y_i = \sum_{j=i}^{i+k-1} a_{i+k-j} X_j = \sum_{s=1}^{k} a_s X_{i+k-s} \quad \text{for } i \in \mathbb{Z}_{>0}. \tag{34}$$

Since the constant coefficients in Equation (34) are the same for each Y_i, the Y_i's are identically distributed. For a fixed constant $c \in \mathbb{R}$, whose value is specified by the practitioner, an "out of control signal" at time $i \in \mathbb{Z}_{>0}$ is indicated by $Y_i > c$, while the event $Y_i \leq c$ indicates that the "the signal is not out of control at time i". We assume that c is chosen so that

$$0 < \mathbb{P}(Y_1 \leq c) < 1 \tag{35}$$

and

$$P(Y_1 \leq c, Y_2 \leq c, \ldots, Y_m \leq c) > 0 \quad \text{for each } m \in \mathbb{Z}_{>0}. \tag{36}$$

The following proposition (whose proof we omit) allows us to frame the "in control at time i" events in terms of the previous theory.

Proposition 2. *For the MA chart $(Y_i)_{i=1}^{\infty}$ defined by Equation (34) and for a fixed $c \in \mathbb{R}$ such that conditions (35) and (36) hold, if we let $E_i := [Y_i \leq c]$ for each $i \in \mathbb{Z}_{>0}$, then the infinite sequence $(E_i)_{i=1}^{\infty}$ satisfies conditions (27), (28), and (31) for each $r \in \mathbb{Z}_{>0}$.*

Unlike Zhang et al. [21], other authors, such as Lai [10] and Ross [17], define the control statistic of an MA chart differently. For example, they define

$$W_s = \sum_{j=s-k+1}^{s} a_{s+1-j} X_j \quad \text{for } s \geq k \text{ and} \tag{37}$$

$$\mathrm{RL}(W) = \inf\{s \in \mathbb{Z}_{>0} : s \geq k \text{ and } W_s > c\}. \tag{38}$$

Denote by $\mathrm{RL}(Y)$ the run length of the sequence of control statistics $(Y_i)_{i=1}^{\infty}$, as defined by Equation (34), and by $\mathrm{RL}(W)$ the run length of the sequence $(W_s)_{s=k}^{\infty}$.

Use similar notation for the average, the factorial moments, and the p.g.f.'s of the run lengths of the two sequences $(Y_i)_{i=1}^{\infty}$ and $(W_s)_{s=k}^{\infty}$. Letting $\mathrm{ARL}_0 := 1$, we then have

$$W_s = Y_{s-k+1} \quad \text{for } s \in \mathbb{Z}_{\geq k}, \quad E_i = [Y_i \leq c] = [W_{i+k-1} \leq c] \quad \text{for } i \in \mathbb{Z}_{>0}, \tag{39}$$

$$\mathrm{RL}(W) = \mathrm{RL}(Y) + (k-1), \quad \mathrm{ARL}(W) = \mathrm{ARL}(Y) + (k-1), \tag{40}$$

$$\mathrm{ARL}_r(W) = r! \sum_{\ell=\max(0,r-k+1)}^{r} \binom{k-1}{r-\ell} \frac{\mathrm{ARL}_\ell(Y)}{\ell!} \quad \text{for } k, r \in \mathbb{Z}_{>0}, \tag{41}$$

and

$$G_W(t) = t^{k-1} G_Y(t) \quad \text{for } |t| < R_Y = R_W.$$

4. The Factorial Moments of the k-step Bernoulli-Fibonacci Distribution

In this section, we generalize an example from Zhang et al. [21] for symmetric Bernoulli processes and we derive the factorial moments of the generalized k-step Bernoulli-Fibonacci distribution. Let $(X_i)_{i=1}^{\infty}$ be a sequence of independent Bernoulli variables with

$$\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2} \quad \text{for } i \in \mathbb{Z}_{>0}. \tag{42}$$

Moreover, consider the MA process given by Equations (34) in Section 3 with $k \in \mathbb{Z}_{>0}$, $a_1 = a_2 = \cdots = a_k = 1$, and $c = k-1$. Then

$$Y_i > c \text{ if and only if } X_i = X_{i+1} = \cdots = X_{i+k-1} = 1. \tag{43}$$

For $i \in \mathbb{Z}_{\geq 0}$, consider the set of binary sequences

$$\Delta_{i,k} = \left\{ (\delta_1, \ldots, \delta_{i+k-1}) \in \{0,1\}^{i+k-1} \mid (\delta_j, \ldots, \delta_{j+k-1}) \neq (1, \ldots, 1) \text{ for } j = 1, \ldots, i \right\}.$$

As in Zhang et al. [21], we obtain the following representation for p_i:

$$\begin{aligned} p_i &= \mathbb{P}(Y_1 \leq k-1, \ldots, Y_i \leq k-1) \\ &= \mathbb{P}\left(\sum_{j=\ell}^{\ell+k-1} X_j \leq k-1 \text{ for } \ell = 1, \ldots, i \right) \\ &= \sum_{(\delta_1, \ldots, \delta_{i+k-1}) \in \Delta_{i,k}} \mathbb{P}\left(X_1 = \delta_1, \ldots, X_{i+k-1} = \delta_{i+k-1} \right) = \frac{\#\Delta_{i,k}}{2^{i+k-1}}. \end{aligned} \tag{44}$$

It is well-known–e.g., see Philippou and Makri [15] or Zhang and Hadjicostas [20]–that

$$\#\Delta_{i,k} = F_{i+k+1}^{(k)} \quad \text{for } i \in \mathbb{Z}_{\geq 0}. \tag{45}$$

Using Equations (2), (6), and (45), we obtain

$$\sum_{i=0}^{\infty}(\#\Delta_{i,k})\,x^i = \frac{2^{k-1} - x - x^2\sum_{\ell=0}^{k-2}(2x)^{\ell}}{(1-x)(1-(x+\cdots+x^k))}. \tag{46}$$

Power series (6) and (46) have the same radius of convergence R_k^*, which satisfies inequality (7).

In the following theorem, we give a simple formula for the r-th factorial moment of the run length $\mathrm{RL}(Y)$ for each $k \in \mathbb{Z}_{>0}$.

Theorem 2. *Let $k \in \mathbb{Z}_{>0}$, let $G_k(t)$ be the p.g.f. of the run length of the sequence of events $(E_i)_{i=1}^{\infty} = ([Y_i \leq k-1])_{i=1}^{\infty}$, and let $\mathrm{ARL}_r^{(k)}$ be the k-th factorial moment of the run length. Then*

$$G_k(t) = \frac{t}{2^k - 2^{k-1}t - 2^{k-2}t^2 - \cdots - t^k} \quad \textit{for } |t| < \frac{1}{1-\frac{1}{2^{k+1}}} \tag{47}$$

$$\textit{and} \quad \mathrm{ARL}_r^{(k)} = \frac{r!F_{(k+1)r+k}^{(k)}}{2^{k-2}} \quad \textit{for } r \in \mathbb{Z}_{>0}. \tag{48}$$

(For $k \geq 2$, we may let $\mathrm{ARL}_0^{(k)} := 1$, and Equation (48) still holds.)

Proof. Clearly, $p_1 = \frac{2^k-1}{2^k} \in (0,1)$ and $p_n > 0$ for each $n \in \mathbb{Z}_{\geq 0}$, and so conditions (35) and (36) hold. By Proposition 2, the infinite sequence $(E_i)_{i=1}^{\infty}$ satisfies conditions (27), (28), and (31) for each $r \in \mathbb{Z}_{>0}$. By Proposition 1 and Equations (44) and (46), the p.g.f. of the run length of the sequence of events $(E_i)_{i=1}^{\infty}$ is

$$\begin{aligned} G_k(t) &= 1 + (t-1)\sum_{\ell=0}^{\infty} p_\ell t^\ell && (49)\\ &= 1 + \frac{t-1}{2^{k-1}}\sum_{\ell=0}^{\infty}\#\Delta_{\ell,k}\left(\frac{t}{2}\right)^{\ell} && (50)\\ &= 1 + \frac{(t-1)\left(2^{k+1} - 2t - t^2\sum_{\ell=0}^{k-2}t^{\ell}\right)}{(2-t)(2^k - 2^{k-1}t - 2^{k-2}t^2 - \cdots - t^k)}. && (51) \end{aligned}$$

After some algebra on Equation (51), we may prove the formula for $G_k(t)$ in Equation (47).

The expansions in the infinite series in (49) and (50) are valid for $\left|\frac{t}{2}\right| < R_k^*$, where R_k^* is the radius of convergence of power series (6) and (46). Because of inequality (7), this means that these power series expansions are valid at least for $|t| < \frac{1}{1-\frac{1}{2^{k+1}}}$.

To prove Equation (48) for $k = 1$, note that

$$1 + \sum_{r=1}^{\infty}\frac{r!F_{2r+1}^{(1)}}{r!2^{1-2}}(t-1)^r = 1 + 2\sum_{r=1}^{\infty}(t-1)^r = \frac{t}{2-t} = G_1(t) \quad \text{for} \quad |t-1| < 1,$$

i.e., for $0 < t < 2$. This means $\mathrm{ARL}_r^{(1)} = \frac{r!F_{2r+1}^{(1)}}{2^{1-2}} = 2r!$ for $r \geq 1$.

Assume now $k \geq 2$. By Theorem 1 in Section 2,

$$\sum_{r=0}^{\infty} F_{(k+1)r+k}^{(k)} x^r = \frac{2^{k-2}(x+1)}{2^k - \sum_{\ell=1}^{k} 2^{k-\ell}(x+1)^\ell} \quad \text{for } |x| < \left(\frac{1}{2-\frac{1}{2^k}}\right)^{k+1}. \tag{52}$$

Thus,

$$\sum_{r=0}^{\infty} \frac{r!F_{(k+1)r+k}^{(k)}}{r!2^{k-2}}(t-1)^r = \frac{t}{2^k - \sum_{\ell=1}^{k} 2^{k-\ell}t^\ell} = G_k(t) \quad \text{for } |t-1| < \left(\frac{1}{2-\frac{1}{2^k}}\right)^{k+1}.$$

Also, $\frac{0!F_{(k+1)0+k}^{(k)}}{2^{k-2}} = \frac{F_k^{(k)}}{2^{k-2}} = 1$ because $k \geq 2$. This proves that

$$\mathrm{ARL}_r^{(k)} = \frac{r!F_{(k+1)r+k}^{(k)}}{2^{k-2}} \quad \text{for } r \geq 0,$$

and the proof of the theorem is complete. □

Remark 1. Medhi [11] and Shane [18] essentially consider the random variables $(W_s : s \geq k)$ defined by Equation (37) with $a_1 = \cdots = a_k = 1$ and the run length $\mathrm{RL}(W)$ defined by Equation (38) with $c = k-1$. This means that, for $s \geq k$,

$$W_s = X_{s-k+1} + \cdots + X_s; \tag{53}$$

i.e., at time $s \geq k$, W_s is the number of 1's we observe in Bernoulli trials $s-k+1$ through s. In contrast, in this section, for $i \geq 1$,

$$Y_i = X_i + \cdots + X_{i+k-1}. \tag{54}$$

That is, at time $i \geq 1$, Y_i is the number of 1's in Bernoulli trials i through $i+k-1$.

Medhi [11, Equation (3.8), p. 218] and Shane [18, Equation (15), p. 521] derived the p.g.f. of $\mathrm{RL}(W)$, which is the number of Bernoulli trials needed to get k successive 1's for the first time starting the counting at time k, while $\mathrm{RL}(Y)$ is the number of Bernoulli trials needed to get k successive 1's for the first time starting the counting at time 1. We see from Equations (39)–(41) that $W_s = Y_{s-k+1}$ for $s \geq k$, and the p.g.f. of $\mathrm{RL}(W)$ is $t^{k-1}G_k(t)$, where $G_k(t)$ is given by Equation (47).

Medhi [11] obtained the p.g.f. of $\mathrm{RL}(W)$, but no higher moments. Shane [18, Equation (9), p. 519], on the other hand, did obtain (in some form) all the factorial moments of $\mathrm{RL}(W)$ for the case $k = 2$, but he stated (on p. 521 of his paper) that no closed form formula is known for a general $k \geq 3$. In addition, because of Equations (41) and (48), it would have been more difficult to identify the higher factorial moments of $\mathrm{RL}(W)$:

$$\mathrm{ARL}_r(W) = \frac{r!}{2^{k-2}} \sum_{\ell=0}^{r} \binom{k-1}{r-\ell} F_{(k+1)\ell+k}^{(k)} \quad \text{for } k \geq 2 \text{ and } r \geq 1. \tag{55}$$

(As usual, $\binom{a}{b} = 0$ for $0 \le a < b$.)

Remark 2. Medhi [11] mentions that his/her formula for the p.g.f. of RL(W) in the symmetric case is a special case of a more general formula by Feller [7, Equations (7.6), Section XIII.7]. If

$$\mathbb{P}(X_i = 1) = p \quad \text{and} \quad \mathbb{P}(X_i = 0) = q \quad \text{for } i \in \mathbb{Z}_{>0}, \tag{56}$$

where $0 < p, q < 1$ with $p + q = 1$, and $(W_s : s \ge k)$ and RL(W) are defined as in Remark 1 above, then the p.g.f. of RL(W) is

$$G_{W,k}(t) = \frac{p^k t^k (1 - pt)}{1 - t + qp^k t^{k+1}} = \frac{p^k t^k}{1 - qt(1 + pt + \cdots + p^{k-1}t^{k-1})}.$$

Feller [7, Equations (7.7), Section XIII.7] proved that

$$\mathbb{E}(\mathrm{RL}(W)) = \frac{1 - p^k}{qp^k} \quad \text{and} \quad \mathrm{Var}(\mathrm{RL}(W)) = \frac{1}{(qp^k)^2} - \frac{2k+1}{qp^k} - \frac{p}{q^2}.$$

In the general case (56), the p.g.f. for the run length RL(Y) of our sequence $(Y_i : i \in \mathbb{Z}_{>0})$ is $t^{-(k-1)} G_{W,k}(t)$. We did not attempt to derive factorial moments for the run length of this (not necessarily symmetric) generalized k-step Bernoulli-Fibonacci distribution, but given our Theorem 2 above, we believe it would be easier to work with the p.g.f. $t^{-(k-1)} G_{W,k}(t)$ (rather than the p.g.f. $G_{W,k}(t)$).

Given related research–see Koutras [9], Philippou and Makri [15], Philippou et al. [14], and Philippou [16]–we strongly believe that the factorial moments of both RL(Y) and RL(W) involve some kind of Fibonacci-like polynomials. For example, Philippou and Makri [15] have calculated the factorial moments of the length L_n of the longest success run in n Bernoulli trials using generalized Fibonacci-like polynomials.

5. Conclusion

Ozdemir and Simsek [12] and Ozdemir et al. [13] defined the Fibonacci-type polynomials $(x, y) \mapsto G_d(x, y; k, m, n)$ through the generating function

$$\sum_{d=0}^{\infty} G_d(x, y; k, m, n)\, t^d = \frac{1}{1 - x^k t - y^m t^{n+m}}, \tag{57}$$

where $k, m, n \in \mathbb{Z}_{\ge 0}$. They proved that

$$G_d(x, y; k, m, n) = \sum_{c=0}^{\lfloor d/(m+n) \rfloor} \binom{d - c(m+n-1)}{c} y^{mc} x^{dk - mck - nck}.$$

Indeed, Equation (4) (due to Dunkel [6]) can be expressed as

$$S(\tilde{k},\tilde{n}) = G_{d=\tilde{n}}(x=2, y=-1; k=1, m=1, n=\tilde{k})$$

and therefore, from Equation (3), we get

$$\begin{aligned} F_{\tilde{n}}^{(\tilde{k})} &= G_{d=\tilde{n}-2}(x=2, y=-1; k=1, m=1, n=\tilde{k}) \\ &\quad - G_{d=\tilde{n}-\tilde{k}-2}(x=2, y=-1; k=1, m=1, n=\tilde{k}). \end{aligned} \tag{58}$$

The integer $\#\Delta_{i,\tilde{k}}$ (see Equation (45)), which equals the number of 0-1 sequences of length $i+\tilde{k}-1$ that avoid $\tilde{k}$ consecutive 1's, can then be expressed as

$$\begin{aligned} \#\Delta_{i,\tilde{k}} = F_{i+\tilde{k}+1}^{(\tilde{k})} &= G_{d=i+\tilde{k}-1}(x=2, y=-1; k=1, m=1, n=\tilde{k}) \\ &\quad - G_{d=i-1}(x=2, y=-1; k=1, m=1, n=\tilde{k}). \end{aligned} \tag{59}$$

Given Equations (58) and (59) above, one might conjecture that the identification of the factorial moments of a non-symmetric Bernoulli-Fibonacci distribution (see Equations (56) in Section 4), or a variation of the Bernoulli-Fibonacci distribution, might be achieved by using the polynomials $(x,y) \mapsto G_d(x,y;k,m,n)$ defined via the generating function in Equation (57). These polynomials are probably related to the Fibonacci-like polynomials in Philippou and Makri [15], who used them to solve related problems. We did not attempt to resolve the related problems for any of these generalizations.

Finally, we mention that the techniques of Section 2 can be modified to give similar results for other subsequences of the generalized k-step Fibonacci numbers. For example, the proof of Lemma 4 can be modified to yield

$$\sum_{r=0}^{\infty} F_{(k+1)(r+1)-m}^{(k)} x^{(k+1)r} = \frac{1}{k+1} \sum_{i=0}^{k} \frac{\omega_{i,k}^{m+1} x^{m-k}}{1-\omega_{i,k}x - \omega_{i,k}^2 x^2 - \cdots \omega_{i,k}^k x^k}$$

for $k,m \in \mathbb{Z}_{\geq 0}$ with $k \geq m+1$ and $|x| < \frac{1}{2-\frac{1}{2^k}}$. As a result, the proof of Theorem 1 can be modified to yield

$$\sum_{r=0}^{\infty} F_{(k+1)(r+1)-m}^{(k)} x^r = \frac{2^{k-1-m}(x+1)^m}{2^k - \sum_{\ell=1}^{k} 2^{k-\ell}(x+1)^\ell} \quad \text{for } |x| < \left(\frac{1}{2-\frac{1}{2^k}}\right)^{k+1}. \tag{60}$$

The above formula is again valid for for $k,m \in \mathbb{Z}_{\geq 0}$ with $k \geq m+1$. We omit the details of the proofs.

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A. Appendix: Proof of Inequality (7) About a Radius of Convergence

Let $k \geq 2$. The radius of convergence R_k^* of power series (6) is the smallest modulus among all k (possibly complex) roots of the polynomial $q_k(x)$, which is defined by Equation (12). Since $q_k(x) = x^k r_k\left(\frac{1}{x}\right)$, where $r_k(y)$ is defined by Equation (13), $\frac{1}{R_k^*}$ is the largest modulus among all k (possibly complex) roots of the polynomial $r_k(y)$.

Capocelli and Cull [4] and Wolfram [19] proved that the characteristic polynomial $r_k(y)$ has a unique positive root $\chi_0^{(k)}$, and this root satisfies

$$2 - \frac{1}{2^{k-1}} < \chi_0^{(k)} < 2 - \frac{1}{2^k}. \tag{61}$$

Furthermore, they proved that any other (possibly complex) root $\chi_i^{(k)}$ of $r_k(y)$ satisfies

$$\frac{1}{\sqrt[k]{3}} < |\chi_i^{(k)}| < 1. \tag{62}$$

It follows from inequalities (61) and (62) that, for $k \geq 2$,

$$\frac{1}{R_k^*} < \max\left(2 - \frac{1}{2^k}, 1\right) = 2 - \frac{1}{2^k} \Longrightarrow R_k^* > \frac{1}{2 - \frac{1}{2^k}}.$$

This proves inequality (7).