SOME DIOPHANTINE PROBLEMS CONCERNING A PAIR OF RATIONAL TRIANGLES WITH A COMMON CIRCUMRADIUS

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Abstract

A triangle with rational sides and rational area is called a rational triangle. In this paper we consider three problems concerned with finding pairs of rational triangles which have a common circumradius as well as a common perimeter or a common inradius or a common area. While several similar problems concerning pairs of rational triangles have been considered in the existing literature, these three problems have not been studied before. For each of these problems, we give a parametric solution and we also indicate how additional parametric solutions of these problems may be obtained.

1. Introduction

A triangle with rational sides and rational area is called a *rational triangle*[1]. Diophantine problems concerning rational triangles have attracted considerable attention. For instance, several mathematicians have considered the problem of finding two rational triangles with a common perimeter and a common area (see [1], [3], [7], [9], [11]). In fact, Choudhry [5] has described a method of generating an arbitrarily large number of scalene rational triangles with a common perimeter and a common area. Skalba and Ulas [10] have considered the problem of finding pairs of Pythagorean triangles with given ratios between catheti. Regarding problems concerning rational triangles with a common circumradius, it has been shown by Lehmer [8, Theorem XI, p. 101] that there exist infinitely many rational triangles with a common circumradius. Further, Andrica and Ţurcaş [2] have recently proved that there are no pairs consisting of a rational right triangle and an isosceles triangle with rational sides having the same circumradius and the same inradius or having the same circumradius and the same perimeter.

[1]Some authors define a rational triangle differently. In this paper, however, all results concerning rational triangles pertain to triangles with rational sides and rational area in accordance with the definition given here.

This paper is concerned with three diophantine problems pertaining to a pair of scalene rational triangles that have the same circumradius. The three problems require that we find pairs of rational triangles with:
(i) a common circumradius and a common perimeter;
(ii) a common circumradius and a common inradius;
(iii) a common circumradius and a common area.

We note that none of the above three problems has been considered earlier in the literature. We obtain parametric solutions of each of the above problems. We also show how more parametric solutions of these problems may be obtained. We note that rational values of the parameters may yield two triangles with rational sides as a solution to any of our three diophantine problems. In each case, we may, after appropriate scaling, readily obtain two triangles whose sides and areas are given by integers and which have the desired properties.

2. Some Basic Formulae Regarding Rational Triangles

In this section we will give basic formulae for the sides, the area, the circumradius and the inradius of a general triangle whose sides and area are rational.

We note that Brahmagupta [6, p. 191] and Euler [6, p. 193] have independently given two sets of formulae for the sides of a general rational triangle. The problem of determining all rational triangles has also been considered by Carmichael [4, pp. 11–13] and by Lehmer [8]. We could try to use these well-known formulae about rational triangles to find pairs of such triangles with a common circumradius and a common perimeter or a common inradius. The resulting diophantine equations are, however, difficult to solve.

We will now derive a new set of formulae for the sides and area of an arbitrary rational triangle. It is interesting to observe that using the new formulae given below we can neatly resolve the problems of finding pairs of rational triangles with a common circumradius and a common perimeter or a common inradius.

Let a, b, c, be the sides of an arbitrary rational triangle. The area, circumradius and inradius of the triangle, denoted by A, R and r, respectively, are given by the following well-known formulae:

$$A = \sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}/4, \tag{2.1}$$

$$R = abc/\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}, \tag{2.2}$$

$$r = \sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}/\{2(a+b+c)\}. \tag{2.3}$$

On making the invertible linear transformation defined by,

$$a = y + z, \quad b = z + x, \quad c = x + y, \tag{2.4}$$

the above formulae may be written as,

$$A = \sqrt{(x+y+z)xyz}, \tag{2.5}$$
$$R = (x+y)(y+z)(x+z)/\{4\sqrt{(x+y+z)xyz}\}, \tag{2.6}$$
$$r = \sqrt{(x+y+z)xyz}/(x+y+z), \tag{2.7}$$

where we note that x, y and z are necessarily nonzero rational numbers.

It follows from (2.5) that the area of the triangle will be rational if and only if there is a nonzero rational number t such that $(x+y+z)x = t^2yz$, so that

$$z = (x+y)x/(t^2y - x), \tag{2.8}$$

and now the values of A, R and r may be written, in terms of nonzero rational numbers x, y and t, as follows:

$$A = txy(x+y)/(t^2y - x), \tag{2.9}$$
$$R = (x^2 + t^2y^2)(t^2+1)/(4(t^2y - x)t), \tag{2.10}$$
$$r = x/t. \tag{2.11}$$

We note that, using the relations (2.4) and the value of z given by (2.8), the sides a, b, c of our triangle may be written, in terms of three arbitrary nonzero parameters x, y and t, as follows:

$$a = (x^2 + t^2y^2)/(t^2y - x), \quad b = xy(t^2+1)/(t^2y - x), \quad c = x + y. \tag{2.12}$$

The perimeter P of the triangle is now given, in terms of x, y and t, by the formula

$$P = 2t^2y(x+y)/(t^2y - x). \tag{2.13}$$

To find pairs of triangles with the desired properties, we will begin with two triangles whose sides may be written, using the formulae (2.12), in terms of arbitrary parameters x_i, y_i, t_i, $i = 1, 2$. We will then impose the desired conditions on the two triangles, and solve the resulting diophantine equations. We will follow this approach in Sections 3.1 and 3.2 to obtain pairs of rational triangles with a common circumradius and a common perimeter or a common inradius.

3. Three Diophantine Problems Concerning Rational Triangles with the Same Circumradius

In the next three subsections we will consider the three problems concerning pairs of rational triangles which have a common circumradius as well as a common perimeter or a common inradius or a common area.

3.1. Pairs of Rational Triangles with a Common Circumradius and a Common Perimeter

We will now obtain examples of pairs of triangles with a common circumradius and a common perimeter.

Let the sides a_1, b_1, c_1 and a_2, b_2, c_2 of the two triangles, respectively, be expressed, using the formulae (2.12), in terms of two sets of arbitrary nonzero rational parameters $x_i, y_i, t_i,\ i = 1, 2$, applicable to the two triangles, respectively.

It follows from the formula (2.10) that if the two triangles have a common circumradius, the parameters $x_i, y_i, t_i,\ i = 1, 2$, must satisfy the following condition:

$$(x_1^2 + t_1^2 y_1^2)(t_1^2 + 1)/(4(t_1^2 y_1 - x_1)t_1) = (x_2^2 + t_2^2 y_2^2)(t_2^2 + 1)/(4(t_2^2 y_2 - x_2)t_2). \quad (3.1)$$

Further, on using the formula (2.13), the condition that the two triangles also have the same perimeter may be be written as follows:

$$2t_1^2 y_1(x_1 + y_1)/(t_1^2 y_1 - x_1) = 2t_2^2 y_2(x_2 + y_2)/(t_2^2 y_2 - x_2). \quad (3.2)$$

We will now solve the simultaneous equations (3.1) and (3.2). On equating either side of Equation (3.2) to m, and solving for x_1 and x_2, we get,

$$x_1 = t_1^2 y_1(m - 2y_1)/(2t_1^2 y_1 + m), \quad x_2 = t_2^2 y_2(m - 2y_2)/(2t_2^2 y_2 + m). \quad (3.3)$$

On substituting the values of x_1 and x_2 given by (3.3), Equation (3.1) may be written as follows:

$$(t_1^2 + 1)(4y_1^2 t_1^2 + m^2)(2y_2 t_2^2 + m)t_2 = (t_2^2 + 1)(4y_2^2 t_2^2 + m^2)(2y_1 t_1^2 + m)t_1. \quad (3.4)$$

Equation (3.4) may be considered as a cubic curve in y_1 and y_2, and a rational point on it, obtained by equating each side of (3.4) to 0, is $(y_1, y_2) = (-m/(2t_1^2), -m/(2t_2^2))$. By drawing a tangent to the curve at this point, and taking its intersection with the curve (3.4), we get a rational solution of (3.4).

We can now obtain a solution of the simultaneous equations (3.1) and (3.2), and using the relations (2.12), we get the sides of the two triangles which have a common circumradius and common perimeter. We omit the tedious details, and simply give below the sides a_1, b_1, c_1, and a_2, b_2, c_2 of the two triangles, obtained after appropriate scaling:

$$\begin{aligned}
a_1 &= t_1(t_2^2+1)(t_1^4t_2^4+3t_1^4t_2^2+4t_1^3t_2^3+3t_1^2t_2^4+t_1^4+2t_1^3t_2+3t_1^2t_2^2\\
&\quad +2t_1t_2^3+t_2^4)(4t_1^6t_2^6+5t_1^6t_2^4-2t_1^5t_2^5+5t_1^4t_2^6+3t_1^6t_2^2\\
&\quad -2t_1^5t_2^3+2t_1^4t_2^4-2t_1^3t_2^5+3t_1^2t_2^6+t_1^6-2t_1^3t_2^3+t_2^6),\\
b_1 &= 2t_1t_2^3(t_1^2+1)^2(t_2^2+1)(t_1^2t_2^2+t_1^2+t_1t_2+t_2^2)\\
&\quad \times(3t_1^5t_2^4+t_1^4t_2^5+3t_1^5t_2^2+t_1^4t_2^3+t_1^3t_2^4-t_1^2t_2^5\\
&\quad +t_1^5+t_1^4t_2+t_1^3t_2^2-t_1^2t_2^3-t_1t_2^4-t_2^5),\\
c_1 &= t_1(t_1+t_2)(t_2^2+1)(3t_1^4t_2^4+3t_1^4t_2^2+3t_1^2t_2^4+t_1^4+t_1^2t_2^2+t_2^4)\\
&\quad \times(2t_1^6t_2^5+2t_1^6t_2^3-t_1^5t_2^4+3t_1^4t_2^5-3t_1^5t_2^2+t_1^4t_2^3\\
&\quad +t_1^3t_2^4+3t_1^2t_2^5-t_1^5-t_1^4t_2-t_1^3t_2^2+t_1^2t_2^3+t_1t_2^4+t_2^5),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
a_2 &= t_2(t_1^2+1)(t_1^4t_2^4+3t_1^4t_2^2+4t_1^3t_2^3+3t_1^2t_2^4+t_1^4+2t_1^3t_2\\
&\quad +3t_1^2t_2^2+2t_1t_2^3+t_2^4)(4t_1^6t_2^6+5t_1^6t_2^4-2t_1^5t_2^5+5t_1^4t_2^6\\
&\quad +3t_1^6t_2^2-2t_1^5t_2^3+2t_1^4t_2^4-2t_1^3t_2^5+3t_1^2t_2^6+t_1^6-2t_1^3t_2^3+t_2^6),\\
b_2 &= 2t_1^3t_2(t_1^2+1)(t_2^2+1)^2(t_1^2t_2^2+t_1^2+t_1t_2+t_2^2)\\
&\quad \times(t_1^5t_2^4+3t_1^4t_2^5-t_1^5t_2^2+t_1^4t_2^3+t_1^3t_2^4+3t_1^2t_2^5\\
&\quad -t_1^5-t_1^4t_2-t_1^3t_2^2+t_1^2t_2^3+t_1t_2^4+t_2^5),\\
c_2 &= t_2(t_1+t_2)(t_1^2+1)(3t_1^4t_2^4+3t_1^4t_2^2+3t_1^2t_2^4+t_1^4+t_1^2t_2^2+t_2^4)\\
&\quad \times(2t_1^5t_2^6+3t_1^5t_2^4-t_1^4t_2^5+2t_1^3t_2^6+3t_1^5t_2^2+t_1^4t_2^3\\
&\quad +t_1^3t_2^4-3t_1^2t_2^5+t_1^5+t_1^4t_2+t_1^3t_2^2-t_1^2t_2^3-t_1t_2^4-t_2^5).
\end{aligned} \tag{3.6}$$

The common circumradius of the above two triangles is

$$\begin{aligned}
&\{(t_1^2+1)(t_2^2+1)(t_1^4t_2^4+3t_1^4t_2^2+4t_1^3t_2^3+3t_1^2t_2^4+t_1^4+2t_1^3t_2+3t_1^2t_2^2+2t_1t_2^3+t_2^4)\\
&\times(4t_1^6t_2^6+5t_1^6t_2^4-2t_1^5t_2^5+5t_1^4t_2^6+3t_1^6t_2^2-2t_1^5t_2^3+2t_1^4t_2^4-2t_1^3t_2^5+3t_1^2t_2^6+t_1^6-2t_1^3t_2^3+t_2^6)\}/4,
\end{aligned}$$

and the common perimeter is

$$4t_1^3t_2^3(t_1+t_2)(t_1^2+1)(t_2^2+1)(t_1^2t_2^2+t_1^2+t_1t_2+t_2^2)(3t_1^4t_2^4+3t_1^4t_2^2+3t_1^2t_2^4+t_1^4+t_1^2t_2^2+t_2^4).$$

As a numerical example, when $t_1 = 2, t_2 = 3$, we get, after appropriate scaling, two triangles with sides $1321940, 1166616, 1636180$ and 991455, $1548096, 1585185$, having a common circumradius $1652425/2$ and a common perimeter 4124736.

It is interesting to observe that when we take $t_1 = 1$, the first triangle becomes a right triangle while taking $t_2 = 1$ makes the second triangle a right triangle. We give below the sides of the two triangles with $t_2 = 1$:

$$\begin{aligned} a_1 &= 2t_1(5t_1^4 + 6t_1^3 + 6t_1^2 + 2t_1 + 1)(13t_1^6 - 4t_1^5 + 7t_1^4 - 4t_1^3 + 3t_1^2 + 1), \\ b_1 &= 4t_1(t_1^2 + 1)^2(2t_1^2 + t_1 + 1)(7t_1^5 + 3t_1^4 + 2t_1^3 - 2t_1^2 - t_1 - 1), \\ c_1 &= 2t_1(t_1 + 1)(7t_1^4 + 4t_1^2 + 1)(4t_1^6 - 5t_1^5 + 3t_1^4 + 4t_1^2 + t_1 + 1), \end{aligned} \tag{3.7}$$

$$\begin{aligned} a_2 &= (t_1^2 + 1)(5t_1^4 + 6t_1^3 + 6t_1^2 + 2t_1 + 1)(13t_1^6 - 4t_1^5 + 7t_1^4 - 4t_1^3 + 3t_1^2 + 1), \\ b_2 &= -8t_1^3(t_1^2 + 1)(2t_1^2 + t_1 + 1)(t_1^5 - 3t_1^4 - 4t_1^2 - t_1 - 1), \\ c_2 &= (t_1 + 1)(t_1^2 + 1)(7t_1^4 + 4t_1^2 + 1)(9t_1^5 + t_1^4 + 4t_1^3 - 4t_1^2 - t_1 - 1). \end{aligned} \tag{3.8}$$

We now have a scalene triangle whose sides are given by (3.7) and a right triangle whose sides are given by (3.8) such that the two triangles have a common circumradius

$$(t_1^2 + 1)(5t_1^4 + 6t_1^3 + 6t_1^2 + 2t_1 + 1)(13t_1^6 - 4t_1^5 + 7t_1^4 - 4t_1^3 + 3t_1^2 + 1)/2$$

and a common perimeter

$$8t_1^3(t_1 + 1)(t_1^2 + 1)(2t_1^2 + t_1 + 1)(7t_1^4 + 4t_1^2 + 1).$$

As a numerical example, when $t_1 = 2$, we get two triangles with sides $500516, 609400, 252324$ and $625645, 123200, 613395$ having common circumradius $625645/2$ and common perimeter 1362240.

We note that for fixed rational values of t_1, t_2 and m, Equation (3.4) reduces to a cubic equation in y_1 and y_2 with a known rational solution, and hence it represents an elliptic curve. Since t_1, t_2 and m are, in fact, arbitrary parameters, Equation (3.4) represents a parametric family of elliptic curves. We note that by simply considering Equation (3.4) as an elliptic curve in y_1 and y_2, and using the known rational point $(y_1, y_2) = (-m/(2t_1^2), -m/(2t_2^2))$ on this curve, we may find more rational points on it by the well-known tangent and chord process or equivalently, by using the group law. These rational points will yield additional parametric solutions of the problem of finding pairs of rational triangles with a common circumradius and a common perimeter.

3.2. Pairs of Rational Triangles with a Common Circumradius and a Common Inradius

We will now obtain two triangles with a common circumradius and a common inradius. As in Section 3.1, let the sides a_i, b_i, c_i, $i = 1, 2$, of the two triangles be expressed, using the formulae (2.12), in terms of arbitrary parameters x_i, y_i, t_i, $i = 1, 2$.

The condition for the two triangles to have a common circumradius is given by (3.1) while, on using the formula (2.11), the condition that they have a common inradius may be written as follows:

$$x_1/t_1 = x_2/t_2. \tag{3.9}$$

We may therefore write $x_1 = mt_1, x_2 = mt_2$, where m is an arbitrary parameter, and now the condition (3.1) reduces to

$$\begin{aligned} t_2(t_1^2+1)y_1^2y_2 - t_1(t_2^2+1)y_1y_2^2 - m(t_1^2+1)y_1^2 + m(t_2^2+1)y_2^2 \\ - m^2t_1(t_2^2+1)y_1 + m^2t_2(t_1^2+1)y_2 - m^3(t_1-t_2)(t_1+t_2) = 0. \end{aligned} \tag{3.10}$$

This is a cubic equation in y_1 and y_2, and it is easily observed that a rational point on the cubic curve defined by (3.10) is given by $(y_1, y_2) = (t_2m, t_1m)$. We now draw a tangent to the cubic curve at the point (t_2m, t_1m), and take its intersection with the curve, and thus obtain a new rational point on the curve (3.10). Using this new rational point, we readily obtain a rational solution of the simultaneous diophantine equations (3.1) and (3.9). Now on using the relations (2.12), we obtain the sides of two triangles with a common circumradius and a common inradius. On appropriate scaling, the sides a_1, b_1, c_1, and a_2, b_2, c_2 of the two triangles may be written as follows:

$$\begin{aligned} a_1 &= t_1(t_2^2+1)(t_1^2t_2^2+t_1^2-8t_1t_2+t_2^2+9), \\ b_1 &= -(t_1t_2^2-t_1-2t_2)(t_1^2+1)(2t_1t_2-t_2^2-3), \\ c_1 &= -2(t_1^2t_2^2-t_1^2-4t_1t_2+t_2^2+3)(t_1^2t_2-2t_1-t_2), \end{aligned} \tag{3.11}$$

$$\begin{aligned} a_2 &= t_2(t_1^2+1)(t_1^2t_2^2+t_1^2-8t_1t_2+t_2^2+9), \\ b_2 &= (t_1^2t_2-2t_1-t_2)(t_2^2+1)(t_1^2-2t_1t_2+3), \\ c_2 &= -2(t_1^2t_2^2+t_1^2-4t_1t_2-t_2^2+3)(t_1t_2^2-t_1-2t_2), \end{aligned} \tag{3.12}$$

where t_1 and t_2 are arbitrary parameters.

The common circumradius of the above two triangles is

$$(t_1^2+1)(t_2^2+1)(t_1^2t_2^2+t_1^2-8t_1t_2+t_2^2+9)/4,$$

while their common inradius is $2(t_1t_2^2-t_1-2t_2)(t_1^2t_2-2t_1-t_2)$.

As a numerical example, when $t_1 = 9/2$ and $t_2 = 7/6$, we get, after appropriate scaling, two triangles with sides $2055, 1105, 3002$, and $4795, 4845, 482$, with their common circumradius and common inradius being $58225/24$ and 228, respectively.

As in the case of the two triangles in Section 3.1 given by (3.5) and (3.6), we note that if we take $t_1 = 1$, the first triangle given by (3.11) becomes a right triangle while on taking $t_2 = 1$, the second triangle given by (3.12) becomes a right triangle.

We give below the sides of the two triangles with $t_2 = 1$:

$$\begin{aligned} a_1 &= 2t_1(t_1^2 - 4t_1 + 5), \\ b_1 &= 2(t_1 - 2)(t_1^2 + 1), \\ c_1 &= 4(t_1 - 1)(t_1^2 - 2t_1 - 1), \end{aligned} \tag{3.13}$$

$$\begin{aligned} a_2 &= (t_1^2 + 1)(t_1^2 - 4t_1 + 5), \\ b_2 &= (t_1^2 - 2t_1 - 1)(t_1^2 - 2t_1 + 3), \\ c_2 &= 4(t_1 - 1)^2. \end{aligned} \tag{3.14}$$

We now have a scalene triangle whose sides are given by (3.13) and a right triangle whose sides are given by (3.14) such that the two triangles have a common circumradius $(t_1^2 + 1)(t_1^2 - 4t_1 + 5)/2$ and a common inradius $2(t_1^2 - 2t_1 - 1)$.

As a numerical example, when $t_1 = 4$, we get two triangles with sides $40, 68, 84$, and $85, 77, 36$, with the common circumradius and common inradius being $85/2$ and 14, respectively.

Equation (3.10) is similar to Equation (3.4) since both of them may be considered as cubic equations in y_1 and y_2 with a known solution while t_1, t_2 and m are considered as arbitrary parameters. Thus, as in the case of Equation (3.4), we note that Equation (3.10) also represents a parametric family of elliptic curves in y_1 and y_2, and, as before, by applying the group law, we can find more rational solutions of Equation (3.10). This will lead to additional parametric solutions of our problem.

3.3. Pairs of Rational Triangles with a Common Circumradius and a Common Area

We will now find pairs of rational triangles with a common circumradius and a common area. It follows from formulae (2.1) and (2.2) that two triangles, with rational sides a_1, b_1, c_1, and a_2, b_2, c_2, respectively, will have a common circumradius and a common area if the following conditions are satisfied:

$$a_1 b_1 c_1 = a_2 b_2 c_2, \tag{3.15}$$

and

$$\begin{aligned} &(a_1 + b_1 + c_1)(a_1 + b_1 - c_1)(b_1 + c_1 - a_1)(c_1 + a_1 - b_1) \\ &\qquad = (a_2 + b_2 + c_2)(a_2 + b_2 - c_2)(b_2 + c_2 - a_2)(c_2 + a_2 - b_2). \end{aligned} \tag{3.16}$$

Further, the common area of the two triangles will be rational if and only if each side of Equation (3.16) is a perfect square.

To solve the simultaneous diophantine equations (3.15) and (3.16), we write,

$$a_1 = pu, \quad b_1 = qv, \quad a_2 = pv, \quad b_2 = qu, \quad c_2 = c_1, \tag{3.17}$$

where p, q, u and v are arbitrary parameters. Now Equation (3.15) is identically satisfied while Equation (3.16) reduces to

$$(u-v)(u+v)(p-q)(p+q)\{2c_1^2-(u^2+v^2)(p^2+q^2)\}=0. \tag{3.18}$$

To obtain a nontrivial solution of Equations (3.15) and (3.16), the last factor on the left-hand side of Equation (3.18) must be equated to 0. We now have a quadratic equation in u, v and c_1, and accordingly, we readily obtain the following solution of Equation (3.18):

$$\begin{aligned} u &= (m^2+2mn-n^2)p-(m^2-2mn-n^2)q, \\ v &= (m^2-2mn-n^2)p+(m^2+2mn-n^2)q, \\ c_1 &= (p^2+q^2)(m^2+n^2), \end{aligned}$$

where m and n are arbitrary parameters.

On substituting the values of u, v and c_1 in the relations (3.17), we obtain the following solution of the simultaneous equations (3.15) and (3.16):

$$\begin{aligned} a_1 &= (m^2+2mn-n^2)p^2-(m^2-2mn-n^2)pq, \\ a_2 &= (m^2-2mn-n^2)p^2+(m^2+2mn-n^2)pq, \\ b_1 &= (m^2-2mn-n^2)pq+(m^2+2mn-n^2)q^2, \\ b_2 &= (m^2+2mn-n^2)pq-(m^2-2mn-n^2)q^2, \\ c_1 &= c_2 = (m^2+n^2)p^2+(m^2+n^2)q^2, \end{aligned} \tag{3.19}$$

where m, n, p and q are arbitrary parameters.

We now have two triangles with a common circumradius and a common area. For the common area to be rational, the values of a_1, b_1, c_1 given by (3.19) must satisfy the condition

$$(a_1+b_1+c_1)(a_1+b_1-c_1)(b_1+c_1-a_1)(c_1+a_1-b_1)=h^2, \tag{3.20}$$

where h is some nonzero rational number.

On using the relations (3.19), the condition (3.20) may be written as follows:

$$\begin{aligned} 16mn(p^2+q^2)^2(m+n)(m-n)(mp+mq-np+nq)(mq-np) \\ \times (mp+nq)(mp-mq+np+nq)=h^2. \end{aligned} \tag{3.21}$$

Now, on writing

$$m=tn, \quad p=uq, \quad h=4v(u^2+1)n^4q^4, \tag{3.22}$$

where t is an arbitrary rational parameter, the condition (3.21) reduces to

$$\begin{aligned} &-(t+1)^2(t-1)^2t^2u^4+(t^2+2t-1)(t^2-2t-1)(t-1)(t+1)tu^3 \\ &\quad +6(t+1)^2(t-1)^2t^2u^2-(t^2+2t-1)(t^2-2t-1)(t-1)(t+1)tu \\ &\quad -(t+1)^2(t-1)^2t^2=v^2. \end{aligned} \tag{3.23}$$

It is readily seen that when $u = 1$, the left-hand side of (3.23) is a perfect square, namely $4(t+1)^2(t-1)^2t^2$. We note that the left-hand side of (3.23) is a quartic function of u, and we know one value of u that makes this quartic function a perfect square. Now on applying a method described by Fermat (as quoted by Dickson [6, p. 639]), we obtain the following value of t that makes the left-hand side of (3.23) a perfect square:

$$\begin{aligned} u = (&t^8 + 8t^7 + 20t^6 - 56t^5 - 26t^4 + 56t^3 + 20t^2 - 8t + 1) \\ &\times (t^8 - 8t^7 + 20t^6 + 56t^5 - 26t^4 - 56t^3 + 20t^2 + 8t + 1)^{-1}, \end{aligned} \tag{3.24}$$

With the value of u given by (3.24), and using the relations (3.19) and (3.22), we get the sides of two rational triangles which have a common circumradius and a common area. On appropriate scaling, the sides a_1, b_1, c_1, and a_2, b_2, c_2 of the two triangles may be written as follows:

$$\begin{aligned} a_1 &= 2t(t^4 - 2t^2 + 5)(5t^4 - 2t^2 + 1)(t^8 + 8t^7 + 20t^6 \\ &\quad - 56t^5 - 26t^4 + 56t^3 + 20t^2 - 8t + 1), \\ b_1 &= (t-1)(t+1)(t^4 - 4t^3 + 10t^2 - 4t + 1)(t^4 + 4t^3 \\ &\quad + 10t^2 + 4t + 1)(t^8 - 8t^7 + 20t^6 + 56t^5 - 26t^4 \\ &\quad - 56t^3 + 20t^2 + 8t + 1), \\ c_1 &= (t^2 + 1)(t^{16} + 104t^{14} - 548t^{12} + 3032t^{10} - 4922t^8 \\ &\quad + 3032t^6 - 548t^4 + 104t^2 + 1), \end{aligned}$$

$$\begin{aligned} a_2 &= (t-1)(t+1)(t^4 - 4t^3 + 10t^2 - 4t + 1)(t^4 + 4t^3 \\ &\quad + 10t^2 + 4t + 1)(t^8 + 8t^7 + 20t^6 - 56t^5 - 26t^4 \\ &\quad + 56t^3 + 20t^2 - 8t + 1), \\ b_2 &= 2t(t^4 - 2t^2 + 5)(5t^4 - 2t^2 + 1)(t^8 - 8t^7 + 20t^6 + 56t^5 \\ &\quad - 26t^4 - 56t^3 + 20t^2 + 8t + 1), \\ c_2 &= (t^2 + 1)(t^{16} + 104t^{14} - 548t^{12} + 3032t^{10} - 4922t^8 \\ &\quad + 3032t^6 - 548t^4 + 104t^2 + 1), \end{aligned}$$

where t is an arbitrary parameter.

The common circumradius of the two triangles is

$$\begin{aligned} &(t^2 + 1)(t^4 - 2t^2 + 5)(5t^4 - 2t^2 + 1)(t^4 - 4t^3 + 10t^2 - 4t + 1)(t^4 + 4t^3 \\ &\quad + 10t^2 + 4t + 1)(t^8 + 8t^7 + 20t^6 - 56t^5 - 26t^4 + 56t^3 + 20t^2 - 8t + 1) \\ &\quad \times (t^8 - 8t^7 + 20t^6 + 56t^5 - 26t^4 - 56t^3 + 20t^2 + 8t + 1)\{2(3t^4 - 6t^2 - 1) \\ &\qquad \times (t^4 + 6t^2 - 3)(t^4 - 4t^3 - 6t^2 - 4t + 1)(t^4 + 4t^3 - 6t^2 + 4t + 1)\}^{-1}, \end{aligned}$$

while their common area is

$$t(t-1)(t+1)(3t^4-6t^2-1)(t^4+6t^2-3)(t^4-4t^3-6t^2-4t+1)(t^4+4t^3 -6t^2+4t+1)(t^{16}+104t^{14}-548t^{12}+3032t^{10}-4922t^8+3032t^6-548t^4+104t^2+1).$$

As a numerical example, when $t=2$ we get two triangles with sides

$$3283540, 7603539, 7776485, \quad \text{and} \quad 4279155, 5834452, 7776485,$$

having a common circumradius 10402718520025/2639802 and also having a common area 12317028393582.

We note that additional solutions of Equation (3.23) may be found by repeated application of the aforementioned method of Fermat. We also note that for a fixed rational value of t, Equation (3.23) represents a quartic model of an elliptic curve. Since t is an arbitrary parameter in Equation (3.23), it follows that Equation (3.23) represents a parametric family of elliptic curves, and using the known rational point on (3.23), we may, by a birational transformation, reduce (3.23) to a parametric family of elliptic curves given by the familiar cubic Weierstrass model. We can find additional rational points on the Weierstrass model of the elliptic curve by using the group law and use these rational points to obtain additional parametric solutions of our problem.

4. Some Open Problems

In each of the three Sections 3.1, 3.2 and 3.3, we have obtained parametric families of elliptic curves. It would be interesting to study various properties of these families of elliptic curves. For instance, it would be interesting to determine whether, for any of the parametric families of elliptic curves obtained in this paper, for fixed values of the parameters, there are additional independent rational points on the related elliptic curve. It would also be of interest to determine the torsion subgroup and the rank of the elliptic curves in each family.

Further, we have obtained pairs of rational triangles with a common circumradius and also having a common perimeter or a common inradius or a common area. It would be of interest to determine whether there are three or more rational triangles with a common circumradius and also having a common perimeter or a common inradius or a common area.

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