GIBONACCI POLYNOMIAL PRODUCTS WITH IMPLICATIONS

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Abstract

We explore infinite products involving gibonacci polynomials, and their Pell and Pell-Lucas implications.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an integer variable; $a(x), b(x), z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas

$$f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \qquad \text{and} \qquad l_n(x) = \alpha^n(x) + \beta^n(x),$$

where $2\alpha(x) = x + \Delta, 2\beta(x) = x - \Delta$, and $\Delta = \sqrt{x^2 + 4}$. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [3].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or $l_n, b_n = p_n$ or q_n.

It follows by the Binet-like formulas that $\lim_{n\to\infty} \frac{l_n}{f_n} = \Delta$, and $\lim_{n\to\infty} \frac{q_n}{p_n} = 2E$, where $E = \sqrt{x^2 + 1}$ and x is fixed.

1.1. Gibonacci Identities

It follows by the Binet-like formulas for f_n and l_n that

$$\begin{aligned} f_{2n} + (-1)^{n+k} f_{2k} &= f_{n+k} l_{n-k}; &(1)\\ f_{2n} - (-1)^{n+k} f_{2k} &= f_{n-k} l_{n+k}. &(2) \end{aligned}$$

These two properties play a major role in our discourse, where we focus on the cases $k = 1$ and $k = 2$.

With this background, we now begin our explorations.

2. Fibonacci Polynomial Products with $k = 1$

When $k = 1$, identities (1) and (2) yield

$$\begin{aligned} f_{2(2n)} + f_2 &= f_{2n-1} l_{2n+1}; &(3)\\ f_{2(2n)} - f_2 &= f_{2n+1} l_{2n-1}; &(4)\\ f_{2(2n+1)} + f_2 &= f_{2n+2} l_{2n}; &(5)\\ f_{2(2n+1)} - f_2 &= f_{2n} l_{2n+2}, &(6) \end{aligned}$$

where $f_2 = x$.

Using identities (3) and (4), we now investigate products involving a special class of even-numbered Fibonacci polynomials.

Theorem 1. *Let $\Delta = \sqrt{x^2 + 4}$. Then*

$$\prod_{n=1}^{\infty} \frac{f_{2(2n)} + f_2}{f_{2(2n)} - f_2} = \frac{f_1}{l_1} \Delta. \tag{7}$$

Proof. Using recursion [3], we will first establish that

$$\prod_{n=1}^{m} \frac{f_{2(2n)} + f_2}{f_{2(2n)} - f_2} = \frac{f_1}{l_1} \cdot \frac{l_{2m+1}}{f_{2m+1}}. \tag{8}$$

To this end, we let A_m denote the left-hand-side of this equation and B_m its right-hand-side. Using identities (3) and (4), we then have

$$\begin{aligned} \frac{B_m}{B_{m-1}} &= \frac{f_{2m-1} l_{2m+1}}{f_{2m+1} l_{2m-1}}\\ &= \frac{f_{2(2m)} + f_2}{f_{2(2m)} - f_2}\\ &= \frac{A_m}{A_{m-1}}. \end{aligned}$$

This implies that $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \cdots = \frac{A_1}{B_1} = \frac{f_4 + f_2}{f_4 - f_2} \cdot \frac{l_1 f_3}{f_1 l_3} = \frac{f_1 l_3}{f_3 l_1} \cdot \frac{l_1 f_3}{f_1 l_3} = 1$. Consequently, $A_m = B_m$, as expected.

Since $\lim_{n\to\infty} \frac{l_n}{f_n} = \Delta$, the given result now follows from Equation (8), as desired. □

It follows from Equation (8) that

$$\begin{aligned} \prod_{n=1}^{m} \frac{F_{2(2n)} + 1}{F_{2(2n)} - 1} &= \frac{L_{2m+1}}{F_{2m+1}}; \\ \prod_{n=1}^{\infty} \frac{F_{2(2n)} + 1}{F_{2(2n)} - 1} &= \sqrt{5}. \end{aligned} \tag{9}$$

Next, we explore products involving another class of even-numbered Fibonacci polynomials.

Theorem 2. *Let* $\Delta = \sqrt{x^2+4}$. *Then*

$$\prod_{n=1}^{\infty} \frac{f_{2(2n+1)} + f_2}{f_{2(2n+1)} - f_2} = \frac{l_2}{f_2} \cdot \frac{1}{\Delta}. \tag{10}$$

Proof. Using recursion [3], we will confirm that

$$\prod_{n=1}^{m} \frac{f_{2(2n+1)} + f_2}{f_{2(2n+1)} - f_2} = \frac{l_2}{f_2} \cdot \frac{f_{2m+2}}{l_{2m+2}}. \tag{11}$$

Suppose A_m denotes the left-hand-side of this equation, and B_m its right-hand side. Using identities (5) and (6), we then have

$$\begin{aligned} \frac{B_m}{B_{m-1}} &= \frac{f_{2m+2} l_{2m}}{f_{2m} l_{2m+2}} \\ &= \frac{f_{2(2m+1)} + f_2}{f_{2(2m+1)} - f_2} \\ &= \frac{A_m}{A_{m-1}}. \end{aligned}$$

Consequently, $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \cdots = \frac{A_1}{B_1} = \frac{f_6 + x}{f_6 - x} \cdot \frac{f_2 l_4}{l_2 f_4} = \frac{l_2 f_4}{f_2 l_4} \cdot \frac{f_2 l_4}{l_2 f_4} = 1$. Thus, $A_m = B_m$.

Clearly, the given result now follows from Equation (11), as expected. □

Equation (10) implies that

$$\prod_{n=1}^{\infty} \frac{F_{2(2n+1)} + 1}{F_{2(2n+1)} - 1} = \frac{3\sqrt{5}}{5}. \tag{12}$$

2.1. A Gibonacci Delight

Equation (8), coupled with Equation (11), yields an interesting consequence:

$$\begin{aligned}\prod_{n=2}^{2m+1}\frac{f_{2n}+f_2}{f_{2n}-f_2} &= \prod_{n=1}^{m}\frac{f_{2(2n)}+f_2}{f_{2(2n)}-f_2}\cdot\prod_{n=1}^{m}\frac{f_{2(2n+1)}+f_2}{f_{2(2n+1)}-f_2}\\ &= \frac{f_1}{l_1}\cdot\frac{l_{2m+1}}{f_{2m+1}}\cdot\frac{l_2}{f_2}\cdot\frac{f_{2m+2}}{l_{2m+2}}\\ &= \frac{f_1}{l_1}\cdot\frac{l_2}{f_2}\cdot\frac{l_{2m+1}}{f_{2m+1}}\cdot\frac{f_{2m+2}}{l_{2m+2}}. \qquad (13)\end{aligned}$$

Consequently, we have ([3])

$$\prod_{n=2}^{\infty}\frac{f_{2n}+f_2}{f_{2n}-f_2}=\frac{f_1}{l_1}\cdot\frac{l_2}{f_2}, \qquad (14)$$

and hence [2, 3]

$$\prod_{n=2}^{\infty}\frac{F_{2n}+1}{F_{2n}-1}=3.$$

2.2. Alternate Forms

Using the identity $l_n^2-\Delta^2 f_n^2=4(-1)^n$ (see identity (31.5) in [3]), if follows from Equations (7), (10), and (14) that

$$\begin{aligned}\prod_{n=1}^{\infty}\frac{l_{2(2n)}^2-\Delta^2x^2-4}{\Delta^2[f_{2(2n)}-f_2]^2} &= \prod_{n=1}^{\infty}\frac{f_{2(2n)}^2-f_2^2}{[f_{2(2n)}-f_2]^2}\\ &= \frac{f_1}{l_1}\Delta; \qquad (15)\\ \prod_{n=1}^{\infty}\frac{l_{2(2n+1)}^2-\Delta^2x^2-4}{\Delta^2[f_{2(2n+1)}-f_2]^2} &= \prod_{n=1}^{\infty}\frac{f_{2(2n+1)}^2-x^2}{[f_{2(2n+1)}-f_2]^2}\\ &= \frac{l_2}{f_2}\cdot\frac{1}{\Delta}; \qquad (16)\\ \prod_{n=2}^{\infty}\frac{l_{2n}^2-\Delta^2x^2-4}{\Delta^2(f_{2n}-f_2)^2} &= \frac{f_1}{l_1}\cdot\frac{l_2}{f_2}, \qquad (17)\end{aligned}$$

respectively.

They yield

$$\prod_{n=1}^{\infty}\frac{L_{2(2n)}^2-9}{5[F_{2(2n)}-1]^2} = \sqrt{5}; \qquad \prod_{n=1}^{\infty}\frac{L_{2(2n+1)}^2-9}{5[F_{2(2n+1)}-1]^2} = \frac{3\sqrt{5}}{5};$$

$$\prod_{n=2}^{\infty}\frac{L_{2n}^2-9}{5(F_{2n}-1)^2} = 3,$$

respectively.

Next, we explore the Pell and Pell-Lucas consequences of Equations (8), (11), and (13).

2.3. Pell and Pell-Lucas Implications

Since $b_n(x) = g_n(2x)$, it follows from Equations (8), (11), and (13) that

$$\begin{aligned}
\prod_{n=1}^{m} \frac{p_{2(2n)} + p_2}{p_{2(2n)} - p_2} &= \frac{p_1}{q_1} \cdot \frac{q_{2m+1}}{p_{2m+1}}; \\
\prod_{n=1}^{m} \frac{p_{2(2n+1)} + p_2}{p_{2(2n+1)} - p_2} &= \frac{q_2}{p_2} \cdot \frac{p_{2m+2}}{q_{2m+2}}; \\
\prod_{n=2}^{2m+1} \frac{p_{2n} + p_2}{p_{2n} - p_2} &= \frac{p_1}{q_1} \cdot \frac{q_2}{p_2} \cdot \frac{q_{2m+1}}{p_{2m+1}} \cdot \frac{p_{2m+2}}{q_{2m+2}},
\end{aligned}$$

respectively, where $p_1 = 1, p_2 = 2x, q_1 = 2x$, and $q_2 = 4x^2 + 2$.

They yield

$$\prod_{n=1}^{\infty} \frac{p_{2(2n)} + p_2}{p_{2(2n)} - p_2} = \frac{E}{x}; \qquad \prod_{n=1}^{\infty} \frac{p_{2(2n+1)} + p_2}{p_{2(2n+1)} - p_2} = \frac{2x^2+1}{2xE};$$

$$\prod_{n=2}^{\infty} \frac{p_{2n} + p_2}{p_{2n} - p_2} = \frac{2x^2+1}{2x^2},$$

respectively.

Consequently, we have

$$\prod_{n=1}^{\infty} \frac{P_{2(2n)} + 2}{P_{2(2n)} - 2} = \sqrt{2}; \qquad \prod_{n=1}^{\infty} \frac{P_{2(2n+1)} + 2}{P_{2(2n+1)} - 2} = \frac{3\sqrt{2}}{4};$$

$$\prod_{n=2}^{\infty} \frac{P_{2n} + 2}{P_{2n} - 2} = \frac{3}{2},$$

again respectively.

Likewise, Equations (15), (16), and (17) yield

$$\begin{aligned}
\prod_{n=1}^{\infty} \frac{q_{2(2n)}^2 - 16E^2x^2 - 4}{4E^2[p_{2(2n)} - p_2]^2} &= \frac{E}{x}; \\
\prod_{n=1}^{\infty} \frac{q_{2(2n+1)}^2 - 16E^2x^2 - 4}{4E^2[p_{2(2n+1)} - p_2]^2} &= \frac{2x+1}{2xE}; \\
\prod_{n=2}^{\infty} \frac{q_{2n}^2 - 16E^2x^2 - 4}{4E^2(p_{2n} - p_2)^2} &= \frac{2x+1}{2x^2},
\end{aligned} \tag{18}$$

respectively.

It follows from Equation (18) that

$$\prod_{n=2}^{\infty} \frac{Q_{2n}^2 - 9}{2(P_{2n} - 2)^2} = \frac{3}{2}.$$

Next, we investigate polynomial products with $k = 2$.

3. Fibonacci Polynomial Products with $k = 2$

With $k = 2$ and $f_4 = x^3 + 2x$, Equations (1) and (2) yield the following identities:

$$\begin{aligned} f_{2(2n)} + f_4 &= f_{2n+2} l_{2n-2}; &(19)\\ f_{2(2n)} - f_4 &= f_{2n-2} l_{2n+2}; &(20)\\ f_{2(2n+1)} + f_4 &= f_{2n-1} l_{2n+3}; &(21)\\ f_{2(2n+1)} - f_4 &= f_{2n+3} l_{2n-1}. &(22)\end{aligned}$$

Theorem 3. *Let $\Delta = \sqrt{x^2+4}$. Then*

$$\prod_{n=2}^{\infty} \frac{f_{2(2n)} + f_4}{f_{2(2n)} - f_4} = \frac{l_2 l_4}{f_2 f_4} \cdot \frac{1}{\Delta^2}. \tag{23}$$

Proof. Using recursion [3], we will first establish that

$$\prod_{n=2}^{m} \frac{f_{2(2n)} + f_4}{f_{2(2n)} - f_4} = \frac{l_2 l_4}{f_2 f_4} \cdot \frac{f_{2m} f_{2m+2}}{l_{2m} l_{2m+2}}. \tag{24}$$

To this end, we let A_m be the left-hand side of this equation and B_m its right-hand side. Using identities (19) and (20), we then have

$$\begin{aligned} \frac{B_m}{B_{m-1}} &= \frac{f_{2m+2} l_{2m-2}}{f_{2m-2} l_{2m+2}}\\ &= \frac{f_{2(2m)} + f_4}{f_{2(2m)} - f_4}\\ &= \frac{A_m}{A_{m-1}}.\end{aligned}$$

This implies that $\dfrac{A_m}{B_m} = \dfrac{A_{m-1}}{B_{m-1}} = \cdots = \dfrac{A_2}{B_2} = \dfrac{f_8 + f_4}{f_8 - f_4} \cdot \dfrac{f_2 f_4}{l_2 l_4} \cdot \dfrac{l_4 l_6}{f_4 f_6} = 1$, again by identities (19) and (20). Consequently, $A_m = B_m$, as expected.

Equation (24) now yields the given result. □

It follows from Equation (24) that

$$\prod_{n=1}^{m} \frac{F_{2(2n)}+3}{F_{2(2n)}-3} = \frac{7F_{2m}F_{2m+2}}{L_{2m}L_{2m+2}}; \qquad \prod_{n=1}^{\infty} \frac{F_{2(2n)}+3}{F_{2(2n)}-3} = \frac{7}{5}.$$

Next, we investigate products involving an equally interesting class of even-numbered Fibonacci polynomials.

Theorem 4. *Let* $\Delta = \sqrt{x^2+4}$. *Then*

$$\prod_{n=1}^{\infty} \frac{f_{2(2n+1)}+f_4}{f_{2(2n+1)}-f_4} = \frac{f_1f_3}{l_1l_3}\Delta^2. \tag{25}$$

Proof. Using recursion [3], we will confirm that

$$\prod_{n=1}^{m} \frac{f_{2(2n+1)}+f_4}{f_{2(2n+1)}-f_4} = \frac{f_1f_3}{l_1l_3} \cdot \frac{l_{2m+1}l_{2m+3}}{f_{2m+1}f_{2m+3}}. \tag{26}$$

Let A_m denote the left side of this equation and B_m its right side. Using identities (21) and (22), we then have

$$\begin{aligned} \frac{B_m}{B_{m-1}} &= \frac{f_{2m-1}l_{2m+3}}{f_{2m+3}l_{2m-1}} \\ &= \frac{f_{2(2m+1)}+f_4}{f_{2(2m+1)}-f_4} \\ &= \frac{A_m}{A_{m-1}}. \end{aligned}$$

Consequently, by identities (21) and (22), we get

$$\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \cdots = \frac{A_1}{B_1} = \frac{f_6+f_4}{f_6-f_4} \cdot \frac{l_1l_3}{f_1f_3} \cdot \frac{f_3f_5}{l_3l_5} = \frac{f_1l_5}{l_1f_5} \cdot \frac{l_1l_3}{f_1f_3} \cdot \frac{f_3f_5}{l_3l_5} = 1.$$

So, $A_m = B_m$, as desired.

The given result now follows from Equation (26). □

Equation (26) yields

$$\prod_{n=1}^{\infty} \frac{F_{2(2n+1)}+3}{F_{2(2n+1)}-3} = \frac{5}{2}.$$

3.1. A Second Gibonacci Delight

It follows by Equations (23) and (25) that

$$\begin{aligned} \prod_{n=3}^{\infty} \frac{f_{2n}+f_4}{f_{2n}-f_4} &= \prod_{n=1}^{\infty} \frac{f_{2(2n)}+f_4}{f_{2(2n)}-f_4} \cdot \prod_{n=1}^{\infty} \frac{f_{2(2n+1)}+f_4}{f_{2(2n+1)}-f_4} \\ &= \frac{l_2l_4}{f_2f_4} \cdot \frac{1}{\Delta^2} \cdot \frac{f_1f_3}{l_1l_3}\Delta^2 \\ &= \frac{f_1f_3}{l_1l_3} \cdot \frac{l_2l_4}{f_2f_4}. \end{aligned} \tag{27}$$

This yields

$$\prod_{n=3}^{\infty} \frac{F_{2n}+3}{F_{2n}-3} = \frac{7}{2}.$$

3.2. Additional Alternate Forms

Using the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [3], we can rewrite Equations (24), (26), and (27) as follows:

$$\begin{aligned}
\prod_{n=2}^{\infty} \frac{l_{2(2n)}^2 - \Delta^2 f_4^2 - 4}{\Delta^2 [f_{2(2n)} - f_4]^2} &= \prod_{n=2}^{\infty} \frac{f_{2(2n)}^2 - f_4^2}{[f_{2(2n)} - f_4]^2} \\
&= \frac{l_2 l_4}{f_2 f_4} \cdot \frac{1}{\Delta^2}; \qquad (28) \\
\prod_{n=1}^{\infty} \frac{l_{2(2n+1)}^2 - \Delta^2 f_4^2 - 4}{\Delta^2 [f_{2(2n+1)} - f_4]^2} &= \prod_{n=1}^{\infty} \frac{f_{2(2n+1)}^2 - f_4^2}{[f_{2(2n+1)} - f_4]^2} \\
&= \frac{f_1 f_3}{l_1 l_3} \Delta^2; \qquad (29) \\
\prod_{n=3}^{\infty} \frac{l_{2n}^2 - \Delta^2 f_4^2 - 4}{\Delta^2 (f_{2n} - f_4)^2} &= \frac{f_1 f_3}{l_1 l_3} \cdot \frac{l_2 l_4}{f_2 f_4}, \qquad (30)
\end{aligned}$$

respectively.

Consequently, we have

$$\prod_{n=2}^{\infty} \frac{L_{2(2n)}^2 - 49}{5[F_{2(2n)} - 3]^2} = \frac{7}{5}; \qquad \prod_{n=1}^{\infty} \frac{L_{2(2n+1)}^2 - 49}{5[F_{2(2n+1)} - 3]^2} = \frac{5}{2};$$

$$\prod_{n=3}^{\infty} \frac{L_{2n}^2 - 49}{5(F_{2n} - 3)^2} = \frac{7}{2},$$

respectively.

3.3. Additional Pell and Pell-Lucas Implications

Since $b_n(x) = g_n(2x)$, it follows from Equations (23), (26), (27), and (30) that

$$\begin{aligned}
\prod_{n=2}^{\infty} \frac{p_{2(2n)} + p_4}{p_{2(2n)} - p_4} &= \frac{q_2 q_4}{p_2 p_4} \cdot \frac{1}{4E^2}; \\
\prod_{n=1}^{\infty} \frac{p_{2(2n+1)} + p_4}{p_{2(2n+1)} - p_4} &= \frac{p_1 p_3}{q_1 q_3} 4E^2; \\
\prod_{n=3}^{\infty} \frac{p_{2n} + p_4}{p_{2n} - p_4} &= \frac{p_1 p_3}{q_1 q_3} \cdot \frac{q_2 q_4}{p_2 p_4};
\end{aligned}$$

$$\prod_{n=3}^{\infty} \frac{q_{2n}^2 - 4E^2p_4^2 - 4}{4E^2(p_{2n} - p_4)^2} = \frac{p_1p_3}{q_1q_3} \cdot \frac{q_2q_4}{p_2p_4},$$

respectively, where $p_3 = 4x^2 + 1, p_4 = 8x^3 + 4x, q_3 = 8x^3 + 6x$, and $q_4 = 16x^4 + 16x^2 + 2$.

They yield

$$\prod_{n=2}^{\infty} \frac{P_{2(2n)} + 12}{P_{2(2n)} - 12} = \frac{17}{16}; \qquad \prod_{n=1}^{\infty} \frac{P_{2(2n+1)} + 12}{P_{2(2n+1)} - 12} = \frac{10}{7};$$
$$\prod_{n=3}^{\infty} \frac{P_{2n} + 12}{P_{2n} - 12} = \frac{85}{56}; \qquad \prod_{n=3}^{\infty} \frac{Q_{2n}^2 - 289}{2(P_{2n} - 12)^2} = \frac{85}{56},$$

again respectively.

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