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MULTIPLICATIVE FUNCTIONS WITH $f(p+q-n_0) = f(p) + f(q) - f(n_0)$

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Abstract

Let n_0 be 1 or 3. If a multiplicative function f satisfies $f(p+q-n_0) = f(p) + f(q) - f(n_0)$ for all primes p and q, then f is the identity function $f(n) = n$ or a constant function $f(n) = 1$.

1. Introduction

In 2016 Chen, Fang, Yuan, and Zheng showed that if a multiplicative function f satisfies $f(p+q+n_0) = f(p) + f(q) + f(n_0)$ with $1 \le n_0 \le 10^6$ then f is the identity function provided $f(p_0) \ne 0$ for some prime p_0 [1]. This is a variation of Spiro's seminal paper in 1992 in which she dealt multiplicative functions satisfying $f(p+q) = f(p) + f(q)$ [7]. She called the set of primes an *additive uniqueness set* for multiplicative functions f with $f(p_0) \ne 0$ for some prime p_0.

A natural question follows about n_0 being negative for the paper of Chen et al. It is natural to consider the condition $f(p+q-n_0) = f(p) + f(q) - f(n_0)$ with $n_0 = 1, 2, 3$ because a multiplicative function is defined on positive integers.

The author already studied a multiplicative function satisfying $f(p+q-2) = f(p) + f(q) - f(2)$, which also yields that the set of numbers 1 less than primes is an additive uniqueness set for multiplicative functions [5].

In this article we classify multiplicative functions satisfying $f(p+q-n_0) = f(p) + f(q) - f(n_0)$ with $n_0 = 1, 3$. For consistency we state the classification for $n_0 = 2$ as well.

Theorem 1. *If a multiplicative function f satisfies $f(p+q-1) = f(p)+f(q)-f(1)$ for all primes p and q, then f is the identity function $f(n) = n$ or a constant function $f(n) = 1$.*

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Theorem 2 ([5])**.** *If a multiplicative function f satisfies $f(p+q-2) = f(p)+f(q)-f(2)$ for all primes p and q, then f is the identity function $f(n) = n$, a constant function $f(n) = 1$, or $f(n) = 0$ for $n \geq 2$ unless n is odd and squareful.*

Theorem 3. *If a multiplicative function f satisfies $f(p+q-3) = f(p)+f(q)-f(3)$ for all primes p and q, then f is the identity function $f(n) = n$ or a constant function $f(n) = 1$.*

Theorem 2 for $n_0 = 2$ has one more option. We give a proof for Theorem 3. We briefly skech the proof of Theorem 1, because it is similar. The proof of Theorem 2 is given in [5, §4].

2. Lemmas

Lemma 1. *Assume a multiplicative function f satisfies $f(p+q-3) = f(p)+f(q)-f(3)$ for all primes p and q. Then, $f(n) = 1$ or $f(n) = n$ for $n = 2, 3, 5, 7,$ and 11.*

Proof. Note that $f(1) = 1$ and the equalities

$$\begin{aligned}
f(1) &= f(2+2-3) = f(2)+f(2)-f(3), \\
f(7) &= f(5+5-3) = f(5)+f(5)-f(3), \\
f(10) &= f(2)\,f(5) \\
&= f(11+2-3) = f(11)+f(2)-f(3), \\
f(11) &= f(7+7-3) = f(7)+f(7)-f(3), \\
f(15) &= f(3)\,f(5) \\
&= f(11+7-3) = f(11)+f(7)-f(3).
\end{aligned}$$

For convenience, let $a = f(2)$, $b = f(3)$, $c = f(5)$, $d = f(7)$, $e = f(11)$. Then,

$$1 = 2a - b \tag{1}$$
$$d = 2c - b \tag{2}$$
$$ac = e + a - b \tag{3}$$
$$e = 2d - b \tag{4}$$
$$bc = e + d - b. \tag{5}$$

Equation (3) becomes
$$ac = 4c - 7a + 4$$
by the Equations (1), (2), and (4). Also, Equation (5) becomes

$$2ac = 7c - 10a + 5.$$

So, $c = 4a - 3$ and we obtain an equation $a^2 - 3a + 2 = 0$.

Thus, $a = 1$ or $a = 2$ and it follows that

$$\begin{array}{lllll} f(2) = 1, & f(3) = 1, & f(5) = 1, & f(7) = 1, & f(11) = 1; \\ f(2) = 2, & f(3) = 3, & f(5) = 5, & f(7) = 7, & f(11) = 11. \end{array}$$

□

Lemma 2. *The results in Lemma 1 can be extended up to n odd and $n < 10^{10}$.*

Proof. We use induction. Let n be odd and $11 < n < 10^{10}$.

If n is prime, then $n = 6k - 1$ or $n = 6k + 1$. Suppose $n = 6k - 1$. Note that

$$f(n + 4) = f(6k + 3) = f(n + 7 - 3) = f(n) + f(7) - f(3).$$

Since $6k + 3$ can be factored into the product of two smaller integers, $f(6k+3) = 1$ or $f(6k + 3) = 6k + 3$ by the induction hypothesis. Thus, $f(n) = 1$ or $f(n) = n$ when $n = 6k - 1$ is prime.

Similarly, if n is a prime of the form $6k + 1$, then $f(n) = 1$ or $f(n) = n$ by

$$f(n + 2) = f(6k + 3) = f(n + 5 - 3) = f(n) + f(5) - f(3).$$

If n is not a prime, n is either a product of two relatively prime integers or a power of a prime. The first case is easy by the multiplicity of f. So the second case remains.

Now, assume that n is a power of a prime with exponent ≥ 2. Then, $n + 3$ is even and can be written as a sum of two primes p and q with $5 \leq p, q < n$ by the numerical verification of the Goldbach Conjecture up to 4×10^{18} [4].

Then, since $f(n) = f(n + 3 - 3) = f(p + q - 3) = f(p) + f(q) - f(3)$, we obtain that $f(n) = 1$ or $f(n) = n$ by the induction hypothesis. □

Indeed, those can be extended up to $n \leq 4 \times 10^{18} - 3$.

Lemma 3. *The results in Lemma 1 can be extended up to n even and $n < 10^{10}$.*

Proof. It is enough to investigate $f(2^r)$ with $r \leq 33$. Note that $k \cdot 2^r + 1$ with $k < 2^r$ is called Proth number. If a Proth number is prime, it is called a Proth prime. It is verified that there exists an odd integer $k \leq 4141$ such that $k \cdot 2^r + 1$ is a Proth prime for $1 \leq r \leq 1000$ in The On-Line Encyclopedia of Integer Sequences (OEIS, `https://oeis.org/A057778`), although the infinitude of Proth primes is not yet proved [6].

Then, $k \cdot 2^r + 1$ is an odd prime and

$$f(k)\, f(2^r) = f\big((k \cdot 2^r + 1) + 2 - 3\big) = f(k \cdot 2^r + 1) + f(2) - f(3).$$

Thus, we are done by Lemmas 1 and 2. □

If the Goldbach Conjecture and the infinitude of Proth primes for all exponents were proved, Theorem 3 could be easily proved. But, neither of them has not yet been proved, so that we need other strategy. In the following lemma, $v_p(n)$ means the exponent of p in the prime factorization of n when p is a prime and n is a positive integer. The set H was defined by Spiro and the numerical verification of the Goldbach Conjecture was up to 2×10^{10} at that time. We would call the set H in the lemma the *Spiro set*.

Lemma 4. *Let*

$$H = \{n \mid v_p(n) \leq 1 \text{ if } p > 1000; v_p(n) \leq \lfloor 9 \log_p 10 \rfloor - 1 \text{ if } p < 1000\}.$$

For any integer $m > 10^{10}$, there is an odd prime $q \leq m - 1$ such that $m + q \in H$.

Proof. This lemma is the consequence of [1, Lemma 2.4] which follows the proof of [7, Lemma 5]. □

Lemma 5 ([2, 3, 8])**.** *Almost every even positive integer is expressible as the sum of two primes.*

Lemma 6. *The restricted function $f|_H$ is the identity function or a constant function on H.*

Proof. Assume $f(n) = n$ for $n = 2, 3, 5, 7, 11$. If $n < 10^{10}$, then $f(n) = n$ from Lemmas 2 and 3. Let $n \in H$ with $n \geq 10^{10}$ and assume that $f(m) = m$ for all $m \in H$ with $m < n$. If n is not a prime power, then $f(n) = f(a)f(b)$ with $(a, b) = 1$ and $a, b > 1$. Since $f(a) = a$ and $f(b) = b$ by the induction hypothesis, $f(n) = n$.

Now, if n is a prime power, then n is a prime by the definition of H. If $n = 6k-1$, then consider $n + 7 - 3 = 6k + 3$. Since

$$f(n + 4) = f(6k + 3) = f(n + 7 - 3) = f(n) + f(7) - f(3)$$

and $6k + 3$ can be factored into the product of two smaller integers, $f(n) = n$.

Similarly, if $n = 6k + 1$, then

$$f(n + 2) = f(6k + 3) = f(n + 5 - 3) = f(n) + f(5) - f(3)$$

yields $f(n) = n$.

By the same reasoning, we can conclude that $f(n) = 1$ if

$$f(2) = f(3) = f(5) = f(7) = f(11) = 1.$$

□

Lemma 7 ([7, Lemma 7]). *For any positive integer n, put*

$$H_n = \begin{cases} \{mn : m \in H, (m,n) = 1\} & \text{if } 2 \mid n; \\ \{2mn : 2m \in H, (m,n) = 1\} & \text{if } 2 \nmid n. \end{cases}$$

Then H_n satisfies the following properties:

1. *Every element of H_n is even.*
2. *The set H_n has positive lower density.*

3. Proofs of Theorems

Let us start to prove Theorem 3. Suppose that there exists n for which $f(n) \neq n$. For $kn \in H_n$, we have that

$$f(kn) = f(k)\, f(n) = k\, f(n).$$

If $f(kn) = kn$, then $f(n) = f(kn)/k = kn/k = n$, which contradicts. So $f(kn) \neq kn$ for every $kn \in H_n$.

But, if $kn + 3$ with k odd can be represented as a sum of two primes p and q, then

$$f(kn) = f(p + q - 3) = f(p) + f(q) - f(3) = p + q - 3 = kn.$$

Thus, this implies that there exist many counterexamples to the Goldbach Conjecture whose density is positive. But, this contradicts Lemma 5. Therefore, $f(n) = n$ for all n.

We can prove Theorem 1 in the similar way. First, we have that

$$\begin{aligned} f(3) &= f(2 + 2 - 1) = f(2) + f(2) - f(1), \\ f(5) &= f(3 + 3 - 1) = f(3) + f(3) - f(1), \\ f(6) &= f(2)\, f(3) \\ &= f(5 + 2 - 1) = f(5) + f(2) - f(1). \end{aligned}$$

Let $a = f(2)$, $b = f(3)$, and $c = f(5)$. Then,

$$b = 2a - 1, \qquad c = 2b - 1, \qquad ab = c + a - 1.$$

Thus,

$$a(2a - 1) = \big(2(2a - 1) - 1\big) + a - 1$$

and it becomes $a^2 - 3a + 2 = 0$. Hence, $a = 1$ or $a = 2$.

Next, we should check $f(2^r)$ as in Lemma 3. We can use $k \cdot 2^r - 1$ instead of $k \cdot 2^r + 1$. The list of prime $k \cdot 2^r - 1$ with $0 \leq r \leq 10000$ is in OEIS (`https://oeis.org/A126717`). See also [6].

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