#A114 INTEGERS 21 (2021)

# FINITE SUMS OF CONSECUTIVE TERMS OF A SECOND ORDER LINEAR RECURRENCE RELATION

**Curtis Cooper**
*School of Computer Science and Mathematics, University of Central Missouri, Warrensburg, Missouri*
cooper@ucmo.edu

*Received: 6/10/21, Accepted: 11/15/21, Published: 11/23/21*

**Abstract**

Let $a$, $b$, $p$, and $q$ be integers. Let $W_0(a, b; p, q) = a$, $W_1(a, b; p, q) = b$, and for $n \geq 2$, let $W_n(a, b; p, q) = pW_{n-1}(a, b; p, q) + qW_{n-2}(a, b; p, q)$. Let $n$ and $m$ be nonnegative integers. We will find a formula for $\sum_{i=n}^{n+m} W_i(a, b; 1, q)$.

## 1. Introduction

We begin by defining a Horadam sequence [1].

**Definition 1.** Let $a$, $b$, $p$, and $q$ be integers. Let $W_0(a, b; p, q) = a$, $W_1(a, b; p, q) = b$, and for $n \geq 2$, let

$$W_n(a, b; p, q) = pW_{n-1}(a, b; p, q) + qW_{n-2}(a, b; p, q).$$

Let $X_n(a, b; p, q) = W_{n+1}(a, b; p, q) + W_{n-1}(a, b; p, q)$ for $n \geq 1$. Let $U_n = W_n(0, 1; p, q)$ and $V_n = X_n(0, 1; p, q)$ for $n \geq 0$. Finally, we define $H_n(a, b) = W_n(a, b; 1, 1)$ for $n \geq 0$.

When there is no confusion regarding the initial values $a$ and $b$ and recurrence coefficients $p$ and $q$, we write $W_n(a, b; p, q)$ as $W_n$. Note that the Fibonacci numbers are $F_n = W_n(0, 1; 1, 1)$, the Lucas numbers are $L_n = W_n(2, 1; 1, 1)$, the Pell numbers are $P_n = W_n(0, 1; 2, 1)$, and the Jacobsthal numbers are $J_n = W_n(0, 1; 1, 2)$.

Several authors have studied this topic. Russell [5, 6] did some initial work. Melham [2] showed that for nonnegative integers $n$ and $m$,

$$\sum_{i=n}^{n+m} W_i(a, b; p, 1)$$
$$= \begin{cases} \frac{1}{p} V_{(m+1)/2}(a, b; p, 1)(W_{(2n+m+1)/2}(a, b; p, 1) + W_{(2n+m-1)/2}(a, b; p, 1)), \\ \qquad \text{if } m \equiv 1 \pmod 4; \\ \frac{1}{p} U_{(m+1)/2}(a, b; p, 1)(X_{(2n+m+1)/2}(a, b; p, 1) + X_{(2n+m-1)/2}(a, b; p, 1)), \\ \qquad \text{if } m \equiv 3 \pmod 4. \end{cases}$$

We will find a formula for

$$\sum_{i=n}^{n+m} W_i(a,b;1,q)$$

for integers $a$, $b$, and $q$.

To state some of our theorems, we need to define the general $r$th order linear recurrence, where $r \geq 2$ is an integer.

**Definition 2.** Let $n \geq 0$ and $r \geq 1$ be integers. Also, let $\{a_i\}_{i=0}^{r-1}$ and $\{c_i\}_{i=1}^{r}$ be two finite sequences of $r$ integers. The general $r$th order Horadam sequence is defined by

$$W_n(a_0,a_1,\ldots,a_{r-1};c_1,c_2,\ldots,c_r) \\ = \begin{cases} a_n, & \text{if } 0 \leq n \leq r-1; \\ \sum_{i=1}^{r} c_i W_{n-i}(a_0,a_1,\ldots,a_{r-1};c_1,c_2,\ldots,c_r), & \text{if } n \geq r. \end{cases}$$

In Section 2, we give a definition and state and prove a helpful lemma. Also, in Section 2, we compute a finite sum of terms involving consecutive $U$s. Then, in Section 3, we will compute a finite sum of consecutive $W$s, where $a$, $b$, and $q$ are integers and $p = 1$.

## 2. Finite Sums Involving Consecutive $U$s

**Definition 3.** Let $n \geq 0$ and $p$ and $q$ be integers. Let

$$Y_n = W_n(0,q,2pq;2p,q-p^2,-pq).$$

Note that when $p = q = 1$, we have $Y_n(0,1,2;2,0,-1) = F_{n+2} - 1$ [3, A000071]. Also, when $p = 2$ and $q = 1$, $Y_n(0,1,4;4,-3,-2)$ is [3, A094706] and is the convolution of $P_n$ and $2^n$. And, when $p = 1$ and $q = 2$, we define $G_n = Y_n(0,2,4;2,1,-2)$, which is [3, A167030].

**Lemma 1.** *Let $p$ and $q$ be integers. Then*

$$U_n = \sum_{i \geq 0} \binom{n-1-i}{i} p^{n-1-2i} q^i \text{ for } n \geq 0, \tag{1}$$

$$Y_n = \sum_{i \geq 0} \binom{n-i}{i+1} p^{n-1-2i} q^{i+1} \text{ for } n \geq 2. \tag{2}$$

We will prove the formula for $Y_n$. The proof of the formula for $U_n$ is similar and will be omitted.

*Proof.* The proof is by induction on $n$.

`Base Step`. For $n = 2$, the left-hand side of (2) is

$$Y_2 = W_2(0, q, 2pq; 2p, q - p^2, -pq) = 2pq.$$

The right-hand side of (2) is

$$\sum_{i\geq 0}\binom{2-i}{i+1}p^{2-1-2i}q^{i+1} = \binom{2}{1}p^1q^1 = 2pq.$$

Therefore, the base step is true for $n = 2$.

For $n = 3$, the left-hand side of (2) is

$$Y_3 = W_3(0, q, 2pq; 2p, q-p^2, -pq) = 2p\cdot 2pq+(q-p^2)\cdot q = 4p^2q-q^2-p^2q = 3p^2q+q^2.$$

The right-hand side of (2) is

$$\sum_{i\geq 0}\binom{3-i}{i+1}p^{3-1-2i}q^{i+1} = \binom{3}{1}p^2q^1 + \binom{2}{2}p^0q^2 = 3p^2q + q^2.$$

Therefore, the base step is true for $n = 3$.

For $n = 4$, the left-hand side of (2) is

$$\begin{aligned}Y_4 &= W_4(0, q, 2pq; 2p, q - p^2, -pq)\\ &= 2p\cdot(3p^2q + q^2) + (q - p^2)\cdot 2pq + (-pq)\cdot q\\ &= 6p^3q + 2pq^2 + 2pq^2 - 2p^3q - pq^2\\ &= 4p^3q + 3pq^2.\end{aligned}$$

The right-hand side of (2) is

$$\sum_{i\geq 0}\binom{4-i}{i+1}p^{4-1-2i}q^{i+1} = \binom{4}{1}p^3q^1 + \binom{3}{2}p^1q^2 = 4p^3q + 3pq^2.$$

Therefore, the base step is true for $n = 4$.

`Induction Step`. Let $n \geq 5$ and assume that the result is true $n-3$, $n-2$, and $n-1$. We will prove the result for $n$. By the induction hypothesis, we know that

$$\begin{aligned}Y_{n-3} &= \sum_{i\geq 0}\binom{n-3-i}{i+1}p^{n-4-2i}q^{i+1},\\ Y_{n-2} &= \sum_{i\geq 0}\binom{n-2-i}{i+1}p^{n-3-2i}q^{i+1},\\ Y_{n-1} &= \sum_{i\geq 0}\binom{n-1-i}{i+1}p^{n-2-2i}q^{i+1}.\end{aligned}$$

Then, using the definition of $Y_n$, we have

$$\begin{aligned}Y_n &= 2pY_{n-1} + (q-p^2)Y_{n-2} - pqY_{n-3}\\ &= 2p\sum_{i\ge 0}\binom{n-1-i}{i+1}p^{n-2-2i}q^{i+1} + (q-p^2)\sum_{i\ge 0}\binom{n-2-i}{i+1}p^{n-3-2i}q^{i+1}\\ &\quad - pq\sum_{i\ge 0}\binom{n-3-i}{i+1}p^{n-4-2i}q^{i+1}.\end{aligned}$$

Rearranging terms and using the binomial coefficient recurrence relation

$$\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1}, \tag{3}$$

twice, we know that for each $i \ge 0$,

$$\begin{aligned}&2\binom{n-1-i}{i+1}p^{n-1-2i}q^{i+1} + \binom{n-2-i}{i+1}p^{n-3-2i}q^{i+2}\\ &\quad - \binom{n-2-i}{i+1}p^{n-1-2i}q^{i+1} - \binom{n-3-i}{i+1}p^{n-3-2i}q^{i+2}\\ &= \binom{n-1-i}{i+1}p^{n-1-2i}q^{i+1}\\ &\quad + \left(\binom{n-1-i}{i+1} - \binom{n-2-i}{i+1}\right)p^{n-1-2i}q^{i+1}\\ &\quad + \left(\binom{n-2-i}{i+1} - \binom{n-3-i}{i+1}\right)p^{n-3-2i}q^{i+2}\\ &= \binom{n-1-i}{i+1}p^{n-1-2i}q^{i+1} + \binom{n-2-i}{i}p^{n-1-2i}q^{i+1}\\ &\quad - \binom{n-3-i}{i}p^{n-3-2i}q^{i+2}.\end{aligned}$$

Replacing $i$ by $i-1$ in the last term of this expression, we have

$$\binom{n-1-i}{i+1}p^{n-1-2i}q^{i+1} + \binom{n-2-i}{i}p^{n-1-2i}q^{i+1} + \binom{n-2-i}{i-1}p^{n-1-2i}q^{i+1}.$$

Combining the last two terms in this expression using (3), we have

$$\binom{n-1-i}{i+1}p^{n-1-2i}q^{i+1} + \binom{n-1-i}{i}p^{n-1-2i}q^{i+1}.$$

Finally, combining the two remaining terms using (3), we have

$$\binom{n-i}{i+1}p^{n-1-2i}q^{i+1}.$$

Summing this last expression for all $i \ge 0$ gives us is what we wanted to prove. Therefore, the induction step is true. This completes the proof by induction. □

We can arrange the terms of the polynomials $U_n$ and $Y_n$ in the following table. The sums of the rows of the table add up to $U_n$. We will show that the terms of $Y_n$ appear in the table.

| $n$ | Sum | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | |
| 1 | 1 | | | | | | |
| 2 | $1p$ | | | | | | |
| 3 | $1p^2$ | $1q$ | | | | | |
| 4 | $1p^3$ | $2pq$ | | | | | |
| 5 | $1p^4$ | $3p^2q$ | $1q^2$ | | | | |
| 6 | $1p^5$ | $4p^3q$ | $3pq^2$ | | | | |
| 7 | $1p^6$ | $5p^4q$ | $6p^2q^2$ | $1q^3$ | | | |
| 8 | $1p^7$ | $6p^5q$ | $10p^3q^2$ | $4pq^3$ | | | |
| 9 | $1p^8$ | $7p^6q$ | $15p^4q^2$ | $10p^2q^3$ | $1q^4$ | | |
| 10 | $1p^9$ | $8p^7q$ | $21p^5q^2$ | $20p^3q^3$ | $5pq^4$ | | |
| 11 | $1p^{10}$ | $9p^8q$ | $28p^6q^2$ | $35p^4q^3$ | $15p^2q^4$ | $1q^5$ | |
| 12 | $1p^{11}$ | $10p^9q$ | $36p^7q^2$ | $56p^5q^3$ | $35p^3q^4$ | $6pq^5$ | |
| 13 | $1p^{12}$ | $11p^{10}q$ | $45p^8q^2$ | $84p^6q^3$ | $70p^4q^4$ | $21p^2q^5$ | $1q^6$ |
| 14 | $1p^{13}$ | $12p^{11}q$ | $55p^9q^2$ | $120p^7q^3$ | $126p^5q^4$ | $56p^3q^5$ | $7pq^6$ |

Now, we have a corollary to Lemma 1.

**Corollary 1.** *Let $n \geq 4$. Then*

$$U_n = p^{n-1} + Y_{n-2}.$$

*Proof.* Let $n \geq 4$, $p$, and $q$ be integers. By Lemma 1,

$$U_n = \sum_{i \geq 0} \binom{n-1-i}{i} p^{n-1-2i} q^i,$$

$$Y_{n-2} = \sum_{i \geq 0} \binom{n-2-i}{i+1} p^{n-3-2i} q^{i+1}.$$

Thus,

$$\begin{aligned} U_n &= p^{n-1} + \sum_{i \geq 1} \binom{n-1-i}{i} p^{n-1-2i} q^i \\ &= p^{n-1} + \sum_{i \geq 0} \binom{n-2-i}{i+1} p^{n-3-2i} q^{i+1} \\ &= p^{n-1} + Y_{n-2}. \end{aligned}$$

This completes the proof. □

**Lemma 2.** *Let $n$ and $m$ be nonnegative integers. Then*

$$W_n(Y_{m+1}, Y_{m+2}; p, q) - pW_n(Y_m, Y_{m+1}; p, q) = qU_{n+m+1}. \tag{4}$$

*Proof.* We fix $m$ and prove the lemma by induction on $n$. Note that by Lemma 1,

$$\begin{aligned} Y_m &= \sum_{i\geq 0} \binom{m-i}{i+1} p^{m-1-2i} q^{i+1}, \\ Y_{m+1} &= \sum_{i\geq 0} \binom{m+1-i}{i+1} p^{m-2i} q^{i+1}, \\ Y_{m+2} &= \sum_{i\geq 0} \binom{m+2-i}{i+1} p^{m+1-2i} q^{i+1}. \end{aligned}$$

`Base Step`. For $n = 0$, the left-hand side of (4) is

$$\begin{aligned} &W_0(Y_{m+1}, Y_{m+2}; p, q) - pW_0(Y_m, Y_{m+1}; p, q) = Y_{m+1} - pY_m \\ &= \sum_{i\geq 0} \binom{m+1-i}{i+1} p^{m-2i} q^{i+1} - p \sum_{i\geq 0} \binom{m-i}{i+1} p^{m-1-2i} q^{i+1} \\ &= \sum_{i\geq 0} \left( \binom{m+1-i}{i+1} - \binom{m-i}{i+1} \right) p^{m-2i} q^{i+1} \\ &= \sum_{i\geq 0} \binom{m-i}{i} p^{m-2i} q^{i+1}. \end{aligned}$$

The right-hand side of (4) is

$$qU_{0+m+1} = qU_{m+1} = q \sum_{i\geq 0} \binom{m-i}{i} p^{m-2i} q^{i} = \sum_{i\geq 0} \binom{m-i}{i} p^{m-2i} q^{i+1}.$$

So the left-hand side and right-hand side are equal for $n = 0$.

For $n = 1$, the left-hand side of (4) is

$$\begin{aligned} &W_1(Y_{m+1}, Y_{m+2}; p, q) - pW_1(Y_m, Y_{m+1}; p, q) = Y_{m+2} - pY_{m+1} \\ &= \sum_{i\geq 0} \binom{m+2-i}{i+1} p^{m+1-2i} q^{i+1} - p \sum_{i\geq 0} \binom{m+1-i}{i+1} p^{m-2i} q^{i+1} \\ &= \sum_{i\geq 0} \left( \binom{m+2-i}{i+1} - \binom{m+1-i}{i+1} \right) p^{m+1-2i} q^{i+1} \\ &= \sum_{i\geq 0} \binom{m+1-i}{i} p^{m+1-2i} q^{i+1}. \end{aligned}$$

The right-hand side of (4) is

$$qU_{1+m+1} = qU_{m+2} = q\sum_{i\geq 0}\binom{m+1-i}{i}p^{m+1-2i}q^i = \sum_{i\geq 0}\binom{m+1-i}{i}p^{m+1-2i}q^{i+1}.$$

So the left-hand side and right-hand side are equal for $n=1$. Therefore, the base step is true.

`Induction Step`. Let $n \geq 2$ and assume that the result is true $n-2$ and $n-1$. We will prove the result for $n$.
By the induction hypothesis, we know that

$$W_{n-2}(Y_{m+1}, Y_{m+2}; p, q) - pW_{n-2}(Y_m, Y_{m+1}; p, q) = qU_{n-2+m+1}$$
$$W_{n-1}(Y_{m+1}, Y_{m+2}; p, q) - pW_{n-1}(Y_m, Y_{m+1}; p, q) = qU_{n-1+m+1}.$$

Then, using the recurrence relation for $W_n$ and the induction hypothesis, we have

$$\begin{aligned}
&W_n(Y_{m+1}, Y_{m+2}; p, q) - pW_n(Y_m, Y_{m+1}; p, q)\\
&= qW_{n-2}(Y_{m+1}, Y_{m+2}; p, q) + pW_{n-1}(Y_{m+1}, Y_{m+2}; p, q)\\
&- p((qW_{n-2}(Y_m, Y_{m+1}; p, q) + pW_{n-1}(Y_m, Y_{m+1}; p, q))\\
&= q(W_{n-2}(Y_{m+1}, Y_{m+2}; p, q) - pW_{n-2}(Y_m, Y_{m+1}; p, q))\\
&+ p(W_{n-1}(Y_{m+1}, Y_{m+2}; p, q) - pW_{n-1}(Y_m, Y_{m+1}; p, q))\\
&= q^2U_{n-2+m+1} + pqU_{n-1+m+1}\\
&= q\left(qU_{n-2+m+1} + pU_{n-1+m+1}\right)\\
&= qU_{n+m+1}.
\end{aligned}$$

Thus, the induction step is true. Therefore, by mathematical induction, the result is true for all integers $n \geq 0$. □

**Theorem 1.** *Let $n$ and $m$ be nonnegative integers. Then*

$$q\sum_{i=n}^{n+m} p^{n+m-i}U_i = W_n(Y_m, Y_{m+1}; p, q). \tag{5}$$

*Proof.* We fix $n$ and prove the result by induction on $m$.

`Base Step`. For $m=0$, the left-hand side of (5) is

$$q\sum_{i=n}^{n+0} p^{n+0-i}U_i = qp^0U_n = qU_n.$$

The right-hand side of (5) is

$$W_n(Y_0, Y_{0+1}; p, q) = W_n(Y_0, Y_1; p, q) = W_n(0, q; p, q) = qW_n(0, 1; p, q) = qU_n.$$

The left-hand side is equal to the right-hand side so the base step is true.

`Induction Step`. We assume the result is true for some $m \geq 0$. We will prove the result is true for $m+1$. Using the induction hypothesis and Lemma 2 we have

$$\begin{aligned} q\sum_{i=n}^{n+m+1} p^{n+m+1-i}U_i &= pq\sum_{i=n}^{n+m} p^{n+m-i}U_i + qU_{n+m+1} \\ &= pW_n(Y_m, Y_{m+1}; p, q) + qU_{n+m+1} \\ &= W_n(Y_{m+1}, Y_{m+2}; p, q). \end{aligned}$$

This is what we wanted to prove. Therefore, by mathematical induction, the result is true. □

Here is a special case of Theorem 1.

**Corollary 2.** *Let $n$ and $m$ be nonnegative integers. Then*

$$\sum_{i=n}^{n+m} F_i = H_n(F_{m+2} - 1, F_{m+3} - 1).$$

## 3. Finite Sums of Consecutive $W(a, b; 1, q)$s

We are now ready to state and prove our main result.

**Theorem 2.** *Let $n$ and $m$ be nonnegative integers. Then*

$$q\sum_{i=n}^{n+m} p^{n+m-i}W_i = bW_n(Y_m, Y_{m+1}; p, q) + qaW_{n-1}(Y_m, Y_{m+1}; p, q).$$

*Proof.* From an identity in Horadam [1] and Rabinowitz [4], we have

$$W_i = bU_i + qaU_{i-1}.$$

Therefore,

$$\begin{aligned} q\sum_{i=n}^{n+m} p^{n+m-i}W_i &= q\sum_{i=n}^{n+m} p^{n+m-i}\left(bU_i + qaU_{i-1}\right) \\ &= qb\sum_{i=n}^{n+m} p^{n+m-i}U_i + q^2a\sum_{i=n}^{n+m} p^{n+m-i}U_{i-1} \\ &= bq\sum_{i=n}^{n+m} p^{n+m-i}U_i + qaq\sum_{i=n-1}^{n-1+m} p^{n-1+m-i}U_i \\ &= bW_n(Y_m, Y_{m+1}; p, q) + qaW_{n-1}(Y_m, Y_{m+1}; p, q). \end{aligned}$$

This completes the proof. □

When $p = 1$, we have the following corollary.

**Corollary 3.** *Let $a$, $b$, $q$, $n \geq 0$, and $m \geq 0$ be integers. Then*

$$\sum_{i=n}^{n+m} W_i(a, b; 1, q) = \frac{b}{q} W_n(Y_m, Y_{m+1}; 1, q) + aW_{n-1}(Y_m, Y_{m+1}; 1, q).$$

In addition, we have three special cases of Corollary 3.

**Corollary 4.** *Let $n$ and $m$ be nonnegative integers. Then*

$$\sum_{i=n}^{n+m} L_i = H_n(F_{m+2} - 1, F_{m+3} - 1) + 2H_{n-1}(F_{m+2} - 1, F_{m+3} - 1),$$

$$\sum_{i=n}^{n+m} J_i = \frac{1}{2} W_n(G_m, G_{m+1}; 1, 2),$$

$$\sum_{i=n}^{n+m} H_i(a, b) = bH_n(F_{m+2} - 1, F_{m+3} - 1) + aH_{n-1}(F_{m+2} - 1, F_{m+3} - 1).$$

**Acknowledgement.** The author thanks an anonymous referee for the helpful suggestions which improved the paper.

## References

[1] A. F. Horadam, Basic properties of a certain generalized sequence of numbers, *Fibonacci Quart.* **3** (1965), 161-176.

[2] R. S. Melham, Certain classes of finite sums that involve generalized Fibonacci and Lucas Numbers, *Fibonacci Quart.* **42** (2004), 47-54.

[3] OEIS Foundation Inc. (2021), *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`.

[4] S. Rabinowitz, Algorithmic manipulation of second-order linear recurrences, *Fibonacci Quart.* **37** (1999), 162-177.

[5] D. L. Russell, Summation of second-order recurrence terms and their squares, *Fibonacci Quart.* **19** (1981), 336-340.

[6] D. L. Russell, Notes on sums of products of generalized Fibonacci numbers, *Fibonacci Quart.* **20** (1982), 114-117.