POLYNOMIAL INTERPOLATION OF MODULAR FORMS FOR HECKE GROUPS

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Abstract
For $m = 3, 4, \ldots$, let $\lambda_m = 2\cos\pi/m$ and let $J_m(m = 3, 4, \ldots)$ be triangle functions for the Hecke groups $G(\lambda_m)$ with Fourier expansions $J_m(\tau) = \sum_{n=-1}^{\infty} a_n(m)q_m^n$, where $q_m(\tau) = \exp 2\pi i\tau/\lambda_m$. (When normalized appropriately, J_3 becomes Klein's j-invariant $j(\tau) = 1/e^{2\pi i\tau} + 744 + \cdots$.) For $n = -1, 0, 1, 2$ and 3, Raleigh gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $m = 3, 4, \ldots$, and conjectured that similar relations hold for all positive integers n. This was proved by Akiyama. We apply work of Hecke to study experimentally similar polynomial interpolations of the J_m Fourier coefficents and the Fourier coefficients of other, positive weight, modular forms for $G(\lambda_m)$. We connect these polynomials (again, only empirically) with variants of Dedekind's eta function, with the Fourier expansions of some standard Hauptmoduln, and, in the case of analogues of Eisenstein series for $SL(2,\mathbb{Z})$, with certain divisor sums.

1. Introduction

Here is an example of a sequence $\{P_n(x)\}$ from $\mathbb{Q}[x]$ and a corresponding sequence of modular forms $\{f_m\}$ having the relationship we examine in this article. Let T_m be the cyclic subgroup of $SL(2,\mathbb{R})$ generated by

$$\begin{pmatrix} 1 & 2\pi/m \\ 0 & 1 \end{pmatrix},$$

let $f_m(x)$ be $\sin(mx)$, and let

$$Q_n(x) := (-1)^{(n-1)/2} x^n/n!.$$

Furthermore let $P_n(x) = Q_n(x)$ if n is odd and $P_n(x) = 0$ if n is even. Members of $SL(2,\mathbb{R})$ act on $\mathbb{R}$ as follows. If

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and x is real, we set

$$M(x) := \frac{ax+b}{cx+d}.$$

Thus the T_m act on $\mathbb{R}$ by translation. From the periodicity and Taylor series of sine, we know that $f_m(x)$ is invariant (weight-0 modular) with respect to the action of the T_m and equal to $\sum_{n=0}^{\infty} P_n(m)x^n$. We say that the elements of $\{f_m\}$ are interpolated by the sequence of polynomials $\{P_n(x)\}_{n=0,1,\ldots}$.

The novel material in the present article is an account of numerical experiments and several conjectures based on them (but no theorems.) Here is a first sketch of the theoretical background of our experiments.

Let $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ and $\mathbb{H}$ denote, respectively, the set of integers $\{0, \pm 1, \pm 2, ...\}$, the set of rational numbers, the set of complex numbers, and the set of complex numbers with positive imaginary parts. (We will reserve the letter τ for elements of the upper half-plane, and z for generic complex numbers.) We write $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$, and we equip $\mathbb{H}^*$ with the Poincaré metric. Figures T made by three geodesics of $\mathbb{H}^*$ are called hyperbolic or circular-arc triangles. Let $\lambda_m = 2\cos \pi/m$. For $m = 3, 4, ...,$ we define the Hecke group $G(\lambda_m)$ as the discrete group generated by the maps $z \to -1/z$ and $z \to z + \lambda_m$. The full modular group $SL(2, \mathbb{Z})$ is identical to $G(\lambda_3)$.

To define modular forms for the Hecke groups, we preview a definition from Berndt [7], which we will quote again in a later section. (We depart occasionally from Berndt's choices of variable to avoid clashes with some of our other notation.)

We say that f belongs to the space $M(\lambda, k, \gamma)$ if

1. $$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda},$$

   where $\lambda > 0$ and $\tau \in \mathbb{H}$, and

2. $f(-1/\tau) = \gamma(\tau/i)^k f(\tau)$, where $k > 0$ and $\gamma = \pm 1$.

We say that f belongs to the space $M_0(\lambda, k, \gamma)$ if f satisfies conditions 1 and 2 and if $a_n = O(n^c)$ for some real number c, as n tends to ∞.

Members of $M(\lambda, k, \gamma)$ are known as modular forms for $G(\lambda)$ of weight k. Condition 1 tells us that they are invariant under translations $\tau \mapsto \tau + \lambda$. Next we preview Berndt's definition of cusp forms for Hecke groups. If $f \in M(\lambda, k, \gamma)$ and $f(i\infty) = 0$, then we call f a cusp form of weight k and multiplier γ with respect to $G(\lambda)$. For cusp forms, the constant terms of condition 1 vanish. We denote by $C(\lambda, k, \gamma)$ the vector space of all cusp forms of this kind.

For our purposes, Schwarz triangles T are hyperbolic triangles in $\mathbb{H}^*$ with certain restrictions on the angles at the vertices. From a Euclidean point of view, their sides are vertical rays, segments of vertical rays, semicircles orthogonal to the real axis and meeting it at points $(r, 0)$ with r rational, or arcs of such semicircles. We choose

λ, μ and ν, all non-negative, such that $\lambda+\mu+\nu < 1$; then the angles of T are $\lambda\pi, \mu\pi$, and $\nu\pi$. By reflecting T across one of its edges, we get another Schwarz triangle. The reflection between two triangles in $\mathbb{H}^*$ is effected by a Möbius transformation, so the orbit of T under repeated reflections is associated to a collection of Möbius transformations. The group generated by these transformations is a triangle group. By the Riemann Mapping Theorem there is a conformal, onto map $\phi : T \mapsto \mathbb{H}^*$ called a triangle function.

Hecke groups are triangle groups H that act properly discontinuously on $\mathbb{H}$ [23]. This means that for compact $K \subset \mathbb{H}$, the set $\{\mu \in H \text{ s.t. } K \cap \mu(K) \neq \varnothing\}$ is finite. Recall that $G(\lambda_m)$ is the Hecke group generated by the maps $z \mapsto -1/z$ and $z \mapsto z + \lambda_m$. Hecke established in [23] that $G(\lambda_m)$ has the structure of a free product of cyclic groups $C_2 * C_m$, generalizing the relation $SL(2,\mathbb{Z}) = C_2 * C_3$ [39].

Let $\rho = -\exp(-\pi i/m) = -\cos(\pi/m) + i\sin(\pi/m)$, and let $T_m \subset \mathbb{H}^*$ denote the hyperbolic triangle with vertices ρ, i, and $i\infty$. The corresponding angles are $\pi/m, \pi/2$ and 0 respectively. Let ϕ_{λ_m} be a triangle function for T_m. The function ϕ_{λ_m} has a pole at $i\infty$ and period λ_m. For $P, Q \in \mathbb{H}^*$, let us us write $P \equiv_H Q$ when $\mu \in H$ and $Q = \mu(P)$. Then ϕ_{λ_m} extends to a function $J_m : \mathbb{H}^* \to \mathbb{H}^*$ by declaring that $J_m(P) = J_m(Q)$ if and only if $P \equiv_H Q$. The function J_m is modular for $G(\lambda_m)$.

Schwarz, Lehner, Raleigh and others studied Schwarz triangle functions, which map hyperbolic triangles T in the extended upper half z-plane onto the extended upper half w-plane [27, 35, 38]. For certain $T = T_m$, a triangle function ϕ_{λ_m} : $T \to \mathbb{H}^*$ extends to a map $J_m : \mathbb{H}^* \to \mathbb{H}^*$ invariant under modular transformations from $G(\lambda_m)$. Suitably normalized, the J_m become analogues j_m of the normalized Klein's modular invariant

$$j(\tau) = 1/q + 744 + 196884q + \ldots$$

where $q = q(\tau) = \exp(2\pi i\tau)$ and $j_3(\tau) = j(\tau)$.[1] The j_m are studied in Conjecture 1 below.

With $\lambda_m = 2\cos\pi/m$ and $q_m(\tau) = \exp(2\pi i\tau/\lambda_m)$, the original J_m have Fourier series $J_m(\tau) = \sum_{n\geqslant -1} a_n(m) q_m(\tau)^n$. For $n = -1, 0, 1, 2$ and 3, Raleigh gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $m = 3, 4, \ldots$, and conjectured that similar relations hold for all positive integers n [35]. Akiyama proved this conjecture in the passage after [1, eq. (6)].

Hecke built families of modular forms f_m for $G(\lambda_m)$ sharing particular properties [7, 23]. Earlier authors, whose work we will also describe, had already built modular functions (meromorphic functions invariant under the action of $G(\lambda)$, thus, of weight zero) from triangle functions.

The plan of the article is as follows: (1) an elaboration of the preceding discussion

[1] For j, see [39, Chapter VII, eq. (23)].

to establish a basis for the code in our experiments;[2] (2) conjectures on polynomials in $\mathbb{Q}[x]$ interpolating the coefficients in Fourier expansions of triangle functions for $G(\lambda_m)$; (3) a survey of Hecke's theory of modular forms for $G(\lambda_m)$, especially the construction of modular forms from modular functions; (4) several conjectures about polynomials in $\mathbb{Q}[x]$ interpolating the coefficients in Fourier expansions of Hecke modular forms on $G(\lambda_m)$; (5) several data plots and tables. Tables at the end of the article focus on the triangle functions since they are the basis of our construction of positive-weight modular forms, but more extensive collections of plots and tables are available within the *Sagemath* and *Mathematica* notebooks on [10].

Our conjectures are based on numerical experiments; here is a little more detail on the way we arrive at them. We begin with a list of modular functions or modular forms f_m for $G(\lambda_m), m = 3, 4, ...$, sharing certain properties picked out by Hecke's theory. Then we make tables of polynomials $Q_n(x)$ generated by Lagrangian interpolation from the values of the coefficient $k_m(n)$ in Fourier expansions $f_m = \sum_n k_m(n) X_m^n$, where X_m is a variable related to $q_m(\tau)$. Thus we are seeking $Q_n(x)$ such that

$$Q_n(m) = k_m(n) \tag{1}$$

for $m = 3, 4, ...$ If the degrees of the $Q_n(x)$ we obtain are linear in n, we take this to be evidence that the $Q_n(x)$ do satisfy Equation (1) for all integers m greater than two. (Typically, the alternative outcome is that the degree of every polynomial $Q_n(x)$ that we generate in a given table is equal to the size of the data set we are trying to interpolate.)

We are indebted to John Leo. Some parts of our exposition of the background material is based on that of [28]. The earliest computer code we located for calculating Fourier expansions of triangle functions for Hecke groups is that of [28]; Leo's code was based on Lehner's construction. Leo also calculates the Fourier coefficients of weight 4 and weight 6 Hecke-analogues of classical Eisenstein series in Chapter 4 of [28]. Our own code for triangle functions comes from the papers of Lehner and Raleigh. (J. Jermann's package [24] is also concerned with modular forms of triangle groups for Hecke groups.)

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[2] We have documention in the data repository [10]. *Mathematica* notebook names end in the suffix ".nb", and *SageMath* notebook names end in the suffix ".ipynb". Numerical data files named in the notebooks is stored in the folder "data" on [10]. A green "Code" button on the top page of the repository contains a drop-down menu with a download option. A *Mathematica* notebook ("mf25.nb") in the repository is a searchable library of functions that may not be defined explicitly within our other notebooks. We used *SageMath* release 9.1.

2. A Glossary

Some special functions in this list are related; different notations for similar objects are used by Lehner and Raleigh, and we included all of them:

1. the digamma function $\psi(z) := \Gamma'(z)/\Gamma(z)$;

2. the Schwarzian derivative

$$\{w, z\} := \frac{2w'w''' - 3w''^2}{2w'^2} \tag{2}$$

for $w = w(z)$; [3]

3. the Pochhammer symbol

$$(a)^0 := 1 \text{ and, for } n \geqslant 1, (a)^n := a(a+1)...(a+n-1) = \Gamma(a+n)/\Gamma(a);$$

4. the function c_ν given by

$$c_\nu = c_\nu(\alpha, \beta, \gamma) := \frac{(\alpha)^\nu (\beta)^\nu}{\nu!(\gamma)^\nu}, \nu \geqslant 0;$$ [4]

5. the function e_ν given by

$$e_\nu = e_\nu(\alpha, \beta) := \sum_{p=0}^{\nu-1} \left(\frac{1}{\alpha+p} + \frac{1}{\beta+p} - \frac{2}{1+p} \right);$$ [5]

[3] [15, page 130, eq. (370.8)]. In section 3 below, we discuss Caratheodory's presentation of a well-known theorem of Schwarz; when stating this theorem in eq. (374.1) of his Section 374, Carathéodory writes "$\{w, z\} = \frac{w' w''' - 3w''^2}{w'^2} = ...$", but we infer that the Schwarzian derivative $\{w, z\}$ is intended from the automorphy property of clause 2 of Schwarz's theorem.

[4] [15, page 138, eq. (377.3)]. To facilitate comparison with Raleigh's [35, eq. (9^1)], we remark that

$$c_\nu = \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\beta+\nu)}{\Gamma(\beta)} \cdot \frac{\Gamma(1)}{\Gamma(1+\nu)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+\nu)}. \tag{3}$$

In the terms of Schwarz's Theorem 1 below, Raleigh is treating the case $\lambda = 0$, for which (by Equation (7) below) $\gamma = 1$ and the expression on the right side of Equation (3) becomes, as in Raleigh,

$$\frac{\Gamma(\alpha+\nu)\Gamma(\beta+\nu)}{\Gamma(\alpha)\Gamma(\beta)(\nu!)^2}.$$

[5] [35, eq. (9^1)]. Here, we are dealing with the same ambiguity present in the definition of c_ν: this is a specialization to the case $\gamma = 1$ of the e_ν for $\nu \geqslant 1$ given by [15, page 153, eq. (387.5)]:

$$e_\nu = e_\nu(\alpha, \beta, \gamma) = \sum_{p=0}^{n-1} \left(\frac{1}{\alpha+p} + \frac{1}{\beta+p} - \frac{2}{\gamma+p} \right).$$

Unless it is explicitly indicated to be otherwise, we intend the former (Raleigh's) definition.

6. (a) Gauss's hypergeometric series

$$F(\alpha,\beta,\gamma;z) := \sum_{\nu=0}^{\infty} c_\nu(\alpha,\beta,\gamma)z^\nu;$$ [6]

(b) we let

$$F_1(\alpha,\beta,\gamma;z) := F(\alpha,\beta,\gamma+1;z);$$ [7]

(c) alternatively, dropping γ [28, eq. (3.5)]:

$$F_1(\alpha,\beta;z) := \sum_{\nu=1}^{\infty} \frac{(\alpha)_k(\beta)_\nu}{(\nu!)^2} e_\nu(\alpha,\beta);$$ [8]

7. with $F = F(\alpha,\beta,\gamma;z)$, a special function

$$F^*(\alpha,\beta,\gamma;z) := \frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial \beta} + 2\frac{\partial F}{\partial \gamma};$$ [9]

8. in [15, page 152, eqns. (386.2) and (386.3)], a special function $\phi_2^*(z)$ is defined as a certain limit, but it is immediately reduced to

$$\phi_2^*(z) = F(\alpha,\beta,1;z)\log z + F^*(\alpha,\beta,1;z);$$

9. the set $\mathscr{Q} = \{2,5,6,8,10,11,14,15,17,18,20,22,23,...\}$ of positive integers not represented by the quadratic form x^2+xy+y^2 [44];[10]

10. the McKay-Thompson series of class 4A, $\{1,24,276,2048,...\}$, which is the sequence of coefficients in the q-series of a certain hauptmodul discussed in [30]; [11]

11. as usual, the cardinality of a finite set S is written $\#S$, the n^{th} prime number is denoted by p_n, the number of primes less than or equal to x is written $\pi(x)$, and $\sigma_k(n) := \sum_{0<d|n} d^k$.

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[6] [15, page 138, eq. (377.4)]. The function F is occasionally written in [15] as ϕ_1.

[7] As defined in the first line of [15, page 142.]

[8] It is in the latter form, defined more cryptically in [27, p. 244], that we will use F_1; to establish his series for the triangle functions, which we will apply below, Lehner uses this definition of F_1, as well as certain theorems from Fricke [20]. Referring to item 4, we see that

$$F_1(\alpha,\beta;z) = \sum_{\nu=1}^{\infty} c_\nu(\alpha,\beta,1)e_\nu(\alpha,\beta).$$

We will derive another form of $F_1(\alpha,\beta;z)$ in item 7.

[9] F^* may be written [15, page 153, eq. (387.4)]

$$F^*(\alpha,\beta,\gamma;z) = \sum_{\nu=1}^{\infty} c_\nu(\alpha,\beta,\gamma)e_\nu(\alpha,\beta,\gamma)z^\nu.$$

It follows that $F^*(\alpha,\beta,1;z) = F_1(\alpha,\beta;z)$.

[10] B. Cloitre asserts on the cited page that $\mathscr{Q}$ is also the set of non-negative integers n such that $\delta(n)$ is non-zero, where η is Dedekind's eta function and $\sum_n \delta(n)x^n = \eta(x^3)/\eta(x)^3$.

[11] We identified it with the sequence $\{\phi_n\}$ of our Conjecture 1 after finding it in on [46].

3. Calculation of Schwarz's Inverse Triangle Function

Schwarz proved the following result.

Theorem 1 ([15, Section 374]).

1. Let the half-plane $\Im z > 0$ be mapped conformally onto an arbitrary circular-arc triangle whose angles at its vertices A, B, and C are $\pi\lambda, \pi\mu$, and $\pi\nu$, and let the vertices A, B, C be the images of the points $z = 0, 1, \infty$, respectively. Then the mapping function $w(z)$ must be a solution of the third-order differential equation
$$\{w, z\} = \frac{1-\lambda^2}{2z^2} + \frac{1-\mu^2}{2(1-z^2)} + \frac{1-\lambda^2-\mu^2+\nu^2}{2z(1-z)}. \tag{4}$$

2. If $w_0(z)$ is any solution of Equation (4) that satisfies $w_0'(z) \neq 0$ at all interior points of the half-plane, then the function
$$w(z) = \frac{aw_0(z)+b}{cw_0(z)+d} \qquad (ad - bc \neq 0)$$
is likewise a solution of Equation (3).

3. Also, every solution of Equation (4) that is regular and non-constant in the half-plane $\Im z > 0$ represents a mapping of this half-plane onto a circular-arc triangle with angles $\pi\lambda, \pi\mu$, and $\pi\nu$.

(In Carathéodory's lexicon, a regular function is one that is differentiable on an open connected set [14, page 124].)

Let us write
$$\alpha = \frac{1}{2}(1-\lambda-\mu+\nu), \tag{5}$$
$$\beta = \frac{1}{2}(1-\lambda-\mu-\nu), \tag{6}$$
and
$$\gamma = 1-\lambda. \tag{7}$$
The solutions w of Equation (4) are inverse to triangle functions; they are quotients of arbitrary solutions of
$$u'' + p(z)u' + q(z)u = 0 \tag{8}$$
when
$$p = \frac{1-\lambda}{z} - \frac{1-\mu}{1-z}$$
and [15, page 136, eq. (376.4)]
$$q = -\frac{\alpha\beta}{z(1-z)}.$$

Equation (8) reduces [15, page 137, eqns. 376.5-7] to the hypergeometric differential equation

$$z(1-z)u'' + (\gamma - (\alpha+\beta+1)z)u' - \alpha\beta u = 0. \tag{9}$$

As long as γ is not a non-positive integer, $u = F(\alpha, \beta, \gamma; z)$ is a solution of Equation (9); it is the only solution regular at $z = 0$, and it satisfies (see the final paragraph of [15, Section 377, page 138]) $F(\alpha, \beta, \gamma; 0) = 1$.

In [15, Sections 386-388, pages 151-155], we find that when $\gamma = 1$ and $\lambda = 0$, another, linearly independent, solution of Equation (8) is $\phi_2^*(z)$. The passage [15, Section 394, pages 165 - 167] is devoted to the case $\lambda = 0$. There [15, page 166, eq. (394.4)] we find that the mapping function w of Theorem 1 satisfies

$$w = \frac{1}{\pi i}\left[\frac{\phi_2^*}{\phi_1} - (2\psi(1) - \psi(1-\alpha) - \psi(1-\beta))\right] + i\frac{\sin \pi\mu}{\cos \pi\mu + \cos \pi\nu}. \tag{10}$$

4. Inversion of Schwarz's Inverse Triangle Function

Following Lehner and Raleigh, we consider the Schwarz triangle T_m with vertices at $\rho = -\exp(-\pi i/m), i$, and $i\infty$. In terms of Theorem 1, T_m has $\lambda = 0$ (an angle 0 at the vertex $i\infty$), $\mu = 1/2$ (an angle $\pi/2$ at i), and $\nu = 1/m$ (an angle π/m at ρ.) In this situation, $\gamma = 1$.

Let J_m be automorphic for $G(\lambda_m)$ with $J_m(\rho) = 0, J_m(i) = 1$, and $J_m(i\infty) = \infty$. In terms of Theorem 1, w and J_m are inverse functions. We are going to write down the Fourier expansion $\sum_{n=-1}^{\infty} a_n q_m(\tau)^n$ of J_m.

By clause 2 of Theorem 1, if w satisfies Equations (4) and (10), so does $\tau = \tau(z) = \lambda_m w(z)/2$, and therefore

$$2\pi i\tau/\lambda_m = \frac{\phi_2^*}{\phi_1} - (2\psi(1) - \psi(1-\alpha) - \psi(1-\beta)) - \pi\sec(\pi/m).$$

Let us write $\log A_m = -2\psi(1) + \psi(1-\alpha) + \psi(1-\beta) - \pi\sec(\pi/m)$. In general, $A_m = a_{-1}(m)$ [35].[12] Recalling the definitions of ϕ_1 and ϕ_2^* from our glossary items 6 and 8, we find (abbreviating $J_m(\tau)$ as J_m) that

$$2\pi i\tau/\lambda_m = -\log J_m + \frac{F^*(\alpha, \beta, 1; 1/J_m)}{F(\alpha, \beta, 1; 1/J_m)} + \log A_m. \tag{11}$$

Equation (11) is [35, eq.(6)], but Raleigh suppresses the subscripts. He also writes $\exp 2\pi i\tau/\lambda_m$ as x_m, so that (in our earlier notation) $x_m = q_m(\tau)$.

In Raleigh's notation, after taking exponentials,

$$x_m/A_m = \frac{1}{J_m}\exp\frac{F^*(\alpha, \beta, 1; 1/J_m)}{F(\alpha, \beta, 1; 1/J_m)}, \tag{12}$$

[12] For example, two lines below eq. (13).

the right side of which has a power series in J_m with rational coefficients. Writing $X_m = x_m/A_m$ we can regard $X_m = X_m(J_m)$ as a power series in J_m with rational coefficients. Following [27] and [35], we inverted this power series to obtain one for the modular function J_m, also with rational coefficients. The Fourier expansion of J_m in X_m is normalized so that the coefficient of $1/X_m$ is 1 [35, eq. (12)]. Let $\mathscr{I}$ be a formal operation taking a power series $\sigma(v)$ to its inverse; that is, if $u = \sigma(v)$ then $v = \mathscr{I}(\sigma)(u)$. Let $Y_m(J)$ be a power series such that

$$Y_m(J_m) = J_m \exp \frac{F^*(\alpha, \beta, 1; J_m)}{F(\alpha, \beta, 1; J_m)} = X_m\,(1/J_m)$$

and hence

$$Y_m(1/J_m) = \frac{1}{J_m} \exp_m \frac{F^*(\alpha, \beta, 1; 1/J_m)}{F(\alpha, \beta, 1; 1/J_m)} = X_m(J_m),$$

so that $\mathscr{I}(Y_m)(X_m(J)) = 1/J_m$ and, therefore, $J_m = 1/\mathscr{I}(Y_m)(X_m)$.

5. Raleigh's Polynomials for Triangle Functions

Let X_m be the variable from the previous section. We define some operators on infinite series in X_m.[13]

Definition 1. Let $f = \sum_{n=a}^{\infty} k_n X_m^n$, where k_n is a rational number for $n = a, a + 1, \ldots$, and $k_a \neq 0$.

1. Let $g = \sum_{n=a}^{\infty} k_n (2^6 m^3 X_m)^n = \sum_{n=a}^{\infty} \tilde{k}_n X_m^n$ (say). Then

$$\overline{f} := g/\tilde{k}_a.$$

2. Let

$$f^* := \frac{1}{k_a} \sum_{n=a}^{\infty} k_n X_m^{n-a}.$$

Recall, from the passage following Equation (12) in the previous section, that the Fourier expansion of J_m in X_m has the form

$$J_m(\tau) = 1/X_m + \sum_{n=0}^{\infty} a_n(m) X_m^n.$$

Definition 2. For the present purpose, we regard J_m as a Laurent series in X_m and write

$$j_m := \overline{J_m}.$$

[13] The substitution involved appears in [28].

Conjecture 1. Let the Fourier expansion of $j_m(\tau)$ be

$$j_m = 1/X_m + \sum_{n\geqslant 0} c_m(n)X_m^n.$$

(Some code for j_m Fourier expansions appearing in *SageMath* notebooks cited below was generated in [10, notebook "j from scratch.ipynb"], which employs a "dictionary" (the definitions at the top of the notebook) distinct from the corresponding dictionaries in the notebooks where it is reproduced.) Then the following statements are true.

1. For each integer n greater than -2, there exists a polynomial $C_n(x) \in \mathbb{Q}[x]$ that satisfies the relation $c_m(n) = C_n(m)$ for $m = 3, 4, ...$ [10, notebook "conjecture 1.nb"].

2. Let $\{\phi_n\}$ be as in item 10 of our glossary. For some degree $2n$, irreducible, monic polynomial $\gamma_n(x)$ in $\mathbb{Q}[x]$ we have [10, notebook "conjecture 1 clause 2.ipynb"]:
$$C_n(x) = \phi_n \cdot (x-2)(x+2)x^{n+1}\gamma_n(x).$$

3. The function j_3 is identical to the modular function on $SL(2,\mathbb{Z})$ usually denoted j [10, notebook "conjecture 1 cause 3.nb"].

4. The complex roots of $\gamma_n(x)$ lie in the disk with center zero and radius $n/\log(n)$. (Pertinent notebooks are [10, notebooks "conjecture1clause4.nb", "conjecture1clause4d.nb", "conjecture 1 clause 4 no2.ipynb", "conjecture 1 clause 4 no3.ipynb", "conjecture 1 clause 4 no4.ipynb", "conjecture 1 clause 4 no5.ipynb", and "conjecture 1 clause 4 no6.ipynb"].)

5. Let G_n be the Galois group of $\gamma_n(x)$ over the rationals. The size of G_n is $2^n n!$ and (if n is greater than two) G_n is isomorphic to a permutation group on $2n$ elements $\{e_1, ..., e_{2n}\}$ with three generators: a transposition (e_j, e_k), a product $(e_j, e_{j'})(e_k, e_{k'})$, and a product $\Gamma_1\Gamma_2$ of disjoint cycles Γ_1 and Γ_2, each of length n, such that Γ_1 sends e_j to $e_{j'}$ and Γ_2 sends e_k to $e_{k'}$ [10, folder "conjecture1clause5"].

6. Let n be larger than one and let π_n be the set of prime numbers dividing the denominator of at least one non-zero coefficient of $C_n(x)$ in its unfactored form. (See [10, notebook "conjecture 1 clause 6.ipynb".]) Then the following claims are true.

   (a) $\pi_2 = \{3\}$ and π_3 is empty.

   (b) If π_n is ordered by size, it contains no gaps. That is, if p and p' are consecutive elements of π_n with $p = p_k$ and $p' = p_j$, then $j = k + 1$.

(c) If n is an odd prime other than 3, then

$$\pi_n = \{3, ..., k, ..., p\}_{k \text{ prime}}$$

where p is the greatest prime less than n.

(d) If n is composite and $n+1$ is prime, then

$$\pi_n = \{3..., k, ..., n+1\}_{k \text{ prime}}.$$

(e) If n and $n+1$ are both composite, then

$$\pi_n = \{3..., k, ..., p\}_{k \text{ prime}}$$

where p is the greatest prime less than n.

Clause 2 implies that, for m greater than or equal to three, $c_n(m)$ is nonzero. It is already known that, for all integers $n \geqslant -1$, the n^{th} Fourier coefficient of $j = j_3$, namely $c(n) = c_n(3)$, is positive. (See, for example, page 199 in [36].) We tested clause 4 in several ways. We approximated the roots of the $\gamma_n(x)$ with root-finding routines and compared their complex moduli with $n/\log(n)$. We used the argument principle to count the zeros in central disks of radius $n/\log(n)$. We superimposed plots of the roots of $\gamma_n(x)$ against plots of circles with radius $n/\log(n)$ and center at the origin. An example is depicted in Figure 1. (See [10, notebook "conjecture1clause4d.nb"], the name of the notebook notwithstanding.) For clause 5, we computed the Galois groups in *Magma*. For clause 6, some sequences we generated in the analysis were identified in [40] and [48].

Conjecture 2. Let the Fourier expansion of $J_m(\tau)$ be

$$J_m = \sum_{n=-1}^{\infty} a_m(n) X_m^n.$$

(Relevant documents in [10] are notebooks "conjecture 2.nb", "conjecture2no1.ipynb", "capital-J make data file1jun21.ipynb" and associated data files.)

1. (For clause 1, see [10, notebooks "conjecture 2.nb", "conjecture 2 clause 1b.ipynb", and "conjecture 2 clause 1b no2.ipynb"].) We have the following.

   (a) There exist polynomials $A_n(x)$ such that $A_{-1}(x) \equiv 1$, $A_0(x) = 3x^2 + 4$, $A_1(x) = 69x^4 - 8x^2 - 48$, and $A_n(m) = m^{2n+2} a_m(n)$ for $m = 3, 4, ...$ [14]

   (b) Let $C_n(x)$ be as in Conjecture 1. We have the following.

$$A_n(x) = 2^{-6n-6} x^{-n-1} C_n(x).$$

[14] The first few polynomials in our table of the polynomials A_n (Table 5) agree with Raleigh's equation-group III in [35].

2. Let π_n be the set of prime numbers dividing the denominator of at least one non-zero coefficient of A_n. Then the following statements are true [10, notebook "conjecture 2 clause 2 w code 14jun21.ipynb"].

   (a) $\pi_2 = \{3\}$.

   (b) If π_n is ordered by size, it contains no gaps. That is, if p and p' are consecutive elements of π_n with $p = p_k$ and $p' = p_j$, then $j = k + 1$.

   (c) If n is an odd prime, then

   $$\pi_n = \{2, ..., k, ..., p\}_{k \text{ prime}}$$

   where p is the greatest prime less than n.

   (d) If n is composite and $n + 1$ is prime, then

   $$\pi_n = \{2..., k, ..., n + 1\}_{k \text{ prime}}.$$

   (e) If n and $n + 1$ are both composite, then

   $$\pi_n = \{2..., k, ..., p\}_{k \text{ prime}}$$

   where p is the greatest prime less than n.

The existence statement in clause 1a of Conjecture 2 is equivalent up to some changes of variable, obviously, to the conjecture of Raleigh proved in [1]. We identified the leading numerical term in clause 1b of Conjecture 2 after looking at [49]. Clause 2 of Conjecture 2 is only a slight refinement of [1, proposition 2].

6. Survey of Hecke's Theory of Modular Forms

When the w-image of $\mathbb{H}^*$ is T_m, the inverse of w is ϕ_{λ_m}. The extension by modularity J_m of ϕ_{λ_m} to $\mathbb{H}^*$, is periodic with period λ_m and maps ρ to 0, i to 1, and $i\infty$ to ∞ [27, eq. (2)]. These mapping properties allow us, following Berndt's exposition of Hecke, to construct positive weight modular forms for $G(\lambda_m)$ from J_m [7]. This section describes results of Hecke that are perhaps most easily accessible for the classical case $m = 3$ in Schoeneberg and, for the general case, in Berndt [37, 7].

6.1. The Case $m = 3$

By keeping track of the weights, zeros and poles of the constituent factors in the numerator and denominator of the fraction defining

$$f_{a,b,c} = \frac{J'^a}{J^b (J-1)^c},$$

Schoeneberg demonstrates that $f_{a,b,c}$ is an entire modular form of weight $2a$ for $SL(2,\mathbb{Z})$ if $a \geqslant 2, 3c \leqslant a, 3b \leqslant 2a$, $b+c \geqslant a$ and a,b,c are integers. (Schoeneberg speaks of "dimension $-2a$" [37, Theorem 16, page 45].) Thus he is able to write down a weight 4 entire modular form $E_4^* = f_{2,1,1}$ for $SL(2,\mathbb{Z})$ with a zero of order $\frac{1}{3}$ at $\rho = e^{2\pi i/3}$ and a weight 6 entire modular form $E_6^* = f_{3,2,1}$ for $SL(2,\mathbb{Z})$ with a zero of order $\frac{1}{2}$ at i. (Schoeneberg writes G_4^*, G_6^*.) It is well known that the (vector space) dimension of the spaces of weight 4 and 6 entire modular forms for $SL(2,\mathbb{Z})$ is equal to one, so E_4^* and E_6^* may be identified with the usual weight 4 and weight 6 Eisenstein series, up to a normalization. Finally, Schoeneberg defines the weight 12 cusp form $\Delta^* = E_4^{*3} - E_6^{*2}$ with a zero of order 1 at $i\infty$. It is a multiple of Δ.

6.2. The Case $m \geqslant 3$

We quote statements from Berndt, which is an exposition of Hecke, [22] and other writings. We depart occasionally from Berndt's choices of variable to avoid clashes with our earlier notation.

Definition 3. We say that f belongs to the space $M(\lambda, k, \gamma)$ if

1.
$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda},$$
where $\lambda > 0$ and $\tau \in \mathbb{H}$, and

2. $f(-1/\tau) = \gamma \cdot (\tau/i)^k f(\tau)$, where $k > 0$ and $\gamma = \pm 1$ [7, Definition 2.2].

We say that f belongs to the space $M_0(\lambda, k, \gamma)$ if f satisfies conditions 1 and 2, and if $a_n = O(n^c)$ for some real number c, as n tends to ∞.

After defining the notion of a fundamental region in the usual way and defining as $G(\lambda)$ the group of linear fractional transformations generated by $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + \lambda$, Berndt states (for $\tau = x + iy$)

Theorem 2 ([7, Theorem 3.1]). Let $B(\lambda) = \{\tau \in \mathbb{H} : x < \lambda/2, |\tau| > 1\}$. If $\lambda \geqslant 2$ or if $\lambda = 2\cos(\pi/m)$, where $m \geqslant 3$ is an integer, then $B(\lambda)$ is a fundamental region for $G(\lambda)$.

Definition 4. Let $T_A = \{\lambda : \lambda = 2\cos(\pi/m), m \geqslant 3, m \in \mathbb{Z}\}$ [7, Definition 3.4].

Berndt states in his Theorem 5.4 that $G(\lambda)$ is discrete if and only if λ belongs to T_A. This discreteness is the premise of the theory of automorphic functions generally. He embeds within the proof of his Lemma 3.1 (which we omit), the following definition.

Definition 5. The symbol τ_λ denotes the intersection in $\mathbb{H}$ of the line $x = -\lambda/2$ and the unit circle $|\tau| = 1$.

(Berndt remarks at the top of page 35 that τ_λ is the lower left corner of $B(\lambda)$). and that $\pi\theta = \pi - \arg(\tau_\lambda)$, so that $\cos(\pi\theta) = \lambda/2$.)

To characterize Eisenstein series, we need to keep track of some analytical properties. The next definition summarizes the second paragraph of Berndt's Chapter 5. (Throughout his Chapter 5, $\lambda < 2$.)

Definition 6. Let f be in $M(\lambda, k, \gamma)$, f not identically zero.

1. $N = N_f$ counts the zeros of f on $\overline{B(\lambda)}$ with multiplicities.
2. N_f does not count zeros at τ_λ, at $\tau_\lambda + \lambda$, at i, or at $i\infty$.
3. If $\tau_0 \in \overline{B(\lambda)}$, $f(\tau_0) = 0$ and $\Re(\tau_0) = -\lambda/2$, then $f(\tau_0 + \lambda) = 0$ and N_f counts only one of the two zeros.
4. If $\tau_0 \in \overline{B(\lambda)}$, $f(\tau_0) = 0$, and $|\tau_0| = 1$, then $f(-1/\tau_0) = 0$, and N_f counts only one of these two zeros.
5. The numbers n_λ, n_i, and n_∞ are the orders of the zeros of f at τ_λ, i and $i\infty$, respectively. The order n_∞ is measured in terms of $\exp(2\pi i\tau/\lambda)$.

The multiplier γ is given by the following theorem.

Theorem 3 ([7, Corollary 5.2]). Let f be in $M(\lambda, k, \gamma)$ and let n_i be the order of the zero of f at $\tau = i$. Then

$$\gamma = (-1)^{n_i}.$$

The next two results tell us that the only nontrivial case in this theory is the one that we are interested in.

Theorem 4 ([7, Lemma 5.1]). If $\dim M(\lambda, k, \gamma) \neq 0$,

$$N_f + n_\infty + \frac{1}{2}n_i + \frac{n_\lambda}{m} = \frac{1}{2}k\left(\frac{1}{2} - \theta\right).$$

By Berndt's eq. (5.16), if $m \geqslant 3$ then the right side can be written as $k(m-2)/4m$.

Theorem 5 ([7, Theorem 5.2)]. If $\dim M(\lambda, k, \gamma) \neq 0$, then $\theta = 1/m$ where $m \geqslant 3$ and $m \in \mathbb{Z}$.

We are concerned with λ in T_A. This makes $\lambda < 2$, as in all the results of Berndt's Chapter 5. One estimate for $\dim M(\lambda, k, \gamma)$ is given in the following theorem.

Theorem 6 ([7, Theorem 5.6]). If λ is not in T_A, then $\dim M(\lambda, k, \gamma) = 0$. If $\lambda = 2\cos(\pi/m)$ is in T_A, then for nontrivial f in $M(\lambda, k, \gamma)$, the weight k has the form

$$k = \frac{4h}{m-2} + 1 - \gamma,$$

where $h \geqslant 1$ is an integer. Furthermore,

$$\dim M(\lambda, k, \gamma) = 1 + \left\lfloor \frac{h + (\gamma - 1)/2}{m} \right\rfloor.$$

Eliminating h, we find that

$$\dim M(\lambda, k, \gamma) = 1 + \left\lfloor k \left(\frac{1}{4} - \frac{1}{2m} \right) + \frac{\gamma}{4} - \frac{1}{4} \right\rfloor. \tag{13}$$

Berndt proves that the dimension formula above holds also when $h = 0$. [7, Remark 5.3]

The existence of certain modular forms is provided by the following theorem.

Theorem 7 ([7, Theorem 5.5]). Let λ lie in T_A. Then there exist functions f_λ, f_i, and f_∞ in $M(\lambda, k, \gamma)$ such that each has a simple zero at τ_λ, i, and $i\infty$, respectively, and no other zeros. Here, γ is given by Theorem 3 of the present article, and k is determined in each case from Theorem 4 of the present article. Thus, f_λ is in $M(\lambda, 4/(m-2), 1)$, f_i is in $M(\lambda, 2m/(m-2), -1)$, and f_∞ is in $M(\lambda, 4m/(m-2), 1)$.

Remark 1. By the Riemann mapping theorem there exists a function $g(\tau)$ that maps the simply connected region $B(\lambda)$ one-to-one and conformally onto $\mathbb{H}$. If we require that $g(\tau_\lambda) = 0, g(i) = 1$, and $g(i\infty) = \infty$, then g is determined uniquely [7], pages 47-48].

Now we can write down f_λ, f_i, and f_∞ explicitly. The next theorem is extracted from the proof of Theorem 7. f_λ and f_i correspond to Eisenstein series and f_∞ to a cusp form. In our code, we take g to be a normalized form of J_m.

Theorem 8 ([7], page 50).

$$f_\lambda(\tau) = \left\{ \frac{g'(\tau)^2}{g(\tau)(g(\tau) - 1)} \right\}^{1/(m-2)},$$

$$f_i(\tau) = \left\{ \frac{g'(\tau)^m}{g(\tau)^{m-1}(g(\tau) - 1)} \right\}^{1/(m-2)},$$

and

$$f_\infty(\tau) = \left\{ \frac{g'(\tau)^{2m}}{g(\tau)^{2m-2}(g(\tau) - 1)^m} \right\}^{1/(m-2)}.$$

In our applications to Lehmer's problem, we will be interested in the dimensions of the weight 12 cusp spaces for $\lambda = \lambda_m = 2 \cos \pi/m$.

Definition 7. If f is in $M(\lambda, k, \gamma)$ and $f(i\infty) = 0$, then we call f a cusp form of weight k and multiplier γ with respect to $G(\lambda)$. We denote by $C(\lambda, k, \gamma)$ the vector space of all cusp forms of this kind. [7, Definition 5.2]

The next remark follows from [7, eq. (5.25)].

Remark 2.

$$\dim C(\lambda, k, \gamma) \geqslant \dim M(\lambda, k, \gamma) - 1.$$

Remark 3. In view of Theorem 6, Equation (12), Remark 2, and the fact that $\gamma = \pm 1$, we see that $\dim C(\lambda_m, 12, \gamma)$ is greater than 1 when m is greater than or equal to 12.

7. Modular Forms Studied in Our Experiments

We are going to write down versions of the functions from Theorem 8 such that, at $m = 3$, they reduce to corresponding functions in the classical theory. Some have fixed weights (four, six and twelve) and others have weights that vary with m. The classical objects (in Serre's notation [39]) are Klein's j-invariant, the weight four Eisenstein series E_2, the weight six Eisenstein series E_3, and the generating function of Ramanujan's tau function, namely the normalized weight twelve cusp form Δ. They all belong to one-dimensional vector spaces of modular forms and the number of zeros each one has in a given fundamental region is small, so the identifications follow by comparison of the initial Fourier coefficients [39, Chapter VII, eqns. (20-21)].

Corresponding to f_λ, we have the following definition.

Definition 8. 1. Let $H_{\lambda,m}(\tau)$ be

$$\left\{ \frac{J_m'(\tau)^2}{J_m(\tau)(J_m(\tau)-1)} \right\}^{1/(m-2)}.$$

2. Let $H_{\lambda,4,m}(\tau)$ equal $H_{\lambda,m}(\tau)^{m-2}$.

Now we state a definition corresponding to f_i.

Definition 9. 1. Let $H_{i,m}(\tau)$ equal

$$\left\{ \frac{J_m'(\tau)^m}{J_m(\tau)^{m-1}(J_m(\tau)-1)} \right\}^{1/(m-2)}.$$

Definition 10. 1. Corresponding to f_∞, let $\Delta_{\infty,m}(\tau)$ equal

$$\left\{ \frac{J_m'(\tau)^{2m}}{J_m(\tau)^{2m-2}(J_m(\tau)-1)^m} \right\}^{1/(m-2)}.$$

2. Let $\Delta_m^\diamond$ equal $H_{\lambda,m}^3/J_m$.

3. Let $\Delta^{\diamond}_{12,m}$ equal $H^3_{\lambda,4,m}/J_m$.

Remark 4. By Berndt's Theorem 7 above, we have the following table of weights:

$H_{\lambda,m}$	$H_{\lambda,4,m}$	$H_{i,m}$	$\Delta^{\diamond}_m$	$\Delta^{\diamond}_{12,m}$	$\Delta_{\infty,m}$
$4/(m-2)$	4	$2m/(m-2)$	$12/(m-2)$	12	$4m/(m-2)$

8. Interpolation by Polynomials

In this section, we state conjectures about polynomials interpolating coefficients of modular forms for Hecke groups. Conjectures 6 and 7 bear on Lehmer's question about the existence of zeros of Ramanujan's tau function.

Berndt's (Hecke's) Theorems 7 and 8 above make it clear that Akiyama's theorem proving Raleigh's conjecture on the interpolation of the coefficients of the Fourier expansions of Hecke triangle functions extends in some way to the modular forms defined in the previous section. We did experiments to explore the details; our observations are summarized in the conjectures below.

8.1. Analogues of $SL(2,\mathbb{Z})$ Eisenstein Series

We found the sequence $\{e_{4,n}\}$ mentioned below on [41].

Conjecture 3. Let the Fourier expansion of $\overline{H_{\lambda,4,m}}(\tau)$ be

$$\overline{H_{\lambda,4,m}}(\tau) = \sum_{n=0}^{\infty} \beta_{4,m}(n) X_m^n.$$

Then the following statements are true ([10, notebooks "conjecture 3.nb" and "conjecture 3.ipynb"]; associated data files are in the data folder on [10]).

1. The Fourier expansion of $\overline{H_{\lambda,4,3}}(\tau)$ reduces to Serre's weight-4 Eisenstein series E_2 in the sense that $\beta_{4,3}(n) = 240\sigma_3(n)$ for $n = 1, 2, 3, ...$ [39, page 93].

2. For each n there is a polynomial $B_{4,n}(x)$ with rational coefficients such that $m^{3n}\beta_{4,m}(n) = B_{4,n}(m)$ for $m = 3, 4, ...$

3. If n is positive, then the degree of $B_{4,n}(x)$ is $6n$.

4. The polynomial $B_{4,0}(x)$ is identically equal to 1 and, if n is positive, then

$$B_{4,n}(x) = e_{4,n}(x^2-4)x^{4n} b_{4,n}(x),$$

where $e_{4,n} = 16 \sum_{\substack{\nu|n \\ \nu \text{odd}}} (-1)^{n-\nu}\nu^3$ and $b_{4,n}(x)$ is a monic irreducible polynomial in $\mathbb{Q}[x]$.

Conjecture 4. Let the Fourier expansion of $\overline{H_{\lambda,m}}$ be

$$\overline{H_{\lambda,m}} = \sum_{n=0}^{\infty} \beta_m(n) X_m^n.$$

Then [10, notebooks "conjecture 4.1-4.3.ipynb", "conjecture 4.4a.ipynb", "conjecture 4.4b.ipynb", and "conjecture 4.5.ipynb"] the following statements are true.[15]

1. For each n there is a polynomial $B_n(x)$ with rational coefficients such that $\beta_m(n) = B_n(m)$ for $m = 3, 4, ...$
2. If n is positive, then the degree of $B_n(x)$ is $3n - 1$.
3. The polynomial $B_0(x)$ is identically equal to 1 and $B_1(x) = 16x(x + 2)$.
4. Let $\mathscr{Q}$ be as in item 9 of our glossary and let $e_n = 16(-1)^{n+1} \sum_{\substack{\nu \mid n \\ \nu \text{odd}}} 1/\nu$. If n is greater than 2 and belongs to $\mathscr{Q}$, then

   $$B_n(x) = e_n(x^2 - 4)(x - 6)x^n b_n(x),$$

   where $b_n(x)$ is a monic irreducible polynomial. Otherwise (for n greater than one) $B_n(x) = e_n(x^2 - 4)x^n b_n(x)$ where, again, $b_n(x)$ is a monic irreducible polynomial in $\mathbb{Q}[x]$.
5. The Fourier expansion of $\overline{H_{\lambda,3}}$ reduces to E_2 in the same sense as in Conjecture 3.1.

(We identified the e_n after reading [44, 45].)

Thus, in the range of our observations ($3 \leqslant m \leqslant 302, 0 \leqslant n \leqslant 100$), the only integer value of m such that $\overline{H_{\lambda,m}}$ has any vanishing coefficients is six, and $\beta_n(6)$ is zero just if n is in $\mathscr{Q}$.

Conjecture 5. Let the Fourier expansion of $\overline{H_{i,m}}$ be

$$\overline{H_{i,m}} = \sum_{n=0}^{\infty} \delta_m(n) X_m^n.$$

Then [10, notebook "conjecture 5.ipynb"] the following statements are true.

1. For each non-negative integer n, there is a polynomial $D_n(x)$ in $\mathbb{Q}[x]$ such that
   (a) The number $D_n(m)$ is $\delta_m(n)$ for $n = 0, 1, ...$ and $m = 3, 4, ...$

[15] Contrary to appearances, the function denoted "H4" in these *SageMath* notebooks is not the function covered in the previous conjecture. "H4" is $H_{\lambda,m}$.

(b) The degree of D_n is $3n$.

(c) The number $D_n(x)$ is rational and is equal to $d_n\times$ a product of monic irreducible polynomials.

(d) The number d_0 is 1 and, for n a positive integer, $d_n = 24(-1)^n \sum^*_{\nu|n} \nu$. Again, the asterisk means that the sum is taken over the odd positive divisors of n.

2. The number $D_n(m)$ is $(-1)^m\delta_n(m)$ for $m = 3, 4, \ldots$.

3. The polynomial $D_0(x)$ is identically equal to 1; $D_1(x) = -24(x-2/3)x^2$, and $D_2(x) =$
$24(x-2/3)(x-2)x^3(x-14)$.

4. For n larger than two, $D_n(x) = d_n(x-2)(x-2/3)x^{n+1}\epsilon_n(x)$ where $\epsilon_n(x)$ is a monic irreducible polynomial in $\mathbb{Q}[x]$.

5. The Fourier expansion of $\overline{H_{i,3}}$ reduces to Serre's weight-6 Eisenstein series E_3 in the sense that $\delta_3(0) = 1$ and $\delta_3(n) = -504\sigma_5(n)$ for $n = 1, 2, 3, \ldots$.

8.2. Analogues of $SL(2,\mathbb{Z})$ Cusp Forms

Let Δ be the usual normalized discriminant, a weight 12 cusp form for $SL(2,\mathbb{Z}) = G(\lambda_3)$ with integer coefficients. Its Fourier expansion is written

$$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n)q^n$$

where $q = e^{2\pi i\tau}$ and $\tau(n)$ is Ramanujan's function. (The reader will not confuse the complex number τ with Ramanujan's function $\tau(n)$ or any of its relatives defined below.) Whether or not the equation $\tau(n) = 0$ has any solutions is, of course, an open question [26]. Several authors have eliminated various classes of integers as values of tau.[16] It will be apparent that each of the conjectures about cusp-form analogues implies that tau has no zeros.

From definition 10.2,

$$\Delta_{\infty,m}(\tau)^{m-2} = \frac{J'_m(\tau)^{2m}}{J_m(\tau)^{2m-2}(J_m(\tau)-1)^m}$$

and, by Theorem 7 in our sketch of Hecke's theory, its weight is $4m$. Since it raises a cusp form beginning with an X^1 term to high powers, we will use the star operator (definition 1.2) to state the following conjecture; it will be evident that the images of power series in X_m under this operator typically have a constant term.

[16] These results are summarized in [25]. Relevant citations are [3, 4, 5, 6, 17, 29, 32] and [25] itself.

Conjecture 6. Let the Fourier expansion of $\overline{(\Delta_{\infty,m}(\tau)^{m-2})^*}$ be written as

$$\overline{(\Delta_{\infty,m}(\tau)^{m-2})^*} = \sum_{n=0}^{\infty} \overline{\tau}_m(n) X_m^n.$$

(See [10, notebooks “conjecture 6Laptop.nb”, “conjecture 6.ipynb” and “conjecture 6 no2.ipynb”].)

1. The number $\overline{\tau}_3(n-1)$ is equal to $\tau(n)$ for $n = 1, 2, \ldots.$
2. There is a set of polynomials $\overline{T}_n(x), n = 1, 2, 3\ldots$ such that, for each $n, \overline{T}_n(m) = \overline{\tau}_m(n)$.
3. The polynomial $\overline{T}_n(x)$ is equal to $(-8)^n(x-2)^3 x^n t_n(x)/n!$ where t_n is a polynomial with rational coefficients that is irreducible over $\mathbb{Q}[x]$.

Conjecture 7. Let the Fourier expansion of $\overline{\Delta_{\infty,m}}(\tau)$ be

$$\overline{\Delta_{\infty,m}}(\tau) = \sum_{n=1}^{\infty} \tau_{\infty,m}(n) X_m^n.$$

(See [10, notebook “conjecture 7.ipnyb”].)

1. For $n = 1, 2, 3, \ldots$, the number $\tau_{\infty,3}(n)$ is equal to $\tau(n)$.
2. There is a set of polynomials $T_{\infty,n}(x)$ with coefficients in $\mathbb{Q}$ such that $\tau_{\infty,m}(n) = T_{\infty,n}(m)$.
3. The polynomial $T_{\infty,1}(x)$ is identically equal to 1 and, if n is greater than one,
   (a) $T_{\infty,n}(x) = s_{\infty,n}(x-2)^2 x^{n-1} t_{\infty,n}(x)$, where $t_{\infty,n}(x)$ is a monic irreducible polynomial over $\mathbb{Q}$ of degree $2n-4$ and
   (b) $s_{\infty,n}$ is (in the notation of [16, Chapter 7, Theorem 7]) the coefficient of q^n in the Fourier expansion of $\Delta_8(z)$ [8].
   (c) Also,
   $$s_{\infty,n} = (-1)^{n+1} \sum_{\substack{\nu \mid n \\ n/\nu \text{odd}}} \nu^3 [9].$$
   This sum is the coefficient of q^n in the Fourier expansion of $E_{\infty,4}$, the unique normalized weight-4 modular form for $\Gamma_0(2)$ with simple zeros at $i\infty$) [11, eq. (2-3)]; it is also the number of representations of $n-1$ as a sum of 8 triangular numbers [33, Theorem 5].
   (d) Finally, $s_{\infty,n}$ is the coefficient of q^n in the expansion of $\eta(2z)^{16}/\eta(z)^{-8}$ where $\eta(z)$ is Dedekind’s function ([11, eq. (2-16)].)

Conjecture 8. Let the Fourier expansion of $\overline{\Delta_m^\diamond}(\tau)$ be

$$\overline{\Delta_m^\diamond} = \sum_{n=1}^{\infty} \tau_m^\diamond(n) X_m^n.$$

(See [10, notebook "conjecture 8.ipynb"].)

1. For $n = 1, 2, 3, ...$, the number $\tau_3^\diamond(n)$ is equal to $\tau(n)$.

2. There is a set of polynomials $T_n^\diamond(x)$ with coefficients in $\mathbb{Q}$ such that $\tau_m^\diamond(n) = T_n^\diamond(m)$.

3. The polynomials $T_1^\diamond(x), T_2^\diamond(x)$, and $T_3^\diamond(x)$ are irreducible over $\mathbb{Q}$ of degrees $3, 6$, and 9, respectively.

4. If n is greater than 3, $T_1^\diamond(x) = s_n^\diamond \cdot (x-2)x^{n-1} t_n^\diamond(x)$, where $s_n^\diamond$ is a rational number and $t_n^\diamond(x)$ is a monic polynomial, irreducible over $\mathbb{Q}$, of degree $2n-3$. Furthermore,

   (a) $\sum_{n=0}^{\infty} s_n^\diamond q(\tau)^n =$

   $$\prod_{n\text{odd}} (1 - q(\tau)^n)^{24} \times \prod_{n \equiv 2(4)} (1 - q(\tau)^n)^{-24} = \eta^{24}(\tau)\eta^{24}(4\tau)\eta^{-48}(2\tau).$$

   (b) $s_n^\diamond = (-1)^{n+1} \times$ the coefficient of $q(\tau)^n$ in $(\eta(2\tau)/\eta(\tau))^{24}$.

5. There is no corresponding set of interpolating polynomials for $\Delta_3^\diamond$.

The product decomposition in clause 4(a) above is a guess based on 43 terms of the series using Euler's method. (See [2, Theorem 14.8]; [19]; a printed English translation of [19] is reproduced in [34]; and [47]. The second decomposition appears in [43].)

Conjecture 9.

Let the Fourier expansion of $\overline{\Delta_{12,m}^\diamond}(\tau)$ be

$$\overline{\Delta_{12,m}^\diamond}(\tau) = \sum_{n=1}^{\infty} \tau_{12,m}^\diamond(n) X_m^n.$$

(See [10, notebook "conjecture 9.ipynb"].)

1. For $n = 1, 2, 3...$, the number $\tau_{12,3}^\diamond(n)$ is equal to $\tau(n)$.

2. There is a set of polynomials $T_{12,n}^\diamond(x), n = 1, 2, ...$ of degree $3n-3$ with coefficients in $\mathbb{Q}$ such that $\tau_{12,m}^\diamond(n) = T_{12,n}^\diamond(m)$ for each $m = 3, 4, ...$

3. For each n, there are zeros of $T^{\diamond}_{12,n}(x)$ on both axes of the complex plane, and there are no other complex zeros.[17] (Figures 2 and 3 illustrate this for $n = 11$ and 24.)

4. The polynomial $T^{\diamond}_{12,n}(x)$ is equal to $(-1)^{n+1}\tau(n)x^{n-1}t^{\diamond}_{12,n}(x)$, where $t^{\diamond}_{12,n}(x)$ is monic and irreducible over $\mathbb{Q}$.

9. Lehmer's Question

Remark 5. By clause 4 of Conjecture 9, for $m = 3, 4, \ldots$: $\tau^{\diamond}_{12,m}(n) = 0$ if and only if $\tau(n) = 0$.

More generally, we have the following conjecture.

Conjecture 10. Letting $T_n(x)$ and τ_m stand for the various polynomials and Fourier coefficients in Conjectures 6 through 9, none of the $T_n(x)$ has an integer root greater than two; consequently, none of the τ_m vanishes for $m = 3, 4, \ldots$.

Let $d(m, n)$ be the minimum Euclidean distance to m of any complex root of $T_n(x)$. We have (in effect) conjectured above that in each case $T_n(3) = \tau(n)$, so the behavior of $d(3, n)$ measures how closely we can come to the assertion that $\tau(n) = 0$ for some n.

Conjecture 11. For any positive real number r, $d(3, n)$ is less that e^{-rn} for sufficiently large n. [18]

10. Other Questions

1. Like G_n in clause 5 of Conjecture 1, the index-n hyperoctahedral group has size $2^n n!$ [18, 21, 31]. Are they isomorphic?

2. In Conjectures 1–9, the n^{th} interpolating polynomial is written as a product of a numerical term and several monic polynomials belonging to $\mathbb{Q}[x]$. In each case, all but one of the monic factors is given explicitly, *i.e.*, in terms of n, but without reference to the Fourier expansion of the underlying modular form. The "inexplicit" factor can, of course, be written in terms of the first n of these coefficients, but can it be expressed in the same way as the other factors: without reference to the Fourier coefficients?

[17] [10, notebook "conjecture 9.nb"] contains plots of the complex zeros for n between 1 and 24.

[18] For this proposal, we depend on graphical evidence which we sample in Figures 10 – 17. More extensive collections of plots are in [10, notebooks "conjecture 6.1.nb", "conjecture 6.2.nb", "conjecture 7.nb", and "conjecture 8.nb"].

3. While checking our calculations, we compared the Fourier expansion of $H_{\lambda,4}(x/A_4)$ (abusing notation in the obvious way) with Leo's expansion of the weight 4 Eisenstein series at $m = 4$ [28, page 54]. (Recall that $A_4 = 1/256$.) Within the range of our observations, they do coincide. The expansions (in our own notation) both begin

$$1 + 48q_4 + 624q_4^2 + 1344q_4^3 + \ldots.$$

Let

$$E_{\gamma,2} = 1 + 24 \sum_{n=1}^{\infty} \sum_{\substack{\nu|n \\ \nu \text{odd}}} \nu q^n.$$

Sloane comments that the sequence $\{1, 48, 624, ...\}$ is the same as that of the coefficients of $E_{\gamma,2}^2$ [42]. $E_{\gamma,2}^2$ is a weight 4, level 2 modular form, that is, a weight 4 modular form for the $SL(2,\mathbb{Z})$ subgroup $\Gamma_0(2)$ [12, eq. (2-1), page 260]. We propose in Conjecture 7.3 (c) above that $s_{\infty,n}$ is the coefficient of q^n in the Fourier expansion of $E_{\infty,4}$, the unique normalized weight-4 modular form for $\Gamma_0(2)$ with simple zeros at $i\infty$. We have also proposed in Conjectures 1, 2, 7 and 8 that interpolating polynomials are products of monic polynomials with rational numbers equal or related to Fourier coefficients of other classical Hauptmoduln. What is the relationship between modular forms for subgroups of $SL(2,\mathbb{Z})$ and modular forms for the other $G(\lambda_m)$?

4. Both J_m and $\overline{J_m}$ (that is, j_m) appear to be interpolated by polynomials. On the other hand, $\overline{\Delta_m^\diamond}$ appears to be interpolated by polynomials, but $\Delta_m^\diamond$ does not. Why are the situations different?

11. Figures

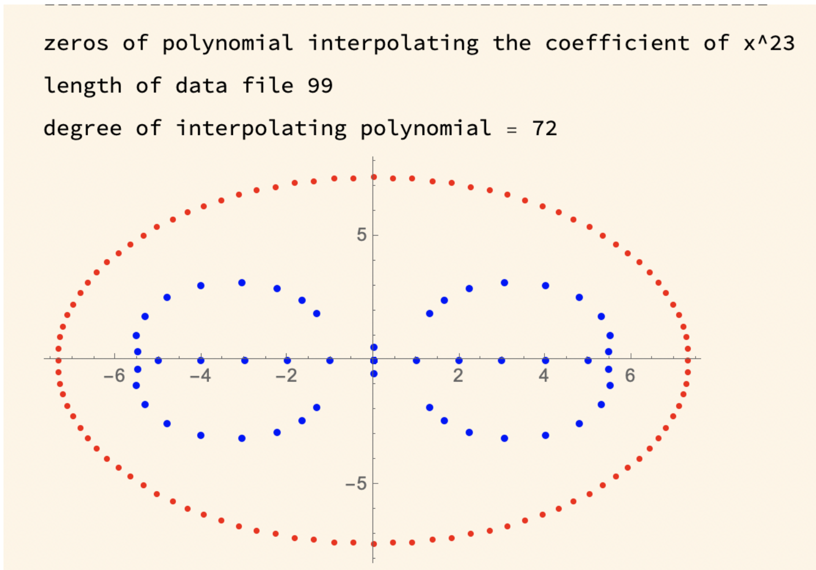

Figure 1: roots of polynomial interpolating the coefficient of q_m^{23} in the Fourier series of $j_m(\tau)$ (Conjecture 1.) See [10, notebook "conjecture1clause4d.nb"].

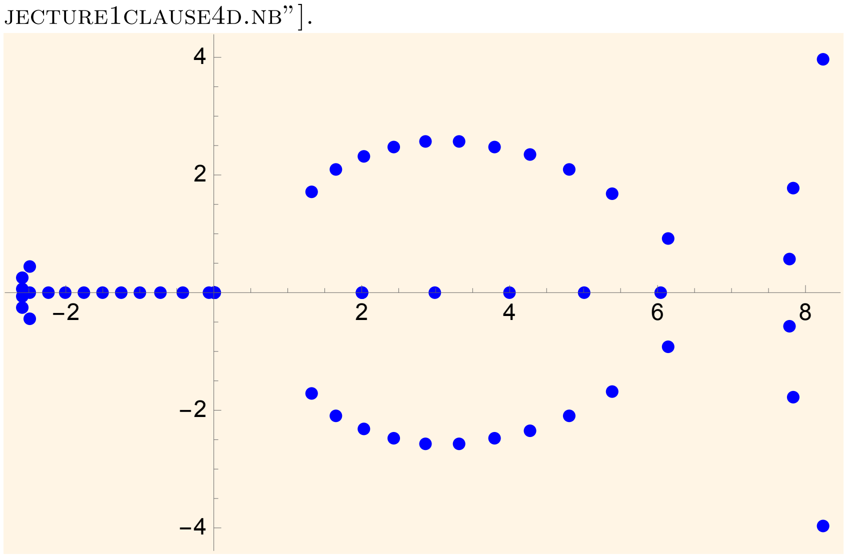

Figure 2: roots of $\overline{T}_{17}$ (Conjecture 6.) See [10, notebook "conjecture 6Laptop.nb"].

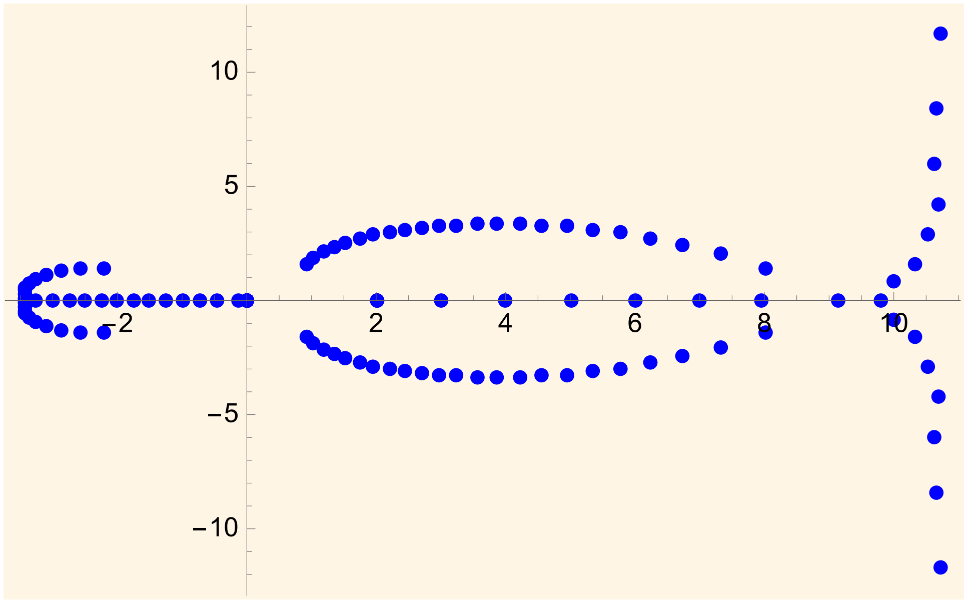

FIGURE 3: ROOTS OF $\overline{T}_{35}$ (CONJECTURE 6.) SEE [10, NOTEBOOK "CONJECTURE 6LAPTOP.NB"].

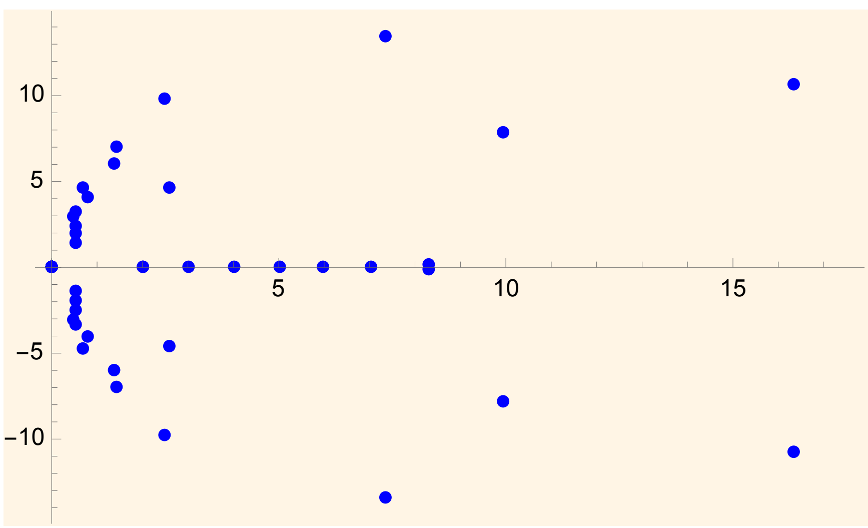

FIGURE 4: ROOTS OF $T_{\infty,20}$ (CONJECTURE 7.) SEE [10, NOTEBOOK "CONJECTURE 7.NB"].

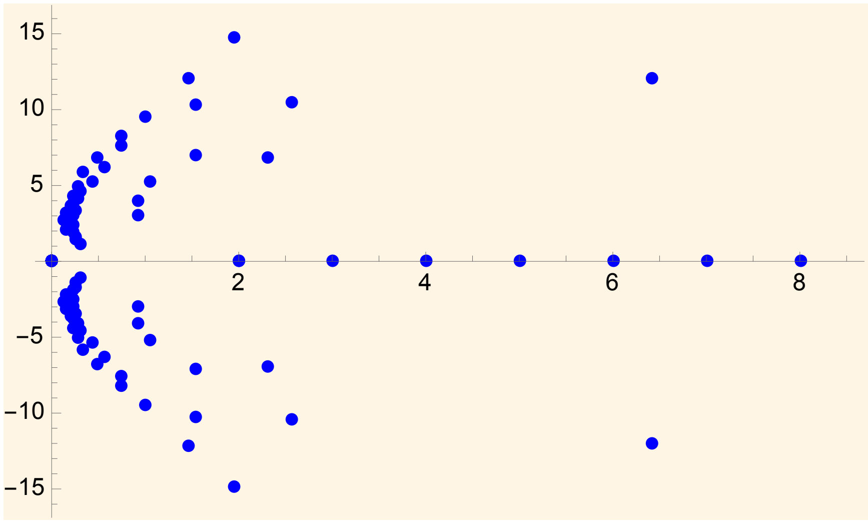

FIGURE 5: ROOTS OF $T_{\infty,50}$ (CONJECTURE 7.) SEE [10, NOTEBOOK "CONJECTURE 7.NB"].

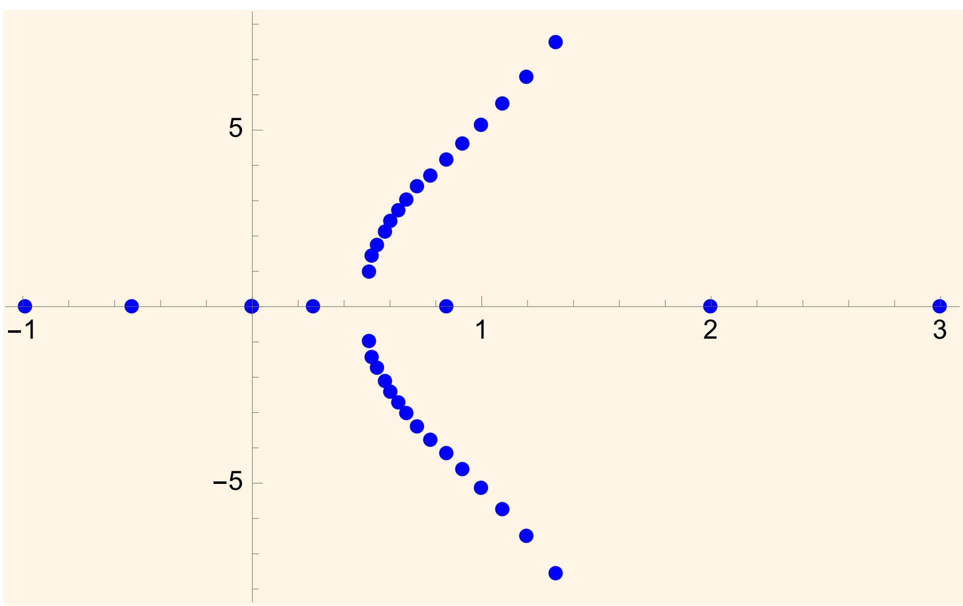

FIGURE 6: ROOTS OF $T^{\diamond}_{19}$ (CONJECTURE 8.) SEE [10, NOTEBOOK "CONJECTURE 8.NB"].

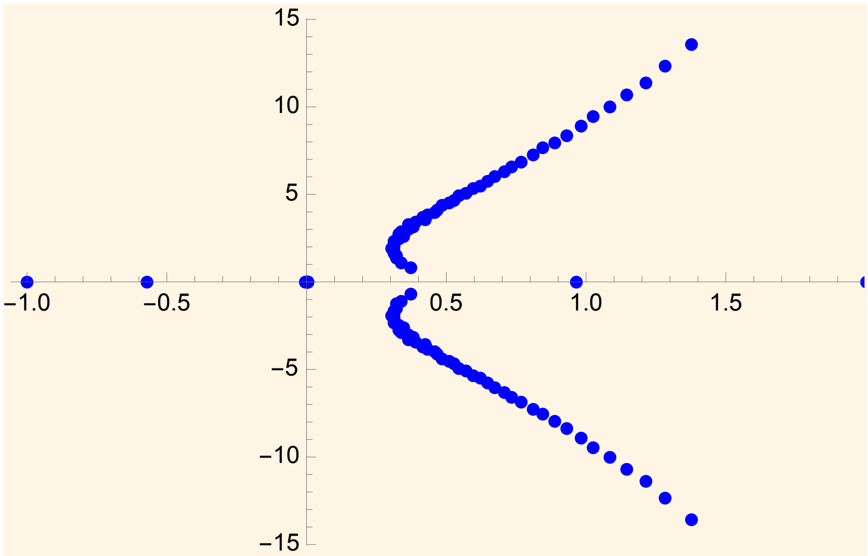

FIGURE 7: ROOTS OF $T^{\diamond}_{50}$ (CONJECTURE 8.) SEE [10, NOTEBOOK "CONJECTURE 8.NB"].

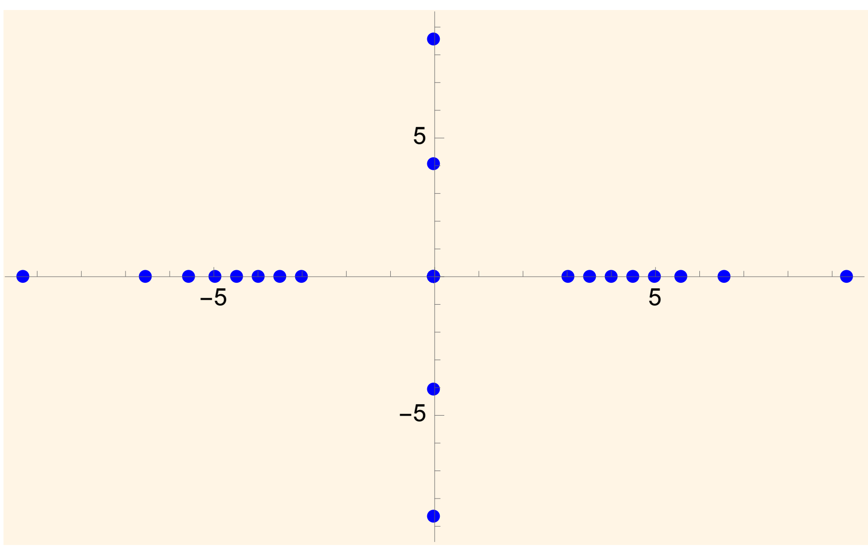

FIGURE 8: ROOTS OF $T^{\diamond}_{12,11}$ (CONJECTURE 9.) SEE [10, NOTEBOOK "CONJECTURE 9.NB"].

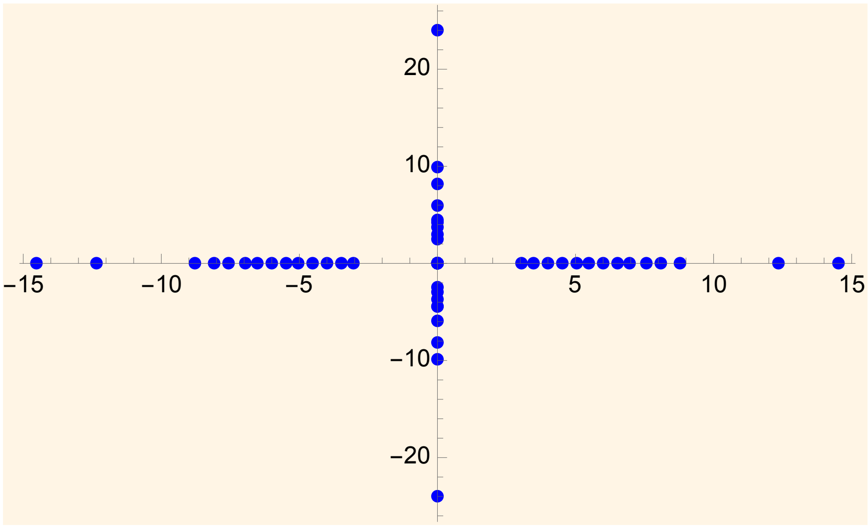

FIGURE 9: ROOTS OF $T^{\diamond}_{12,24}$ (CONJECTURE 9.) SEE [10, NOTEBOOK "CONJECTURE 9.NB"].

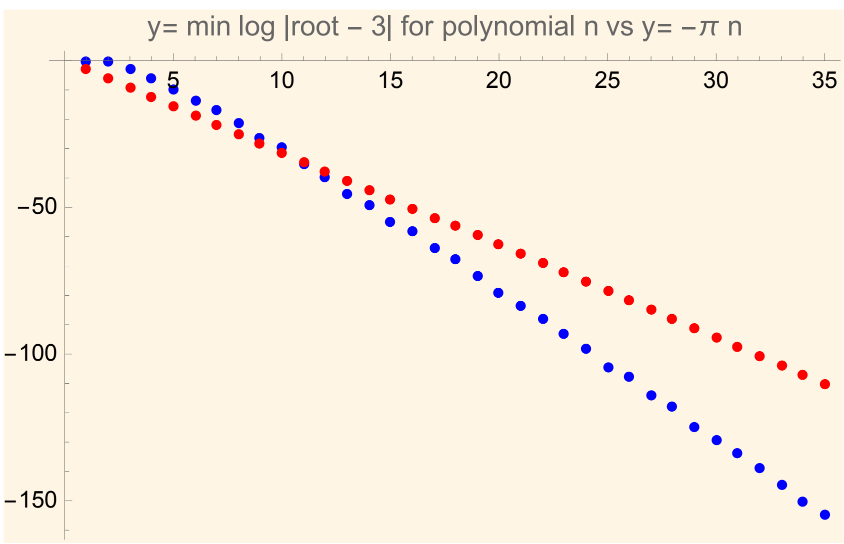

FIGURE 10: $y = $ LOG(MINIMUM DISTANCE OF ROOTS OF $\overline{T}_n$ FROM 3) IN BLUE VS $y = -\pi n$ IN RED (CONJECTURES 6 AND 11.) SEE [10, NOTEBOOK "CONJECTURE 6LAPTOP.NB"].

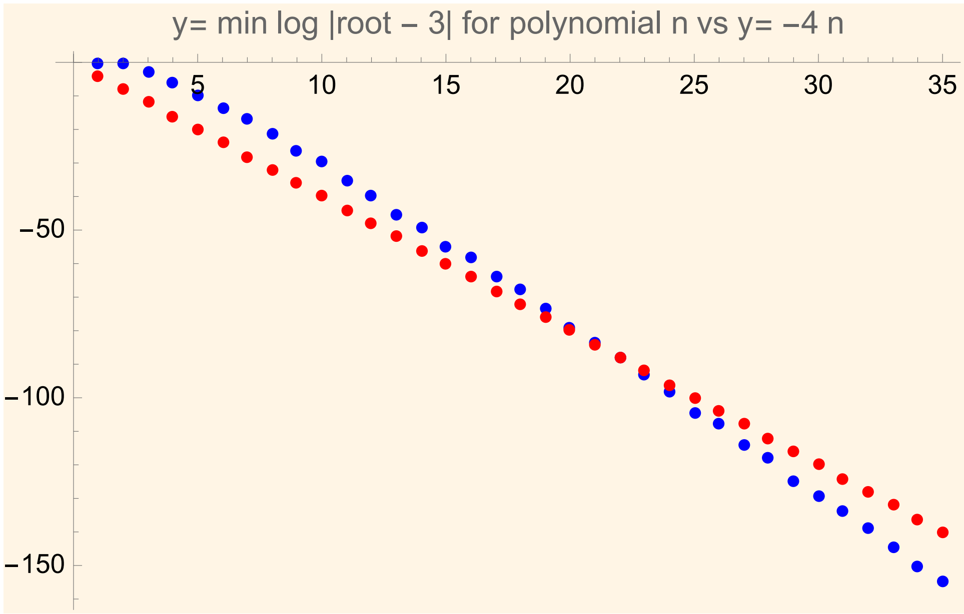

FIGURE 11: $y = $ LOG(MINIMUM DISTANCE OF ROOTS OF $\overline{T}_n$ FROM 3) IN BLUE VS $y = -4n$ IN RED (CONJECTURES 6 AND 11.) SEE [10, NOTEBOOK "CONJECTURE 6LAPTOP.NB"].

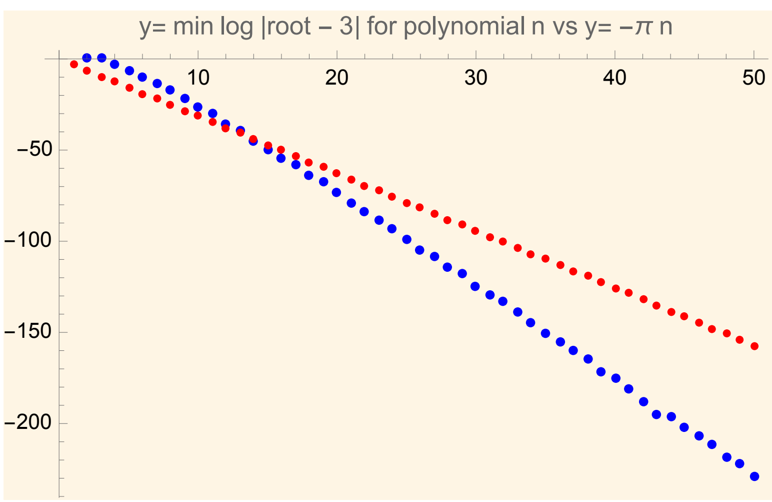

FIGURE 12: $y = $ LOG(MINIMUM DISTANCE OF ROOTS OF $T_{\infty,n}$ FROM 3) IN BLUE VS $y = -\pi n$ IN RED (CONJECTURES 7 AND 11.) SEE [10, NOTEBOOK "CONJECTURE 7.NB"].

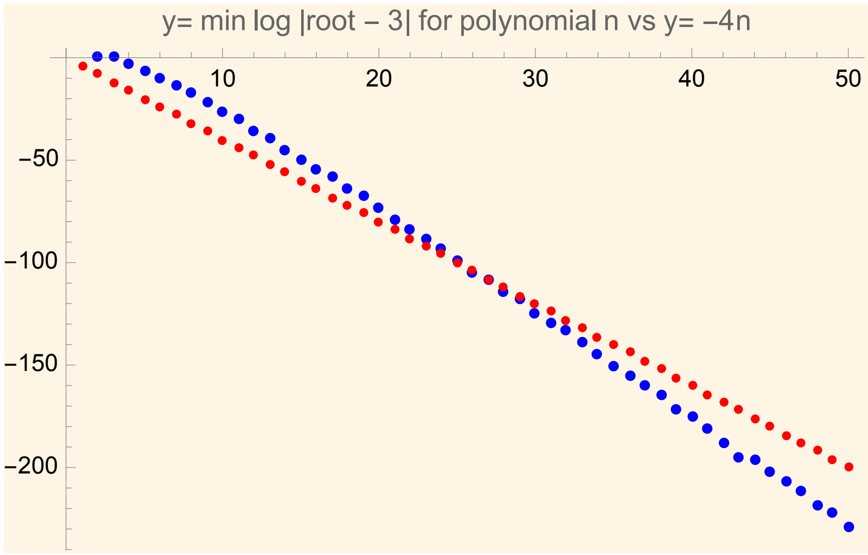

FIGURE 13: $y = \log$(minimum distance of roots of $T_{\infty,n}$ from 3) in blue vs $y = -4n$ in red (Conjectures 7 and 11.) See [10, notebook "conjecture 7.nb"].

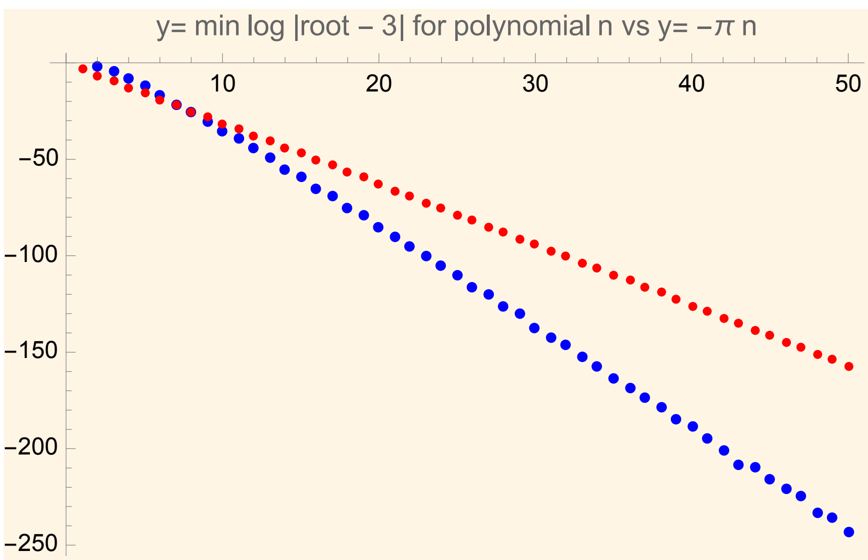

FIGURE 14: $y = \log$(minimum distance of roots of $T_n^{\diamond}$ from 3) in blue vs $y = -\pi n$ in red (Conjectures 8 and 11.) See [10, notebook "conjecture 8.nb"].

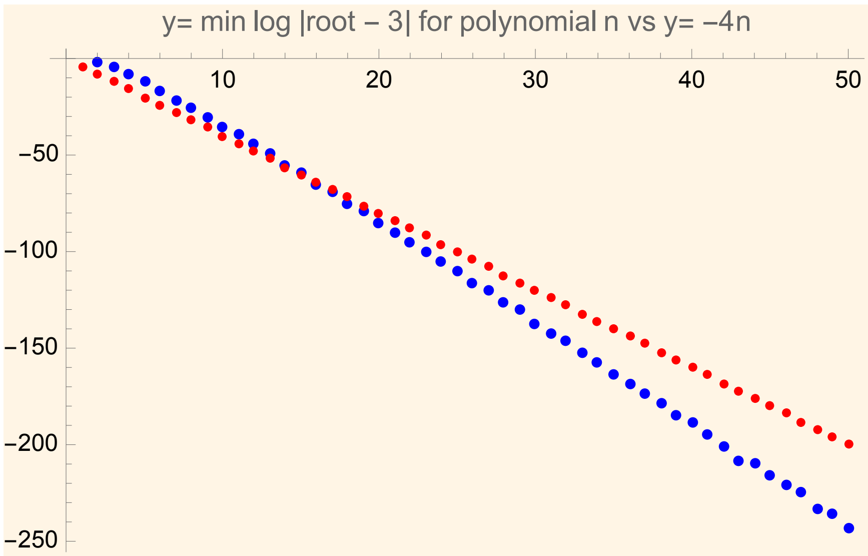

FIGURE 15: $y = $ LOG(MINIMUM DISTANCE OF ROOTS OF $T_n^{\diamond}$ FROM 3) IN BLUE VS $y = -4n$ IN RED (CONJECTURES 8 AND 11.) SEE [10, NOTEBOOK "CONJECTURE 8.NB"].

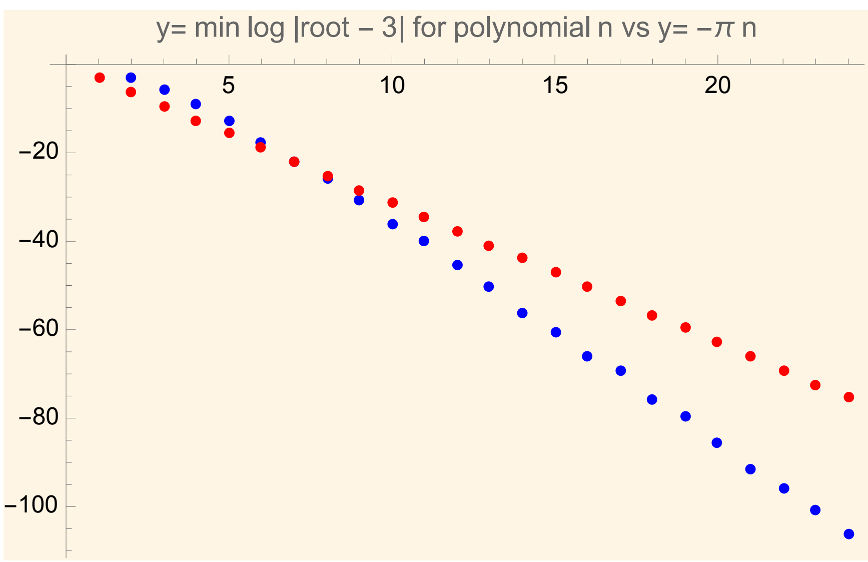

FIGURE 16: $y = $ LOG(MINIMUM DISTANCE OF ROOTS OF $T_{12,n}^{\diamond}$ FROM 3) IN BLUE VS $y = -\pi n$ IN RED (CONJECTURES 9 AND 11.) SEE [10, NOTEBOOK "CONJECTURE 9.NB"].

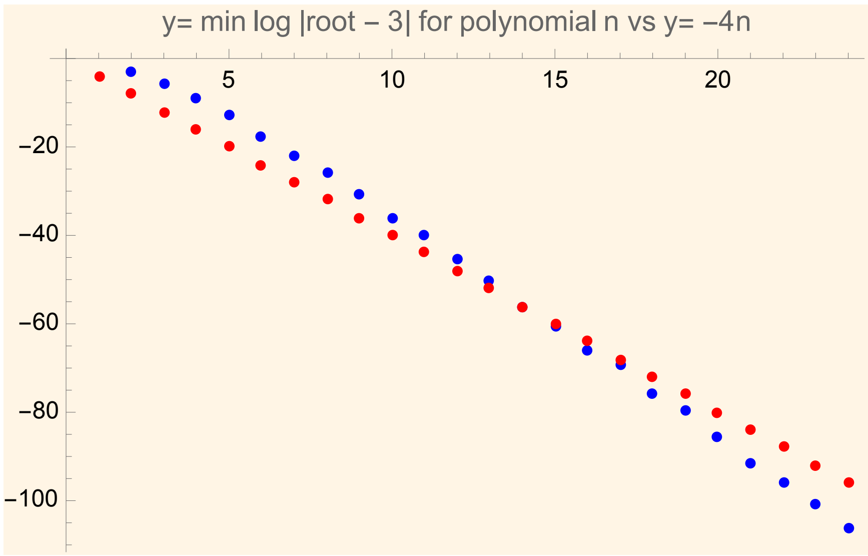

FIGURE 17: $y = \log$(MINIMUM DISTANCE OF ROOTS OF $T^{\diamond}_{12,n}$ FROM 3) IN BLUE VS $y = -4n$ IN RED (CONJECTURES 9 AND 11.) SEE [10, NOTEBOOK "CONJECTURE 9.NB"].

12. Tables

12.1. Table 1

```
m:  3
[1, 744, 196884, 21493760, 864299970, 20245856256, 333202640600, 4252023300096, 44656994071935]

m:  4
[1, 1664, 1119232, 394264576, 81264771072, 11218454577152, 1248696645713920, 115837948032712704, 947788423537
8327552]

m:  5
[1, 3160, 4287700, 3316992000, 1669770821250, 611783148748800, 182311990830375000, 331082591561671680000/7, 1
0920021527700042909375]

m:  6
[1, 5376, 12828672, 18186502144, 17546820452352, 12765329998479360, 7659879696296837120, 40316764296175964651
52, 1914422267289591355539456]

m:  7
[1, 8456, 32384884, 75420047360, 122421510269730, 763277744265838592/5, 158451050063635522200, 72433296238511
8166556672/5, 1201696957016428623788444447]

m:  8
[1, 12544, 72208384, 769403125760/3, 644070915440640, 18847818977487355904/15, 18474189128405614592000/9, 148
09640947694988624396288/5, 350048632201721335293138669824/9]

m:  9
[1, 17784, 146452212, 751427905536, 2751723239454882, 39426163455019106304/5, 18997742593479521191320, 141775
99719109530714812989 44/35, 39373050735713829163067599 1067/5]

m:  10
[1, 24320, 275660800, 5880414208000/3, 10012474736640000, 120667058077486284800/3, 12262001750268968960000000/
9, 28630432282497411081830400000/7, 1008748798729537148231352320000 0/9]

m:  11
[1, 32296, 488459092, 4657814779904, 32056430053661442, 870684224968695627776/5, 799294542465481997797720, 11
392253630287460115406604697 6/35, 605519658989119051458542859067 47/5]

m:  12
[1, 41856, 823440384, 10252263096320, 92450199949148160, 3298990680702153916416/5, 3985163925711416931123200,
74803043009994337477291356979 2/35, 104790308403795592053429519777792]

m:  13
[1, 53144, 1331252884, 21166363412480, 244399876142003010, 11190259716337492803584/5, 17366247753115182553546
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m:  14
[1, 66304, 2076884992, 41386975952896, 600207736273108992, 34572394432541086449664/5, 67562583266770425709854
720, 293417688630571953330895139635 2/5, 233209588178604993748214172530769 92/5]

m:  15
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000, 179655703910920277658283161600 00/7, 252730737159863805742504357434093 75]

m:  16
[1, 98816, 4628365312, 138376866955264, 3020070316337528832, 262411526961908842233856/5, 774664686021625638 10
0664320, 35601453105291992883982470060441 6/35, 61143501653457979662455060741436 2112/5]

m:  17
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98342124311000/9, 129474296579708535593141350266470 4/35, 482453631252394965546606128659937 9159/9]

m:  18
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36041779200, 436519395936750157514629036218777 6/35, 215418797923359071468883024173806 3872]

m:  19
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15013741741117080/9, 393139994684825658986354734571520, 721246348222191334455305361751584 10487/9]

m:  20
[1, 192640, 17658803200, 1037317439488000, 44639834873610240000, 153376581323454540677 1200, 4483525812623587 7
769216000000, 816851839617614117184064192512 0000/7, 278258504051670528692179160793088 00000]
```

Fourier coefficients $c_m(n)$ (Conjecture 1.) See [10, notebook "conjecture 1 tables.ipynb"].

12.2. Table 2

```
n:  0
24*x^3 + 32*x

n:  1
276*x^6 - 32*x^4 - 192*x^2

n:  2
2048*x^9 - 237568/27*x^7 + 32768/27*x^5 + 131072/27*x^3

n:  3
11202*x^12 - 122272*x^10 + 332480*x^8 - 51712*x^6 - 155136*x^4

n:  4
49152*x^15 - 1072627712/1125*x^13 + 4173856768/675*x^11 - 45736787968/3375*x^9 + 1564475392/675*x^7 + 6257901
568/1125*x^5

n:  5
184024*x^18 - 144266176/27*x^16 + 43337949824/729*x^14 - 217405085696/729*x^12 + 140124878848/243*x^10 - 7790
0890112/729*x^8 - 155801780224/729*x^6

n:  6
614400*x^21 - 3105702576128/128625*x^19 + 3087576727552/7875*x^17 - 181127478771712/55125*x^15 + 605256780087
296/42875*x^13 - 39732717289472/1575*x^11 + 276526671069184/55125*x^9 + 1106106684276736/128625*x^7

n:  7
1881471*x^24 - 105854663648/1125*x^22 + 61647569321408/30375*x^20 - 2214841137463808/91125*x^18 + 15555313881
637376/91125*x^16 - 4024369502937088/6075*x^14 + 102528260099096576/91125*x^12 - 21727588469571584/91125*x^10
- 10863794234785792/30375*x^8

n:  8
5373952*x^27 - 10273321450471424/31255875*x^25 + 2504376772162748416/281302875*x^23 - 39204345181884121088/28
1302875*x^21 + 128732665203233128448/93767625*x^19 - 2404056704249503940608/281302875*x^17 + 2901573412174312
767488/93767625*x^15 - 1590811983944125251584/31255875*x^13 + 1072440905902422228992/93767625*x^11 + 42897636
23609688915968/281302875*x^9

n:  9
14478180*x^30 - 135380572197152/128625*x^28 + 49971491504672789 12/144703125*x^26 - 58282744523269031936/86821
875*x^24 + 1233199636594239219712/144703125*x^22 - 70555734228780674252 8/9646875*x^20 + 18123673665010426206 6
176/434109375*x^18 - 1390633318492274111283 2/9646875*x^16 + 33675952843327111246643 2/144703125*x^14 - 4796417
6875082926260224/86821875*x^12 - 95928353750165852520448/144703125*x^10

n:  10
37122048*x^33 - 8700800730301448192/2773437975*x^31 + 275524637447268663296/2269176525*x^29 - 193050946826125
80220928/6807529575*x^27 + 4306202211369503581798 4/972504225*x^25 - 1097244752820562334580736/2269176525*x^23
+ 2802994405000673051835105 28/74882825325*x^21 - 1364184034084454111935201 28/6807529575*x^19 + 15209668785017
5801795280896/2269176525*x^17 - 9018954917651823196110848/84043575*x^15 + 60883733248483864447287296/22691765
25*x^13 + 24353493299393545778914918 4/8320313925*x^11

n:  11
91231550*x^36 - 275311453932678496/31255875*x^34 + 415438005083265837278 08/105488578125*x^32 - 1023195699428 7
088820478464/949397203125*x^30 + 2722879473922321167043942 4/135628171875*x^28 - 2539126818467141230257455104/
949397203125*x^26 + 8241939235790494063111929856/316465734375*x^24 - 17664156650231009741036349030 4/949397203
125*x^22 + 900980586709588453508522835968/949397203125*x^20 - 10941801917483878204106289971 2/3516285937 5*x^18
+ 4729632263755748417927672496128/949397203125*x^16 - 1242638610598866834802452987904/949397203125*x^14 - 124
263861059886683480245298790 4/949397203125*x^12

n:  12
216072192*x^39 - 238749387742101240020992/10155405385125*x^37 + 10499395751226728221423697 92/878833158328125*
x^35 - 297041305554441632386346123264/7909498424953125*x^33 + 1757458215387066419623762067456/215713593407812
5*x^31 - 2775638125717669984486035973734 4/2157135934078125*x^29 + 36018993007393609204581480225832 96/23728495
274859375*x^27 - 4165099235276834505316495788836126 72/308470438573171875*x^25 + 2145990902868033640710450920 0
4069376/23728495274859375*x^23 - 1369885410423803830444408530219827 2/308162276296875*x^21 + 3111676456781325 3
4837954679423369216/2157135934078125*x^19 - 551751760130944132733234802011301478 4/23728495274859375*x^17 + 15
20585638035255818410888487136919552/23728495274859375*x^15 + 60823425521410232736435539485476782 08/1028234795
24390625*x^13

n:  13
495248952*x^42 - 1665494863962368652 16/2773437975*x^40 + 5497433496942319604306362662 4/16053005679046875*x^38
- 35911093777780324418002130944/2948511247171875*x^36 + 1869641349258146187285389556797 44/619187361906093 75*
x^34 - 652798134055141506270356797500620 8/11820849636389 0625*x^32 + 2037930341171858230942398773169356 8/26536
601224546875*x^30 - 15283087934396853356736737053968957 44/185756208571828125*x^28 + 884242924564141669068432 7
3018515914752/130029346000279687 5*x^26 - 114613273191428752404378846216763473 92/26536601224546875*x^24 + 3483
7153800953430313205261294712651776/16886928051984375*x^22 - 868146287539724809228717939024271822028 8/13002934
60002796875*x^20 + 5904162372782415220171498466289123328/541563290296875*x^18 - 58458891119085835154359003572
8390684672/185756208571828125*x^16 - 1169177822381716703087180071456781369344/433431153334265625*x^14
```

Polynomials $C_n(x)$ (Conjecture 1.) See [10, notebook "conjecture 1 tables.ipynb"].

12.3. Table 3

```
n:  0
(24) * x * (x^2 + 4/3)

n:  1
(276) * x^2 * (x^4 - 8/69*x^2 - 16/23)

n:  2
(2048) * (x - 2) * (x + 2) * x^3 * (x^4 - 8/27*x^2 - 16/27)

n:  3
(11202) * (x - 2) * (x + 2) * x^4 * (x^6 - 38732/5601*x^4 + 11312/5601*x^2 + 6464/1867)

n:  4
(49152) * (x - 2) * (x + 2) * x^5 * (x^8 - 51968/3375*x^6 + 650144/10125*x^4 - 190976/10125*x^2 - 95488/3375)

n:  5
(184024) * (x - 2) * (x + 2) * x^6 * (x^10 - 15548948/621081*x^8 + 3737957344/16769187*x^6 - 452733568/621081
*x^4 + 1217201408/5589729*x^2 + 4868805632/16769187)

n:  6
(614400) * (x - 2) * (x + 2) * x^7 * (x^12 - 340526584/9646875*x^10 + 14381852336/28940625*x^8 - 32414868736/
9646875*x^6 + 18398435584/1929375*x^4 - 16877848576/5788125*x^2 - 33755697152/9646875)

n:  7
(1881471) * (x - 2) * (x + 2) * x^8 * (x^14 - 97388044148/2116654875*x^12 + 51129660553424/57149681625*x^10 -
320257042164544/34289808975*x^8 + 3050057679448832/57149681625*x^6 - 2640538932296704/19049893875*x^4 + 74688
58536415232/171449044875*x^2 + 2715948558696448/57149681625)

n:  8
(5373952) * (x - 2) * (x + 2) * x^9 * (x^16 - 73253258992/1281490875*x^14 + 16469761126016/11533417875*x^12 -
46645274445568/2306683575*x^10 + 2013551386772992/11533417875*x^8 - 1469613075017728/1647631125*x^6 + 2526257
8868092928/11533417875*x^4 - 8182074782580736/11533417875*x^2 - 8182074782580736/11533417875)

n:  9
(14478180) * (x - 2) * (x + 2) * x^10 * (x^18 - 31982887146788/465563975625*x^16 + 1105364295456273728/523759
472578125*x^14 - 5958905910861l005184/1571278417734375*x^12 + 68654349101l235394048/1571278417734375*x^10 - 5
19134613669288427724 8/1571278417734375*x^8 + 2454379961575452840 7552/1571278417734375*x^6 - 58271049867362723
889152/1571278417734375*x^4 + 19485446855502438793216/1571278417734375*x^2 + 59955221093853 65782528/523759472
578125)

n:  10
(37122048) * (x - 2) * (x + 2) * x^11 * (x^20 - 38182570845784/474257893725*x^18 + 667081983908264464/2262210
15306825*x^16 - 1252564368273433088/19390372740585*x^14 + 1437996691319776768/1538918471475*x^12 - 2101160162
551379873792/226221015306825*x^10 + 6174079805745740849152/96951863702925*x^8 - 2149819804501768654028 8/75407
005102275*x^6 + 3009688164273557084569 6/45244203061365*x^4 - 7432096343809065484288/32317287900975*x^2 - 1486
4192687618130968576/75407005102275)

n:  11
(91231550) * (x - 2) * (x + 2) * x^12 * (x^22 - 131952683120626748/1425760961428125*x^20 + 271293414743354725
2272/6874204635457 03125*x^18 - 4432319091990290502666688/433074892033792968 75*x^16 + 861905613548000875954124
8/48119432448199218 75*x^14 - 9592773883562902997852426 24/43307489203379296875*x^12 + 852579930026057989552692
4288/43307489203379296875*x^10 - 619629554858431189977989 12/49494273375290625*x^8 + 778733163847811034206550 7
5328/1443582973445976 5625*x^6 - 18088782074765010550216274 7392/14435829734459765625*x^4 + 3883245658121458858
7576655872/8661497840675859375*x^2 + 15532982632485835435030662 3488/43307489203379296875)

n:  12
(216072192) * (x - 2) * (x + 2) * x^13 * (x^24 - 3509097060138717472/33482371554757125*x^22 + 192479547796556
354381536/37667667999101765625*x^20 - 519931143952868194634332 16/339009011991915890625*x^18 + 321086846710565
634158220313 6/1017027035975747671875*x^16 - 47721139775624433926503088128/1017027035975747671875*x^14 + 17453
4507468009441928745009152/339009011991915890625*x^12 - 4261022975930814347789654294528/1017027035975747671875
*x^10 + 25524697368256169685922471018496/1017027035975747671875*x^8 - 35712721208893488548797959110656/339009
011991915890625*x^6 + 35773896497934458041423781429248/145289576567963953125*x^4 - 9280918200898778188543020 5
513728/1017027035975747671875*x^2 - 23202295502246945471357551378432/339009011991915890625)

n:  13
(495248952) * (x - 2) * (x + 2) * x^14 * (x^26 - 2013191467425173205 2/171692781319469025*x^24 + 6405687716682
286279049023328/99377927987475165032 8125*x^22 - 1968949736088342181014498212224/894401351887276485295312 5*x^2
0 + 12724201929729789565027055243008/243927641423802677807812 5*x^18 - 1206013171658489832832038789 0176/133051
440776619642440625*x^16 + 956377146426408626906865861623 97184/804961216698548836765781 25*x^14 - 6364795571261
074786344787650484633 6/536640811132365891177187 5*x^12 + 72341592142951259915485365370237 09184/804961216698548
83676578125*x^10 - 750254417683083761481598130175868928/14635658485428160668468 75*x^8 + 469012764308259887709
777000889581568/22175240129436607073437 5*x^6 - 16167053025962506183252048739546693632/3219844866794195347063 1
25*x^4 + 15528142953507174962876610324035377561 6/804961216698548836765781 25*x^2 + 365368069494286469714743772
33024417792/268320405566182945588593 75)
```

Factored $C_n(x)$ (Conjecture 1.) See [10, notebook "conjecture 1 tables.ipynb"].

12.4. Table 4

m: 3

$$\frac{31}{72}+\frac{1}{X}+\frac{1823\,X}{27\,648}+\frac{10\,495\,X^2}{2\,519\,424}+\frac{1\,778\,395\,X^3}{18\,345\,885\,696}+\frac{45\,767\,X^4}{34\,828\,517\,376}+\frac{41\,650\,330\,075\,X^5}{3\,327\,916\,660\,110\,655\,488}+$$
$$\frac{711\,997\,X^6}{7\,703\,510\,787\,293\,184}+\frac{1\,653\,962\,743\,405\,X^7}{2\,944\,327\,674\,199\,660\,893\,831\,168}+\frac{1\,021\,044\,125\,X^8}{349\,351\,379\,311\,776\,170\,508\,288}$$

m: 4

$$\frac{13}{32}+\frac{1}{X}+\frac{1093\,X}{16\,384}+\frac{47\,X^2}{8192}+\frac{620\,001\,X^3}{2\,147\,483\,648}+\frac{653\,X^4}{67\,108\,864}+$$
$$\frac{9\,303\,515\,X^5}{35\,184\,372\,088\,832}+\frac{52\,677\,X^6}{8\,796\,093\,022\,208}+\frac{2\,206\,741\,887\,X^7}{18\,446\,744\,073\,709\,551\,616}+\frac{77\,191\,X^8}{36\,028\,797\,018\,963\,968}$$

m: 5

$$\frac{79}{200}+\frac{1}{X}+\frac{42\,877\,X}{640\,000}+\frac{12\,957\,X^2}{2\,000\,000}+\frac{1\,335\,816\,657\,X^3}{3\,276\,800\,000\,000}+\frac{1\,493\,611\,203\,X^4}{80\,000\,000\,000\,000}+\frac{1\,458\,495\,926\,643\,X^5}{2\,097\,152\,000\,000\,000\,000}+$$
$$\frac{64\,664\,568\,664\,389\,X^6}{2\,867\,200\,000\,000\,000\,000\,000}+\frac{3\,494\,406\,888\,864\,013\,731\,X^7}{5\,368\,709\,120\,000\,000\,000\,000\,000\,000}+\frac{23\,644\,062\,224\,068\,813\,X^8}{1\,376\,256\,000\,000\,000\,000\,000\,000\,000}$$

m: 6

$$\frac{7}{18}+\frac{1}{X}+\frac{29\,X}{432}+\frac{271\,X^2}{39\,366}+\frac{269\,X^3}{559\,872}+\frac{215\,X^4}{8\,503\,056}+$$
$$\frac{1\,741\,655\,X^5}{1\,586\,874\,322\,944}+\frac{307\,X^6}{7\,346\,640\,384}+\frac{491\,999\,X^7}{342\,764\,853\,755\,904}+\frac{235\,733\,X^8}{5\,205\,741\,216\,417\,792}$$

m: 7

$$\frac{151}{392}+\frac{1}{X}+\frac{165\,229\,X}{2\,458\,624}+\frac{107\,365\,X^2}{15\,059\,072}+\frac{25\,493\,858\,865\,X^3}{48\,358\,655\,787\,008}+\frac{2\,771\,867\,459\,X^4}{92\,561\,489\,592\,320}+\frac{168\,351\,462\,893\,475\,X^5}{118\,895\,751\,725\,676\,756\,992}+$$
$$\frac{36\,826\,135\,390\,421\,541\,X^6}{624\,462\,781\,016\,702\,892\,113\,920}+\frac{20\,845\,419\,590\,657\,658\,847\,X^7}{9\,354\,238\,358\,105\,289\,311\,446\,368\,256}+\frac{53\,674\,329\,840\,187\,738\,667\,X^8}{690\,893\,631\,231\,484\,559\,139\,312\,500\,736}$$

m: 8

$$\frac{49}{128}+\frac{1}{X}+\frac{17\,629\,X}{262\,144}+\frac{11\,465\,X^2}{1\,572\,864}+\frac{307\,116\,945\,X^3}{549\,755\,813\,888}+\frac{34\,283\,983\,X^4}{1\,030\,792\,151\,040}+\frac{2\,150\,678\,672\,875\,X^5}{1\,297\,036\,692\,682\,702\,848}+$$
$$\frac{52\,614\,413\,973\,X^6}{720\,575\,940\,379\,279\,360}+\frac{31\,836\,737\,635\,032\,599\,X^7}{10\,880\,332\,376\,531\,662\,572\,355\,584}+\frac{83\,000\,417\,975\,587\,X^8}{765\,023\,370\,224\,882\,524\,618\,752}$$

m: 9

$$\frac{247}{648}+\frac{1}{X}+\frac{150\,671\,X}{2\,239\,488}+\frac{13\,589\,191\,X^2}{1\,836\,660\,096}+\frac{69\,901\,012\,027\,X^3}{120\,367\,356\,051\,456}+$$
$$\frac{366\,770\,621\,371\,X^4}{10\,282\,945\,612\,677\,120}+\frac{3\,257\,500\,444\,698\,134\,635\,X^5}{1\,768\,591\,357\,765\,866\,863\,198\,208}+\frac{108\,551\,656\,609\,834\,559\,X^6}{1\,289\,597\,865\,037\,611\,254\,415\,360}+$$
$$\frac{2\,222\,620\,238\,316\,981\,329\,803\,361\,X^7}{633\,718\,259\,619\,258\,503\,956\,804\,550\,000\,640}+\frac{641\,719\,347\,824\,464\,135\,620\,559\,X^8}{4\,737\,105\,877\,202\,748\,260\,290\,391\,042\,949\,120}$$

m: 10

$$\frac{19}{50}+\frac{1}{X}+\frac{673\,X}{10\,000}+\frac{701\,X^2}{93\,750}+\frac{59\,679\,X^3}{100\,000\,000}+\frac{2\,194\,921\,X^4}{58\,593\,750\,000}+\frac{17\,843\,561\,X^5}{9\,000\,000\,000\,000}+$$
$$\frac{254\,289\,321\,X^6}{2\,734\,375\,000\,000\,000}+\frac{89\,594\,891\,393\,X^7}{22\,500\,000\,000\,000\,000\,000}+\frac{87\,629\,178\,911\,X^8}{553\,710\,937\,500\,000\,000\,000}$$

FOURIER COEFFICIENTS $a_m(n)$ (CONJECTURE 2.) SEE [10, NOTEBOOK "CONJECTURE 2.NB"].

12.5. Table 5

n: −1

$$1$$

n: 0

$$\frac{1}{2}+\frac{3x^2}{8}$$

n: 1

$$-\frac{3}{64}-\frac{x^2}{128}+\frac{69x^4}{1024}$$

n: 2

$$\frac{1}{54}+\frac{x^2}{216}-\frac{29x^4}{864}+\frac{x^6}{128}$$

n: 3

$$-\frac{303}{32\,768}-\frac{101x^2}{32\,768}+\frac{5195x^4}{262\,144}-\frac{3821x^6}{524\,288}+\frac{5601x^8}{8\,388\,608}$$

n: 4

$$\frac{373}{72\,000}+\frac{373x^2}{172\,800}-\frac{21\,809x^4}{1\,728\,000}+\frac{7961x^6}{1\,382\,400}-\frac{16\,367x^8}{18\,432\,000}+\frac{3x^{10}}{65\,536}$$

n: 5

$$-\frac{4\,754\,693}{1\,528\,823\,808}-\frac{4\,754\,693x^2}{3\,057\,647\,616}+\frac{68\,420\,351x^4}{8\,153\,726\,976}-\frac{106\,154\,827x^6}{24\,461\,180\,928}+\frac{338\,577\,733x^8}{391\,378\,894\,848}-\frac{2\,254\,159x^{10}}{28\,991\,029\,248}+\frac{23\,003x^{12}}{8\,589\,934\,592}$$

n: 6

$$\frac{8\,241\,137}{4\,214\,784\,000}+\frac{8\,241\,137x^2}{7\,225\,344\,000}-\frac{4\,736\,509x^4}{825\,753\,600}+$$
$$\frac{288\,608\,923x^6}{89\,915\,392\,000}-\frac{345\,473\,249x^8}{462\,422\,016\,000}+\frac{23\,556\,341x^{10}}{264\,241\,152\,000}-\frac{94\,778\,521x^{12}}{17\,263\,755\,264\,000}+\frac{75x^{14}}{536\,870\,912}$$

n: 7

$$-\frac{165\,768\,344\,647}{130\,459\,631\,616\,000}-\frac{165\,768\,344\,647x^2}{195\,689\,447\,424\,000}+\frac{6\,257\,828\,375\,189x^4}{1\,565\,515\,579\,392\,000}-\frac{491\,256\,042\,839x^6}{208\,735\,410\,585\,600}+\frac{30\,381\,472\,425\,073x^8}{50\,096\,498\,540\,544\,000}-$$
$$\frac{4\,325\,861\,596\,609x^{10}}{50\,096\,498\,540\,544\,000}+\frac{963\,243\,270\,647x^{12}}{133\,590\,662\,774\,784\,000}-\frac{3\,307\,958\,239x^{14}}{9\,895\,604\,649\,984\,000}+\frac{1\,881\,471x^{16}}{281\,474\,976\,710\,656}$$

n: 8

$$\frac{124\,848\,553\,201}{147\,483\,721\,728\,000}+\frac{124\,848\,553\,201x^2}{196\,644\,962\,304\,000}-\frac{185\,194\,889\,077x^4}{65\,548\,320\,768\,000}+\frac{1\,351\,150\,410\,331x^6}{786\,579\,849\,216\,000}-\frac{8\,955\,809\,117\,293x^8}{18\,877\,916\,381\,184\,000}+$$
$$\frac{1\,918\,266\,195\,107x^{10}}{25\,170\,555\,174\,912\,000}-\frac{1\,168\,380\,534\,109x^{12}}{151\,023\,331\,049\,472\,000}+\frac{298\,544\,975\,777x^{14}}{604\,093\,324\,197\,888\,000}-\frac{19\,594\,805\,623x^{16}}{1\,073\,943\,687\,462\,912\,000}+\frac{41x^{18}}{137\,438\,953\,472}$$

Polynomials $A_n(x)$ (Conjecture 2.) See [10, notebook "conjecture 2.nb"].

12.6. Table 6

n: -1

$$\{\{1, 1\}\}$$

---

n: 0

$$\left\{\left\{\frac{3}{8}, 1\right\}, \left\{\frac{4}{3} + x^2, 1\right\}\right\}$$

---

n: 1

$$\left\{\left\{\frac{69}{1024}, 1\right\}, \left\{-\frac{16}{23} - \frac{8\,x^2}{69} + x^4, 1\right\}\right\}$$

---

n: 2

$$\left\{\left\{\frac{1}{128}, 1\right\}, \{-2 + x, 1\}, \{2 + x, 1\}, \left\{-\frac{16}{27} - \frac{8\,x^2}{27} + x^4, 1\right\}\right\}$$

---

n: 3

$$\left\{\left\{\frac{5601}{8\,388\,608}, 1\right\}, \{-2 + x, 1\}, \{2 + x, 1\}, \left\{\frac{6464}{1867} + \frac{11\,312\,x^2}{5601} - \frac{38\,732\,x^4}{5601} + x^6, 1\right\}\right\}$$

---

n: 4

$$\left\{\left\{\frac{3}{65\,536}, 1\right\}, \{-2 + x, 1\}, \{2 + x, 1\}, \left\{-\frac{95\,488}{3375} - \frac{190\,976\,x^2}{10\,125} + \frac{650\,144\,x^4}{10\,125} - \frac{51\,968\,x^6}{3375} + x^8, 1\right\}\right\}$$

---

n: 5

$$\left\{\left\{\frac{23\,003}{8\,589\,934\,592}, 1\right\}, \{-2 + x, 1\}, \{2 + x, 1\},\right.$$
$$\left.\left\{\frac{4\,868\,805\,632}{16\,769\,187} + \frac{1\,217\,201\,408\,x^2}{5\,589\,729} - \frac{452\,733\,568\,x^4}{621\,081} + \frac{3\,737\,957\,344\,x^6}{16\,769\,187} - \frac{15\,548\,948\,x^8}{621\,081} + x^{10}, 1\right\}\right\}$$

---

n: 6

$$\left\{\left\{\frac{75}{536\,870\,912}, 1\right\}, \{-2 + x, 1\}, \{2 + x, 1\}, \left\{-\frac{33\,755\,697\,152}{9\,646\,875} - \frac{16\,877\,848\,576\,x^2}{5\,788\,125} +\right.\right.$$
$$\left.\left.\frac{18\,398\,435\,584\,x^4}{1\,929\,375} - \frac{32\,414\,868\,736\,x^6}{9\,646\,875} + \frac{14\,381\,852\,336\,x^8}{28\,940\,625} - \frac{340\,526\,584\,x^{10}}{9\,646\,875} + x^{12}, 1\right\}\right\}$$

---

n: 7

$$\left\{\left\{\frac{1\,881\,471}{281\,474\,976\,710\,656}, 1\right\}, \{-2 + x, 1\}, \{2 + x, 1\},\right.$$
$$\left\{\frac{2\,715\,948\,558\,696\,448}{57\,149\,681\,625} + \frac{7\,468\,858\,536\,415\,232\,x^2}{171\,449\,044\,875} - \frac{2\,640\,538\,932\,296\,704\,x^4}{19\,049\,893\,875} +\right.$$
$$\left.\left.\frac{3\,050\,057\,679\,448\,832\,x^6}{57\,149\,681\,625} - \frac{320\,257\,042\,164\,544\,x^8}{34\,289\,808\,975} + \frac{51\,129\,660\,553\,424\,x^{10}}{57\,149\,681\,625} - \frac{97\,388\,044\,148\,x^{12}}{2\,116\,654\,875} + x^{14}, 1\right\}\right\}$$

---

n: 8

$$\left\{\left\{\frac{41}{137\,438\,953\,472}, 1\right\}, \{-2 + x, 1\}, \{2 + x, 1\},\right.$$
$$\left\{-\frac{8\,182\,074\,782\,580\,736}{11\,533\,417\,875} - \frac{8\,182\,074\,782\,580\,736\,x^2}{11\,533\,417\,875} + \frac{25\,262\,578\,868\,092\,928\,x^4}{11\,533\,417\,875} - \frac{1\,469\,613\,075\,017\,728\,x^6}{1\,647\,631\,125} +\right.$$
$$\left.\left.\frac{2\,013\,551\,386\,772\,992\,x^8}{11\,533\,417\,875} - \frac{46\,645\,274\,445\,568\,x^{10}}{2\,306\,683\,575} + \frac{16\,469\,761\,126\,016\,x^{12}}{11\,533\,417\,875} - \frac{73\,253\,258\,992\,x^{14}}{1\,281\,490\,875} + x^{16}, 1\right\}\right\}$$

$A_n(x)$ FACTORED IN *Mathematica* (Conjecture 2.) See [10, notebook "conjecture 2.nb"].

12.7. Table 7

```
n:  -1
1

n:  0
(3/8) * (x^2 + 4/3)

n:  1
(69/1024) * (x^4 - 8/69*x^2 - 16/23)

n:  2
(1/128) * (x - 2) * (x + 2) * (x^4 - 8/27*x^2 - 16/27)

n:  3
(5601/8388608) * (x - 2) * (x + 2) * (x^6 - 38732/5601*x^4 + 11312/5601*x^2 + 6464/1867)

n:  4
(3/65536) * (x - 2) * (x + 2) * (x^8 - 51968/3375*x^6 + 650144/10125*x^4 - 190976/10125*x^2 - 95488/3375)

n:  5
(23003/8589934592) * (x - 2) * (x + 2) * (x^10 - 15548948/621081*x^8 + 3737957344/16769187*x^6 - 452733568/621081*x
^4 + 1217201408/5589729*x^2 + 4868805632/16769187)

n:  6
(75/536870912) * (x - 2) * (x + 2) * (x^12 - 340526584/9646875*x^10 + 14381852336/28940625*x^8 - 32414868736/964687
5*x^6 + 18398435584/1929375*x^4 - 16877848576/5788125*x^2 - 33755697152/9646875)

n:  7
(1881471/281474976710656) * (x - 2) * (x + 2) * (x^14 - 97388044148/2116654875*x^12 + 51129660553424/57149681625*x^
10 - 320257042164544/34289808975*x^8 + 3050057679448832/57149681625*x^6 - 2640538932296704/19049893875*x^4 + 746885
8536415232/171449044875*x^2 + 2715948558696448/57149681625)

n:  8
(41/137438953472) * (x - 2) * (x + 2) * (x^16 - 73253258992/1281490875*x^14 + 16469761126016/11533417875*x^12 - 466
45274445568/2306683575*x^10 + 2013551386772992/11533417875*x^8 - 1469613075017728/1647631125*x^6 + 2526257886809292
8/11533417875*x^4 - 8182074782580736/11533417875*x^2 - 8182074782580736/11533417875)
```

$A_n(x)$ FACTORED IN *SageMath* (Conjecture 2.) (See [10, notebook "conjecture 2 clause 1b.ipynb"].)

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