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A NEW CLASS OF POLYNOMIALS RELATED TO THE STIRLING NUMBERS AND SERIES REPRESENTATIONS FOR SOME MATHEMATICAL CONSTANTS

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Abstract

The polynomials

$$P_n(x) = \int_0^1 (1-2t)(x-t)^{\overline{n}} \mathrm{d}t = \sum_{k=0}^{n-1} A_{n,k} x^k \quad (n = 1, 2, 3, ...)$$

appear in an asymptotic expansion for Euler's gamma function. We investigate the properties of the coefficients $A_{n,k}$ and show that the coefficients can be expressed in terms of the unsigned Stirling numbers of the first kind. We also show that these numbers appear in series representations for some mathematical constants, like, for instance, Euler's constant, $\log(2)$ and $\zeta(3)$.

1. Introduction

The Stirling numbers of the first kind have interesting applications in various fields, like, calculus of finite differences, number theory, numerical analysis and they play a role in combinatorics. These numbers can be defined by the recurrence relation

$$s(0,0) = 1, \quad s(n,0) = s(0,k) = 0 \quad (n, k \geq 1),$$
$$s(n+1,k) = s(n,k-1) - ns(n,k) \quad (n \geq 0, k \geq 1).$$

A combinatorial interpretation states that $s(n,k)$ is $(-1)^{n-k}$ times the number of permutations of $\{1, 2, ..., n\}$ with precisely k cycles. For more information on this subject we refer to Roman [22].

The Stirling numbers appear in the expansions of the falling factorial polynomials and the rising factorial polynomials,

$$x^{\underline{n}} = \prod_{j=0}^{n-1}(x-j) = \sum_{k=0}^{n} s(n,k)x^k$$

and

$$x^{\overline{n}} = \prod_{j=0}^{n-1}(x+j) = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k. \tag{1}$$

Our work was inspired by an interesting paper published in 2006 by Shi et al. [23]. The authors presented the following new asymptotic series for Euler's gamma function:

$$\log\Gamma(s+1) = \frac{1}{2}\log(2\pi) + \Big(s+\frac{1}{2}\Big)\log s - s + \sum_{n=1}^{\infty}\frac{P_n(x)}{2n\prod_{j=0}^{n-1}(s+x+j)}.$$

Here, x is a nonnegative real number, s is a complex number with $\Re(s) \geq 1$, and P_n is the function

$$P_n(x) = \int_0^1 (1-2t)\prod_{j=0}^{n-1}(x+j-t)\,\mathrm{d}t = \int_0^1 (1-2t)\,\frac{\Gamma(x+n-t)}{\Gamma(x-t)}\,\mathrm{d}t. \tag{2}$$

It is not difficult to show that P_n (with $n \geq 1$) is a polynomial of degree $n-1$ with leading coefficient $n/6$. For $n = 1, 2, 3, 4$ we have

$$P_1(x) = \frac{1}{6}, \qquad P_2(x) = \frac{1}{3}\,x, \qquad P_3(x) = \frac{1}{2}\,x^2 + \frac{1}{2}\,x - \frac{1}{60},$$
$$P_4(x) = \frac{2}{3}\,x^3 + 2\,x^2 + \frac{19}{15}\,x - \frac{1}{15}.$$

Throughout this paper, we set

$$P_n(x) = \sum_{k=0}^{n-1} A_{n,k}\,x^k \quad \text{and} \quad A_n = A_{n,0} = P_n(0). \tag{3}$$

A result discovered by Yu and Yang [25] states that if $n \geq 3$, then the constant coefficient of P_n is negative, whereas all other coefficients are positive. Thus,

$$A_n < 0 \quad \text{and} \quad A_{n,k} > 0 \qquad (k = 1, ..., n-1; n \geq 3). \tag{4}$$

The aim of this paper is to study further properties of $A_{n,k}$.

The paper is organized as follows. In the next section, we provide some representations for $A_{n,k}$. It turns out that $A_{n,k}$ can be expressed in terms of the Stirling numbers of the first kind. In Section 3, we prove that the sequence $(A_{n,k})_{1\le k\le n-1}$ is strictly log-concave with respect to k, and in Section 4, we present new series representations for some mathematical constants, like, for example, $1/\log 2$, $\log 3$ and Euler's constant γ. The terms of our series involve $A_{n,k}$. Finally, we show that the numbers A_n and $A_{n,1}$ can be expressed in terms of the Cauchy numbers and the Gregory coefficients.

2. Representations for $A_{n,k}$

In what follows, we use the notation

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} s(n,k) = |s(n,k)|. \tag{5}$$

The numbers $\left[{n \atop k}\right]$ are called the unsigned Stirling numbers of the first kind; see [15, section 6]. Our first theorem provides an explicit formula for $A_{n,k}$.

Theorem 1. *We have*

$$A_{n,k} = \sum_{j=k+1}^{n} (-1)^{j-k+1} \begin{bmatrix} n \\ j \end{bmatrix} \binom{j}{k} \frac{j-k}{(j-k+1)(j-k+2)} \qquad (k = 0,1,...,n-1;\ n \ge 1). \tag{6}$$

Proof. Using (1), (2) and (5), we obtain

$$\begin{aligned} P_n(x) &= \int_0^1 (1-2t)(x-t)^{\overline{n}} \mathrm{d}t \\ &= \frac{1}{2}\int_{-1}^{1} u\Big(x+\frac{u-1}{2}\Big)^{\overline{n}} \mathrm{d}u \\ &= \frac{1}{2}\int_{-1}^{1} u \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \Big(x+\frac{u-1}{2}\Big)^{k} \mathrm{d}u \\ &= \frac{1}{2}\int_{-1}^{1} u \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{j=0}^{k} \binom{k}{j} x^j \Big(\frac{u-1}{2}\Big)^{k-j} \mathrm{d}u \\ &= \sum_{k=0}^{n}\sum_{j=0}^{k} \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{j} x^j \int_{-1}^{1} \frac{u}{2}\Big(\frac{u-1}{2}\Big)^{k-j} \mathrm{d}u. \end{aligned}$$

The fact

$$\int_{-1}^{1} \frac{u}{2}\Big(\frac{u-1}{2}\Big)^{m} \mathrm{d}u = (-1)^{m+1} \frac{m}{(m+1)(m+2)} \quad (0 \le m \in \mathbb{Z})$$

implies

$$
\begin{aligned}
P_n(x) &= \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k-j+1}\begin{bmatrix} n \\ k \end{bmatrix}\binom{k}{j}x^j\frac{k-j}{(k-j+1)(k-j+2)} \\
&= \sum_{k=0}^{n-1}\sum_{j=k+1}^{n}(-1)^{j-k+1}\begin{bmatrix} n \\ j \end{bmatrix}\binom{j}{k}\frac{j-k}{(j-k+1)(j-k+2)}x^k.
\end{aligned} \tag{7}
$$

Comparing the coefficients we conclude from (3) and (7) that (6) holds. □

Remark 1. The special case $k = 0$ leads to the simple formula

$$A_n = \sum_{j=1}^{n}(-1)^{j+1}\begin{bmatrix} n \\ j \end{bmatrix}\frac{j}{(j+1)(j+2)}.$$

Inverting this identity gives

$$\sum_{k=0}^{n}(-1)^{k+1}\begin{Bmatrix} n \\ k \end{Bmatrix}A_k = \frac{n}{(n+1)(n+2)},$$

where

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

denotes the Stirling numbers of the second kind; see [15, Section 6]. From (6) with $k = 1, 2, 3$ we obtain

$$A_{n,1} = \sum_{j=2}^{n}(-1)^j\begin{bmatrix} n \\ j \end{bmatrix}\frac{j-1}{j+1}, \qquad A_{n,2} = \frac{1}{2}\sum_{j=3}^{n}(-1)^{j+1}\begin{bmatrix} n \\ j \end{bmatrix}(j-2),$$

and

$$A_{n,3} = \frac{1}{6}\sum_{j=4}^{n}(-1)^j\begin{bmatrix} n \\ j \end{bmatrix}j(j-3).$$

In Section 4, we give several series representations for various mathematical constants which involve these expressions.

The next theorem provides two additional representations for $A_{n,k}$.

Theorem 2. *We have*

$$A_{n,k} = \sum_{j=k}^{n}\begin{bmatrix} j \\ k \end{bmatrix}\binom{n}{j}A_{n-j} \tag{8}$$

and

$$A_{n,k} = \frac{1}{n+1}\sum_{j=k+1}^{n}(-1)^{j-k-1}\binom{j}{k}A_{n+1,j}. \tag{9}$$

Proof. (i) Using the formula

$$(a+b)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} a^{\overline{k}} b^{\overline{n-k}}$$

(see [1, p. 105]) together with (1), (2) and (5), we obtain

$$\begin{aligned}
P_n(x) &= \int_0^1 (1-2t)(x-t)^{\overline{n}} \mathrm{d}t \\
&= \int_0^1 (1-2t) \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} (-t)^{\overline{n-k}} \mathrm{d}t \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} \int_0^1 (1-2t)(-t)^{\overline{n-k}} \mathrm{d}t \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} P_{n-k}(0) \\
&= \sum_{k=0}^{n} \binom{n}{k} A_{n-k} x^{\overline{k}} \\
&= \sum_{k=0}^{n} \binom{n}{k} A_{n-k} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} x^j \\
&= \sum_{k=0}^{n} \sum_{j=k}^{n} \begin{bmatrix} j \\ k \end{bmatrix} \binom{n}{j} A_{n-j} x^k
\end{aligned}$$

which concludes (8).

(ii) We have

$$\begin{aligned}
P_{n+1}(x) &= \int_0^1 (1-2t)(x-t)^{\overline{n+1}} \mathrm{d}t \\
&= \int_0^1 (1-2t)(x-t)^{\overline{n}} (x-t+n) \mathrm{d}t \\
&= (x+n) \int_0^1 (1-2t)(x-t)^{\overline{n}} dt - \int_0^1 (1-2t)t(x-t)^{\overline{n}} \mathrm{d}t \\
&= (x+n) P_n(x) - \int_0^1 (1-2t)t(x-t)^{\overline{n}} \mathrm{d}t
\end{aligned}$$

and

$$P_{n+1}(x-1) = \int_0^1 (1-2t)(x-1-t)^{\overline{n+1}} \mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^1 (1-2t)(x-t)^{\overline{n}}(x-1-t)\mathrm{d}t \\
&= (x-1)\int_0^1 (1-2t)(x-t)^{\overline{n}}\mathrm{d}t - \int_0^1 (1-2t)t(x-t)^{\overline{n}}\mathrm{d}t \\
&= (x-1)P_n(x) - \int_0^1 (1-2t)t(x-t)^{\overline{n}}\mathrm{d}t.
\end{aligned}
$$

It follows that

$$P_{n+1}(x) - P_{n+1}(x-1) = (n+1)P_n(x). \tag{10}$$

Thus,

$$
\begin{aligned}
(n+1)\sum_{k=0}^{n-1} A_{n,k}x^k &= \sum_{k=0}^{n} A_{n+1,k}\big(x^k - (x-1)^k\big) \\
&= \sum_{k=0}^{n} A_{n+1,k}\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k}{j}x^j \\
&= \sum_{k=0}^{n-1}\sum_{j=k+1}^{n}(-1)^{j-k-1}\binom{j}{k}A_{n+1,j}x^k.
\end{aligned}
$$

This leads to (9). □

Remark 2. From (10) with $x = 1$ and $x = 1/2$ we obtain

$$(n+1)\sum_{k=0}^{n-1} A_{n,k} = \sum_{k=1}^{n} A_{n+1,k} \qquad \text{and} \qquad (n+1)\sum_{k=0}^{n-1}\frac{A_{n,k}}{2^k} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{n}\frac{A_{n+1,k}}{2^{k-1}},$$

respectively.

3. A Log-concavity Property

A positive sequence $(a_k)_{1\le k\le m}$ is called strictly log-concave, if

$$a_{k-1}a_{k+1} < a_k^2 \quad (k = 2, ..., m-1).$$

In this section, we present a log-concavity property of the sequence $(A_{n,k})_{1\le k\le n-1}$. We need the following theorem which might be of independent interest.

Theorem 3. *The polynomial P_n $(n \ge 3)$ has only simple zeros. More precisely, P_n has one positive and $n-2$ negative zeros.*

Proof. Let $m \in \{0, 1, ..., n-2\}$. From (2) we obtain

$$P_n(-m+1/2) = 2\int_{-m-1/2}^{-m+1/2} \prod_{j=0}^{m-1}(t+j) \cdot (t+m)^2 \cdot \prod_{j=m+1}^{n-1}(t+j)\mathrm{d}t.$$

If $-m-1/2 < t < -m+1/2$, then $t+j < 0$ for $j = 0, 1, ..., m-1$, and $t+j > 0$ for $j = m+1, ..., n-1$. This implies that

$$P_n(-m+1/2) > 0, \quad \text{if } m \text{ is even,} \tag{11}$$

and

$$P_n(-m+1/2) < 0, \quad \text{if } m \text{ is odd.} \tag{12}$$

We have

$$P_n(-m-1/2) = 2\int_{-m-3/2}^{-m+1/2} \prod_{j=0}^{m}(t+j) \cdot (t+m+1)^2 \cdot \prod_{j=m+2}^{n-1}(t+j)\mathrm{d}t.$$

If $-m-3/2 < t < -m-1/2$, then $t+j < 0$ for $j = 0, 1, ..., m$, and $t+j > 0$ for $j = m+2, ..., n-1$. Hence,

$$P_n(-m-1/2) < 0, \quad \text{if } m \text{ is even,} \tag{13}$$

and

$$P_n(-m-1/2) > 0, \quad \text{if } m \text{ is odd.} \tag{14}$$

Applying (11)-(14) gives that P_n has a zero in the interval

$$(-m-1/2, -m+1/2) \quad \text{for} \quad m = 0, 1, ..., n-2.$$

This implies that P_n has at least $n-2$ negative zeros. From (4) and (11) with $m = 0$ we obtain $P_n(0) < 0 < P_n(1/2)$. Thus, P_n has a positive zero. Since P_n has the degree $n-1$, we conclude that P_n has one positive zero and $n-2$ negative zeros. □

The following lemma is due to Newton. A proof can be found, for instance, in [16, pp. 104-105].

Lemma 1. *If the real polynomial*

$$\sum_{j=0}^{m} \frac{a_j}{j!(m-j)!} x^{m-j} \quad (m \geq 2; a_0 a_m \neq 0)$$

has only real zeros, then

$$a_{j-1}a_{j+1} < a_j^2 \quad (j = 1, 2, ..., m-1),$$

unless all zeros are equal.

We are now in a position to prove that $(A_{n,k})_{1\leq k\leq n-1}$ is strictly log-concave with respect to k.

Theorem 4. *Let $n \geq 3$ be an integer. Then,*

$$A_{n,k-1}A_{n,k+1} < \left(1 - \frac{n}{(n-k)(k+1)}\right) A_{n,k}^2 \quad (k = 1, 2, ..., n-2). \tag{15}$$

Proof. We apply Theorem 3 and Lemma 1 with

$$m = n - 1 \quad \text{and} \quad a_j = j!(n-1-j)!A_{n,n-1-j} \quad (j = 0, 1, ..., n-1).$$

Then we obtain (15). □

4. Series Representations

Finding representations for mathematical constants by series, products, integrals and continued fractions has attracted the attention of researchers for many years. Detailed information on this subject with many interesting historical comments can be found in Finch's monographs *Mathematical Constants* and *Mathematical Constants II* [13, 14]. Here, we give three examples:

$$\frac{1}{\log 2} = \sum_{n=0}^{\infty} \int_0^1 \binom{x}{n} \mathrm{d}x, \quad \log 2 = \frac{1}{6} \sum_{n=1}^{\infty} \frac{p_n + q_n}{n},$$

where

$$p_n = \prod_{k=1}^{n} \left(1 - \frac{1}{4k}\right) \quad \text{and} \quad q_n = \prod_{k=1}^{n} \left(1 - \frac{3}{4k}\right),$$

$$\log 3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + + - \cdots .$$

The above representations can be found in [19], [4] and [18, p. 312], respectively. We show that the numbers $A_{n,k}$ can be used to obtain new series representations for several mathematical constants. Throughout, we use the notations $A_n^* = A_n/n!$ and $A_{n,k}^* = A_{n,k}/n!$.

The following theorem presents the exponential generating series for A_n.

Theorem 5. *For $|t| < 1$, we have*

$$A(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!} = \frac{(t-2)\log(1-t) - 2t}{(\log(1-t))^2}. \tag{16}$$

Proof. Let $|t|<1$ and let N be a nonnegative integer. Then,

$$\begin{aligned}\sum_{n=0}^{N} A_n \frac{t^n}{n!} &= \sum_{n=0}^{N} \int_0^1 (1-2u)(-u)^{\overline{n}} \mathrm{d}u \frac{t^n}{n!}\\ &= \sum_{n=0}^{N} \int_0^1 (1-2u)\binom{u}{n} \mathrm{d}u (-t)^n\\ &= \int_0^1 (1-2u) \sum_{n=0}^{N} \binom{u}{n} (-t)^n \mathrm{d}u.\end{aligned}$$

We let N tend to ∞ and obtain

$$\begin{aligned}\sum_{n=0}^{\infty} A_n \frac{t^n}{n!} &= \int_0^1 (1-2u)(1-t)^u \mathrm{d}u\\ &= \left[\frac{(1-2u)(1-t)^u}{\log(1-t)} + \frac{2(1-t)^u}{(\log(1-t))^2}\right]_{u=0}^{u=1}\\ &= \frac{(t-2)\log(1-t)-2t}{(\log(1-t))^2}.\end{aligned}$$

This settles (16). $\square$

As a first consequence of Theorem 5, we obtain the following result.

Corollary 1. *We have*

$$\frac{2}{(\log 2)^2} - \frac{3}{\log 2} = \sum_{n=0}^{\infty} (-1)^n A_n^*. \tag{17}$$

Proof. Let $x_n = (-1)^n A_n^*$. We obtain for $n \geq 1$:

$$|x_n| \leq \frac{1}{n!} \int_0^1 |1-2t| t \prod_{j=1}^{n-1} (j-t) \mathrm{d}t \leq \frac{1}{n!} \int_0^1 |1-2t| t (n-1)! \mathrm{d}t = \frac{1}{4n}.$$

Using (4) we get for $n \geq 3$:

$$(n+1)!(|x_n| - |x_{n+1}|) = A_{n+1} - (n+1)A_n = \int_0^1 u(t) v_n(t) \mathrm{d}t$$

with

$$u(t) = t(1-2t)(2-t)(1-t^2) \quad \text{and} \quad v_n(t) = \prod_{j=3}^{n-1} (j-t).$$

We have $u \geq 0$ on $[0,1/2]$, $u \leq 0$ on $[1/2,1]$, and v_n is decreasing on $[0,1]$. This gives

$$u(t)\big(v_n(t) - v_n(1/2)\big) \geq 0 \quad (0 \leq t \leq 1).$$

Thus,

$$\int_0^1 u(t)v_n(t)\mathrm{d}t \geq v_n(1/2)\int_0^1 u(t)\mathrm{d}t = 0.$$

It follows that $|x_n|$ is decreasing for $n \geq 3$ and tends to 0 as $n \to \infty$, so that the Leibniz criterion for alternating series reveals that the series in (17) is convergent. Applying Abel's limit theorem and (16) gives

$$\sum_{n=0}^{\infty}(-1)^n A_n^* = \lim_{t\to -1}\sum_{n=0}^{\infty} A_n^* t^n = \lim_{t\to -1}\frac{(t-2)\log(1-t)-2t}{(\log(1-t))^2} = \frac{2}{(\log 2)^2} - \frac{3}{\log 2}.$$

□

Remark 3. From (16) with $t = m/(m+1)$ ($m = 1, 2, 3, 4$), we obtain the following series representations:

$$\frac{1}{(\log 2)^2} = \frac{3}{2\log 2} - \sum_{n=0}^{\infty}\frac{A_n^*}{2^n} = \frac{5}{3\log 2} - \frac{8}{3}\sum_{n=0}^{\infty} A_n^*\Big(\frac{3}{4}\Big)^n, \tag{18}$$

$$\frac{1}{(\log 3)^2} = \frac{1}{\log 3} - \frac{3}{4}\sum_{n=0}^{\infty} A_n^*\Big(\frac{2}{3}\Big)^n, \tag{19}$$

$$\frac{1}{(\log 5)^2} = \frac{3}{4\log 5} - \frac{5}{8}\sum_{n=0}^{\infty} A_n^*\Big(\frac{4}{5}\Big)^n. \tag{20}$$

Remark 4. A Sheffer matrix [2, p. 309], [20] (or exponential Riordan array) is an infinite matrix $S = [s_{n,k}]_{n,k\geq 0} = (g(t), f(t))$, whose columns are generated, for every $k \in \mathbb{N}$, by the exponential series

$$\sum_{n=0}^{\infty} s_{n,k}\frac{t^n}{n!} = g(t)\frac{(f(t))^k}{k!},$$

where $g(t) = \sum_{n=0}^{\infty} g_n t^n/n!$ and $f(t) = \sum_{n=0}^{\infty} f_n t^n/n!$ are two exponential series with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. If $g_0 = 0$, then we have an improper Sheffer matrix.

Moreover, a polynomial sequence $(s_n(x))_{n\geq 0}$ is a Sheffer sequence [8, 20, 21, 22] if the polynomials $s_n(x) = \sum_{k=0}^{n} s_{n,k}x^k$ are the row polynomials of a Sheffer matrix $S = (g(t), f(t))$. So, in particular, we have the exponential generating series

$$\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!} = g(t)\,\mathrm{e}^{xf(t)}.$$

The ordinary powers, the falling and the rising factorials, the Bernoulli and the Euler polynomials, the Hermite polynomials, the Laguerre polynomials and the Mittag-Leffler polynomials are all classical examples of Sheffer sequences. Many other examples of this kind arise in the context of enumerative combinatorics.

In our case, the matrix $A = [A_{n,k}]_{n,k\geq 0}$ turns out to be an improper Sheffer matrix, and the polynomials $P_n(x)$ form an improper Sheffer sequence. Applying (16) and (8) with the fact that

$$\frac{1}{k!}\Big(\log\frac{1}{1-t}\Big)^k = \sum_{n=k}^{\infty}\begin{bmatrix} n\\ k\end{bmatrix}\frac{t^n}{n!} \quad (|t|<1; 0\leq k\in\mathbb{Z}),$$

(see [15, p. 337]), we obtain for $|t|<1$ and nonnegative integers k:

$$\frac{A(t)}{k!}\Big(\log\frac{1}{1-t}\Big)^k = \sum_{n=0}^{\infty}\sum_{j=k}^{n}\begin{bmatrix} j\\ k\end{bmatrix}\binom{n}{j}A_{n-j}\frac{t^n}{n!} = \sum_{n=k+1}^{\infty}A^*_{n,k}t^n. \tag{21}$$

Hence, we have the Sheffer matrix $A = (A(t), -\log(1-t))$ and the exponential generating series

$$\frac{A(t)}{(1-t)^x} = \sum_{n=0}^{\infty}P_n(x)\frac{t^n}{n!}.$$

In particular, $P_n(x)$ can be expressed in terms of the Narumi polynomials; see [8, p. 37], [22, p. 127].

Remark 5. From (4) and (21) we conclude that the function

$$F_k(t) = A(t)\Big(\log\frac{1}{1-t}\Big)^k \quad (2\leq k\in\mathbb{N})$$

is absolutely monotonic on $[0,1)$, that is, we have $F_k^{(\nu)}(t)\geq 0$ $(0\leq t<1; \nu = 0,1,2,...)$. Absolutely monotonic functions have applications in the theory of analytic functions and other fields. For more information on this subject we refer to [9] and [24, chapter IV].

Remark 6. Applying the same technique as in the proof of Corollary 1 we obtain from (16) and (21) with $k=1$ the following counterpart of (17):

$$\frac{1}{\log 2} = \frac{3}{2} - \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n A^*_{n,1}.$$

Next, we apply (21) with $k=1,2,3$ and $t = m/(m+1)$ $(m=1,2,3,4)$. For $k=1$ we find

$$\frac{1}{\log 2} = \frac{3}{2} - \sum_{n=2}^{\infty}\frac{A^*_{n,1}}{2^n} = \frac{5}{3} - \frac{4}{3}\sum_{n=2}^{\infty}A^*_{n,1}\Big(\frac{3}{4}\Big)^n,$$

$$\frac{1}{\log 3} = 1 - \frac{3}{4}\sum_{n=2}^{\infty}A^*_{n,1}\Big(\frac{2}{3}\Big)^n,$$

$$\frac{1}{\log 5} = \frac{3}{4} - \frac{5}{8}\sum_{n=2}^{\infty}A^*_{n,1}\Big(\frac{4}{5}\Big)^n.$$

Setting $k = 2$ leads to

$$\log 2 = \frac{2}{3} + \frac{4}{3}\sum_{n=3}^{\infty}\frac{A^*_{n,2}}{2^n} = \frac{3}{5} + \frac{4}{5}\sum_{n=3}^{\infty}A^*_{n,2}\Big(\frac{3}{4}\Big)^n,$$

$$\log 3 = 1 + \frac{3}{2}\sum_{n=3}^{\infty}A^*_{n,2}\Big(\frac{2}{3}\Big)^n,$$

$$\log 5 = \frac{4}{3} + \frac{5}{3}\sum_{n=3}^{\infty}A^*_{n,2}\Big(\frac{4}{5}\Big)^n,$$

and for $k = 3$ we find

$$(\log 2)^2 = \frac{2}{3}\log 2 + 4\sum_{n=4}^{\infty}\frac{A^*_{n,3}}{2^n} = \frac{3}{5}\log 2 + \frac{6}{5}\sum_{n=4}^{\infty}A^*_{n,3}\Big(\frac{3}{4}\Big)^n,$$

$$(\log 3)^2 = \log 3 + \frac{9}{2}\sum_{n=4}^{\infty}A^*_{n,3}\Big(\frac{2}{3}\Big)^n,$$

$$(\log 5)^2 = \frac{4}{3}\log 5 + 5\sum_{n=4}^{\infty}A^*_{n,3}\Big(\frac{4}{5}\Big)^n.$$

From (21) with $k = 2$ and $k = 3$ we obtain

$$\sum_{n=3}^{\infty}A^*_{n,2}t^n = \frac{1}{2}(t-2)\log(1-t) - t = \frac{1}{2}\sum_{n=3}^{\infty}\frac{n-2}{(n-1)n}t^n$$

and

$$\begin{aligned}\sum_{n=4}^{\infty}A^*_{n,3}t^n &= \frac{1}{3}t\log(1-t) - \frac{1}{6}(t-2)(\log(1-t))^2\\ &= \frac{1}{6}\sum_{n=4}^{\infty}\sum_{j=0}^{n-4}\frac{n-j-3}{(j+1)(n-j-1)(n-j-2)}\,t^n,\end{aligned}$$

respectively. This implies that for $A^*_{n,2}$ and $A^*_{n,3}$ we have the formulas

$$A^*_{n,2} = \frac{n-2}{2(n-1)n} \quad (n \ge 3)$$

and

$$A^*_{n,3} = \frac{1}{6}\sum_{j=0}^{n-4}\frac{n-j-3}{(j+1)(n-j-1)(n-j-2)} \quad (n \ge 4).$$

Remark 7. Further applications of (21) provide more series representations for mathematical constants. For instance, from (21) with $k = 1$ we obtain

$$\frac{1}{\log(1-t)} = -\frac{1}{t} + \frac{1}{2} + \frac{1}{2}\sum_{n=2}^{\infty} A^*_{n,1} t^{n-1}. \tag{22}$$

Differentiating both sides of the above equation gives

$$\frac{1}{(\log(1-t))^2} = \frac{1-t}{t^2}\Big(1 + \frac{1}{2}\sum_{n=2}^{\infty}(n-1)A^*_{n,1} t^n\Big). \tag{23}$$

Setting $t = m/(m+1)$ $(m = 1,2,3,4)$ in (23) leads to counterparts of (18), (19) and (20). Here, we just state the representations for $1/(\log 2)^2$. For $m = 1,3$ we get

$$\frac{1}{(\log 2)^2} = 2 + \sum_{n=2}^{\infty} \frac{(n-1)}{2^n} A^*_{n,1} = \frac{16}{9} + \frac{8}{9}\sum_{n=2}^{\infty}(n-1)A^*_{n,1}\Big(\frac{3}{4}\Big)^n.$$

Remark 8. Formula (22) can be applied to find various new series representations for Euler's constant γ. We give two examples. Let

$$H(y) = \int_0^y \Big(\frac{1}{\log(1-t)} + \frac{1}{t}\Big)\mathrm{d}t \quad (0 < y < 1). \tag{24}$$

Using (22) gives

$$H(y) = \frac{1}{2}\int_0^y \Big(1 + \sum_{n=2}^{\infty} A^*_{n,1} t^{n-1}\Big)\mathrm{d}t = \frac{1}{2}y + \frac{1}{2}\sum_{n=2}^{\infty} A^*_{n,1}\frac{y^n}{n}. \tag{25}$$

From (24) we obtain

$$\begin{aligned} H(1/2) &= \gamma - \mathrm{Ei}(-\log 2) - \log 2, \\ H(1/3) &= \gamma - \mathrm{Ei}(-\log(3/2)) - \log 3, \end{aligned} \tag{26}$$

where

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^s}{s}\,\mathrm{d}s \quad (x < 0)$$

denotes the exponential integral. Applying (25) with $y = 1/2$, $y = 1/3$ and (26) leads to

$$\begin{aligned} \gamma &= \frac{1}{4} + \log 2 + \mathrm{Ei}(-\log 2) + \sum_{n=2}^{\infty} \frac{A^*_{n,1}}{2^{n+1} n} \\ &= \frac{1}{6} + \log 3 + \mathrm{Ei}(-\log(3/2)) + \frac{1}{2}\sum_{n=2}^{\infty} \frac{A^*_{n,1}}{3^n n}. \end{aligned}$$

Next, we supplement Theorem 5. We obtain a formula for the m-th derivative of the generating series $A(t)$.

Theorem 6. *Let $m \geq 0$ be an integer and $|t| < 1$. Then,*

$$\sum_{n=0}^{\infty} A_{n+m} \frac{t^n}{n!} = \frac{1}{(1-t)^{m-1}} \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \sum_{j=0}^{k} (-1)^{k-j+1} \binom{k}{j} \frac{k+j+1}{k-j+1} \frac{j!}{(\log(1-t))^{j+1}}$$
$$- \frac{1}{(1-t)^m} \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \frac{k!}{(\log(1-t))^{k+1}} - \frac{2t}{(1-t)^m} \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \frac{(k+1)!}{(\log(1-t))^{k+2}}. \tag{27}$$

Proof. It follows from (2), (3) and (16) that

$$\begin{aligned}
A^{(m)}(t) &= \sum_{n=0}^{\infty} A_{n+m} \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} (-1)^{n+m} \frac{t^n}{n!} \int_0^1 (1-2x) x^{\underline{n+m}} \mathrm{d}x \\
&= \sum_{n=0}^{\infty} (-1)^{n+m} \frac{t^n}{n!} \int_0^1 (1-2x) x^{\underline{m}} (x-m)^{\underline{n}} \mathrm{d}x \\
&= (-1)^m \int_0^1 (1-2x) x^{\underline{m}} \sum_{n=0}^{\infty} (x-m)^{\underline{n}} \frac{(-t)^n}{n!} \mathrm{d}x \\
&= (-1)^m \int_0^1 (1-2x) x^{\underline{m}} \sum_{n=0}^{\infty} \binom{x-m}{n} (-t)^n \mathrm{d}x \\
&= (-1)^m \int_0^1 (1-2x) x^{\underline{m}} (1-t)^{x-m} \mathrm{d}x \\
&= \frac{(-1)^m}{(1-t)^m} \int_0^1 (1-2x) x^{\underline{m}} (1-t)^x \mathrm{d}x \\
&= \frac{(-1)^m}{(1-t)^m} \int_0^1 \sum_{k=0}^{m} (-1)^{m-k} \begin{bmatrix} m \\ k \end{bmatrix} (x^k - 2x^{k+1})(1-t)^x \mathrm{d}x \\
&= \frac{1}{(1-t)^m} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix} \int_0^1 (x^k - 2x^{k+1})(1-t)^x \mathrm{d}x. \qquad (28)
\end{aligned}$$

Next, we apply the integral formula

$$\int_0^1 x^n s^x \mathrm{d}x = s \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k!}{(\log s)^{k+1}} - (-1)^n \frac{n!}{(\log s)^{n+1}} \quad (n = 0, 1, 2, ..., s > 0)$$

with $s = 1-t$. Then,

$$\int_0^1 (x^k - 2x^{k+1})(1-t)^x \mathrm{d}x = (1-t)\sum_{j=0}^{k}(-1)^{j+1}\binom{k}{j}\frac{k+j+1}{k-j+1}\frac{j!}{(\log(1-t))^{j+1}} - (-1)^k\frac{k!}{(\log(1-t))^{k+1}} - (-1)^k\frac{2t(k+1)!}{(\log(1-t))^{k+2}}.$$

Inserting this formula in (28) we conclude that (27) is valid. □

Remark 9. For $t = 1/2$ and $m \geq 0$, (27) yields

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{A_{n+m}}{2^n n!} &= 2^{m-1}\sum_{k=0}^{m}(-1)^k\begin{bmatrix}m\\k\end{bmatrix}\sum_{j=0}^{k}\binom{k}{j}\frac{k+j+1}{k-j+1}\frac{j!}{(\log 2)^{j+1}}\\ &\quad +2^m\sum_{k=0}^{m}(-1)^k\begin{bmatrix}m\\k\end{bmatrix}\frac{k!}{(\log 2)^{k+1}} - 2^m\sum_{k=0}^{m}(-1)^k\begin{bmatrix}m\\k\end{bmatrix}\frac{(k+1)!}{(\log 2)^{k+2}}.\end{aligned}$$

In particular, for $m = 1, 2, 3, 4$ we obtain

$$\sum_{n=0}^{\infty}\frac{A_{n+1}}{2^n n!} = -\frac{1}{\log 2} - \frac{5}{(\log 2)^2} + \frac{4}{(\log 2)^3},$$

$$\sum_{n=0}^{\infty}\frac{A_{n+2}}{2^n n!} = -\frac{2}{(\log 2)^2} + \frac{36}{(\log 2)^3} - \frac{24}{(\log 2)^4},$$

$$\sum_{n=0}^{\infty}\frac{A_{n+3}}{2^n n!} = -\frac{12}{(\log 2)^2} + \frac{128}{(\log 2)^3} - \frac{360}{(\log 2)^4} + \frac{192}{(\log 2)^5},$$

$$\sum_{n=0}^{\infty}\frac{A_{n+4}}{2^n n!} = -\frac{80}{(\log 2)^2} + \frac{784}{(\log 2)^3} - \frac{2880}{(\log 2)^4} + \frac{4416}{(\log 2)^5} - \frac{1920}{(\log 2)^6}.$$

5. Relations with Cauchy Numbers and Gregory Coefficients

Comtet [11, p. 294] introduced the Cauchy numbers of the first kind C_n and the Cauchy numbers of the second kind $\widehat{C}_n$ by

$$C_n = \int_0^1 x^{\underline{n}}\,\mathrm{d}x = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^{n-k}}{k+1}, \qquad \widehat{C}_n = \int_0^1 x^{\overline{n}}\,\mathrm{d}x = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{1}{k+1}$$

with the exponential generating series

$$C(t) = \sum_{n=0}^{\infty} C_n\frac{t^n}{n!} = \frac{t}{\log(1+t)}, \tag{29}$$

$$\widehat{C}(t) = \sum_{n=0}^{\infty} \widehat{C}_n \frac{t^n}{n!} = \frac{-t}{(1-t)\log(1-t)}.$$

These numbers are related by the identity

$$C_n = (-1)^n(\widehat{C}_n - n\widehat{C}_{n-1}) \quad (n = 1, 2, ...). \tag{30}$$

The Gregory coefficients G_n (also known as Bernoulli numbers of the second kind [11, p. 294], [22, p. 114] or logarithmic numbers [12]) are given by

$$G_n = \int_0^1 \binom{x}{n} \mathrm{d}x$$

with the ordinary generating series

$$G(t) = \sum_{n=0}^{\infty} G_n t^n = \frac{t}{\log(1+t)}.$$

Obviously, we have $C_n = n!\, G_n$. We also consider the generalized Cauchy numbers $C_n^{(\nu)}$. They are defined by

$$(C(t))^{\nu} = \sum_{n=0}^{\infty} C_n^{(\nu)} \frac{t^n}{n!} = \Big(\frac{t}{\log(1+t)}\Big)^{\nu}.$$

In particular, we obtain

$$C_n^{(2)} = \sum_{k=0}^{n} \binom{n}{k} C_k C_{n-k}.$$

Moreover, as proved in [26] we have

$$C_n^{(2)} = -(n-1)C_n - n(n-2)C_{n-1} \quad (n = 1, 2, ...). \tag{31}$$

Next, we show that A_n and $A_{n,1}$ can be expressed in terms of the Cauchy numbers or, equivalently, in terms of the absolute value of the Gregory coefficients.

Theorem 7. *We have for $n \geq 1$,*

$$A_n = (-1)^{n+1}\big((2n-1)C_n + 2C_{n+1}\big), \tag{32}$$

$$A_n = (2n-1)n\widehat{C}_{n-1} - (4n+1)\widehat{C}_n + 2\widehat{C}_{n+1}, \tag{33}$$

and

$$A_n^* = (2n-1)|G_n| - 2(n+1)|G_{n+1}|. \tag{34}$$

Moreover, we have for $n \geq 2$,

$$A_{n,1} = 2(-1)^{n-1}C_n, \quad A_{n,1}^* = 2|G_n|. \tag{35}$$

Proof. From (16) and (29) we obtain

$$tA(t) = (2-t)C(-t) - 2(C(-t))^2.$$

Comparing the coefficients of the series gives

$$(n+1)A_n = (-1)^{n+1}\big((n+1)C_n + 2C_{n+1} - 2C_{n+1}^{(2)}\big).$$

Applying (31) yields

$$\begin{aligned}(n+1)A_n &= (-1)^{n+1}\big((n+1)C_n + 2C_{n+1} + 2(nC_{n+1} + (n^2-1)C_n)\big)\\ &= (n+1)(-1)^{n+1}\big((2n-1)C_n + 2C_{n+1}\big).\end{aligned}$$

This gives (32). From (30) and (32) we obtain (33). Using $G_n = (-1)^{n-1}|G_n|$ $(n \geq 1)$ we conclude from (32) that (34) is valid.

Applying (21) with $k = 1$ and (29) we find

$$\sum_{n=2}^{\infty} A_{n,1}\frac{t^n}{n!} = 2 - t + \frac{2t}{\log(1-t)} = 2 - t - 2C(-t) = -2\sum_{n=2}^{\infty}(-1)^n C_n \frac{t^n}{n!}.$$

This leads to (35). □

Remark 10. Alabdulmohsin [3], Blagouchine [5, 6], Blagouchine and Coppo [7], Candelpergher and Coppo [10], Kowalenko [17] and others presented several interesting series with Gregory coefficients, like, for instance,

$$\sum_{n=1}^{\infty}|G_n| = 1, \quad \sum_{n=1}^{\infty}\frac{|G_n|}{n} = \gamma,$$

$$\sum_{n=2}^{\infty}\frac{|G_n|}{n-1} = -\frac{1}{2} + \frac{1}{2}\log(2\pi) - \frac{\gamma}{2},$$

$$\sum_{n=3}^{\infty}\frac{|G_n|}{n-2} = -\frac{1}{8} + \frac{1}{12}\log(2\pi) - \frac{\zeta'(2)}{2\pi^2},$$

$$\sum_{n=4}^{\infty}\frac{|G_n|}{n-3} = -\frac{1}{16} + \frac{1}{24}\log(2\pi) + \frac{\zeta(3)}{8\pi^2} - \frac{\zeta'(2)}{4\pi^2},$$

$$\sum_{n=1}^{\infty}\frac{|G_n|}{n+m} = \frac{1}{m} + \sum_{j=1}^{m}(-1)^j\binom{m}{j}\log(j+1) \quad (m = 1, 2, 3, \ldots).$$

Using these formulas and (34) we obtain the following identities involving the numbers A_n^*:

$$\sum_{n=1}^{\infty}\frac{A_n^*}{n} = 2 - \log(2\pi),$$

$$\sum_{n=1}^{\infty} \frac{A_n^*}{n+1} = 3\log 2 - 2,$$

$$\sum_{n=2}^{\infty} \frac{A_n^*}{n-1} = \frac{1}{6} - \frac{1}{2}\gamma + \frac{1}{6}\log(2\pi) + 2\frac{\zeta'(2)}{\pi^2},$$

$$\sum_{n=3}^{\infty} \frac{A_n^*}{n-2} = \frac{1}{12} - \frac{3}{4}\frac{\zeta(3)}{\pi^2},$$

and for integers $m \geq 2$, we have

$$\sum_{n=1}^{\infty} \frac{A_n^*}{n+m} = \frac{1}{m}\sum_{j=1}^{m}(-1)^{j-1}\binom{m}{j}\big(2(m-1)j+3m\big)\log(j+1).$$

In particular, for $m = 2, 3, 4$,

$$\sum_{n=1}^{\infty} \frac{A_n^*}{n+2} = 8\log 2 - 5\log 3,$$

$$\sum_{n=1}^{\infty} \frac{A_n^*}{n+3} = 27\log 2 - 17\log 3,$$

$$\sum_{n=1}^{\infty} \frac{A_n^*}{n+4} = 78\log 2 - 36\log 3 - 9\log 5.$$

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