#A123 INTEGERS 21 (2021)

A NOTE ON INTEGERS THAT ARE UNIQUELY EXPRESSIBLE BY INTEGRAL GEOMETRIC SEQUENCES

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Received: 6/22/21, Accepted: 12/8/21, Published: 12/13/21

Abstract

Let a, b be positive and coprime integers. By an integral geometric sequence we mean a finite set of the form $\{a^k, a^{k-1}b, a^{k-2}b^2, \ldots, b^k\}$. We characterize integers that can be uniquely expressed as a linear combination of an integral geometric sequence over nonnegative integers.

1. The Result

The problem of determining the number of solutions of linear Diophantine equations $a_1x_1+\cdots+a_kx_k = n$ over nonnegative integers $x_1, \ldots, x_k$ has a long and rich history; see [3]. The method of generating functions is a basic tool for this problem, and can be found in many standard textbooks on Combinatorics; for instance, see [1, 4].

We explore the following variant of this problem that arises naturally. Given a set $A = \{a_1, \ldots, a_k\}$ of positive integers, determine all $n \in \mathbb{Z}_{\geq 0}$ for which there is a unique k-tuple $(x_1, \ldots, x_k)$ of nonnegative integers such that

$$a_1x_1 + \cdots + a_kx_k = n. \tag{1}$$

We denote the set of all $n \in \mathbb{Z}_{\geq 0}$ such that Equation (1) has a unique solution by $S_1(A)$. This problem has been resolved in the case when A is a modified arithmetic sequence, i.e., $A = \{a, ha + d, ha + 2d, \ldots, ha + kd\}$, with a, d, h, k are positive integers and $\gcd(a, d) = 1$ in [2]. Observe that an arithmetic sequence is the special case $h = 1$.

The case of a geometric sequence: $A = \{a, ar, ar^2, \ldots, ar^k\}$, a, r, k are positive integers, $r > 1$, is easily dealt with. If $n \in S_1(A)$, then $a \mid n$. With $n = ma$,

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Equation (1) can be rewritten as $x_0 + rx_1 + r^2x_2 + \cdots + r^kx_k = m$. If $0 \le m < r$, each $x_i = 0$ for $i > 0$, so that $x_0 = m$. Thus, each such m is uniquely representable by $\{1, r, r^2, \ldots, r^k\}$. On the other hand, the above equation has at least the two solutions, viz., (i) $x_0 = m$, $x_i = 0$ for $i > 0$, and (ii) $x_0 = m - r$, $x_1 = 1$, $x_i = 0$ for $i > 1$ when $m \ge r$. Therefore, $S_1(A) = \{0, a, 2a, \ldots, (r-1)a\}$ in this case.

Since the problem of determining $S_1(A)$ when A is a geometric sequence is easily resolved, we consider a relaxation on the requirement that r be a positive integer in a geometric sequence while maintaining that the elements in the set A are positive integers. One way to achieve this is to consider a two parameter family, parametrized by positive and coprime integers a and b, with first term a^k and ratio b/a. For positive and coprime integers a, b, with $a < b$, and any positive integer k, by an integral geometric sequence we mean a sequence of the form

$$\mathcal{A}_k(a, b) = \{a^k, a^{k-1}b, a^{k-2}b^2, \ldots, b^k\}.$$

Note that the condition on coprimality of a, b can be assumed without loss of generality since $\gcd(a, b) = d$ can be easily linked to the case where $\gcd(a, b) = 1$. The purpose of this brief note is determine $S_1(\mathcal{A}_k)$ when $\mathcal{A}_k$ is an integral geometric sequence.

Let $\Gamma_k(a, b) = \left\{a^kx_0 + a^{k-1}bx_1 + \cdots + b^kx_k : x_i \in \mathbb{Z}_{\ge 0}\right\}$. Each integer in $\Gamma_k(a, b)$ is of the form $\mathbf{v}(x_0, \ldots, x_k) := \sum_{i=0}^{k} a^{k-i}b^ix_i$, with each $x_i \in \mathbb{Z}_{\ge 0}$. The transformation $(x_{k-1}, x_k) \mapsto (x_{k-1} + b, x_k - a)$ maintains the value of $\mathbf{v}(x_0, \ldots, x_k)$, and we repeatedly apply this until $0 \le x_k \le a - 1$. Note that the corresponding $x_{k-1} > 0$. Next we repeatedly apply the transformation $(x_{k-2}, x_{k-1}) \mapsto (x_{k-2} + b, x_{k-1} - a)$ until $0 \le x_{k-1} \le a - 1$. The corresponding $x_{k-2} > 0$ while maintaining the value of $\mathbf{v}(x_0, \ldots, x_k)$. Continuing with this process with successive transformations $(x_{i-1}, x_i) \mapsto (x_{i-1} + b, x_i - a)$, $i > 0$ leads to the same value of $\mathbf{v}(x_0, \ldots, x_k)$, but with $0 \le x_i \le a - 1$ for each $i > 0$ and $x_0 \ge 0$. Therefore, each integer in $\Gamma_k(a, b)$ is of the form $\sum_{i=0}^{k} c_ix_i$, with $0 \le x_i \le a - 1$ for each $i > 0$ and $x_0 \ge 0$.

Definition 1. We say that the expression $n = \sum_{i=0}^{k} c_ix_i$ is in *standard form* if we can write $0 \le x_i \le a - 1$ for each $i > 0$ and $x_0 \ge 0$.

For brevity, let us denote $(x_0, \ldots, x_k)$ and $(y_0, \ldots, y_k)$ by $\mathbf{x}$ and $\mathbf{y}$, respectively.

Lemma 1. *If* $\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{y})$ *with* $\mathbf{x}, \mathbf{y}$ *in standard form, then* $\mathbf{x} = \mathbf{y}$.

Proof. Suppose $\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{y})$ with $\mathbf{x}, \mathbf{y}$ in standard form. Then

$$a^kx_0 + a^{k-1}bx_1 + \cdots + b^kx_k = a^ky_0 + a^{k-1}by_1 + \cdots + b^ky_k. \qquad (2)$$

Reducing Equation (2) modulo a gives $x_k \equiv y_k \pmod{a}$ since $\gcd(a, b) = 1$. Therefore $x_k = y_k$, and Equation (2) reduces to

$$a^{k-1}x_0 + a^{k-2}bx_1 + \cdots + b^{k-1}x_{k-1} = a^{k-1}y_0 + a^{k-2}by_1 + \cdots + b^{k-1}y_{k-1}. \qquad (3)$$

Note that Equation (3) is of the form of Equation (2) with k replaced by $k-1$. Reducing Equation (3) modulo a leads to $x_{k-1} = y_{k-1}$, and continuing this argument shows $x_i = y_i$, $i \in \{0, \ldots, k\}$, so that $\mathbf{x} = \mathbf{y}$. □

Theorem 1. *A nonnegative integer n is uniquely representable by elements of $\mathcal{A}_k(a,b)$ if and only if*

$$n = \sum_{i=0}^{k} a^{k-i} b^i x_i, \text{ where } 0 \le x_0 \le b-1 \text{ and } 0 \le x_i \le a-1, i = 1, \ldots, k.$$

In particular, the number of such integers equals $a^k b$.

Proof. Let $X = \{(x_0, \ldots, x_k) : 0 \le x_0 \le b-1, 0 \le x_i \le a-1, i = 1, \ldots, k\}$. Let $n = \mathbf{v}(\mathbf{x})$, with $\mathbf{x} \in X$. We must show that $\mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{y})$ with $\mathbf{y} \in \mathbb{Z}_{\ge 0}^{k+1}$ implies $\mathbf{x} = \mathbf{y}$. Repeated applications of the transformations described in the paragraph above Lemma 1 leads to $\mathbf{v}(\mathbf{y}) = \mathbf{v}(\mathbf{y}')$, where $\mathbf{y}'$ is in standard form. Since $\mathbf{x}$ is in standard form, we have $\mathbf{x} = \mathbf{y}'$ by Lemma 1. Applying the inverse transformations in reverse order to $\mathbf{y}'$ must lead back to $\mathbf{y}$. However, such transformations are not applicable to $\mathbf{x}$. Thus, $\mathbf{y}' = \mathbf{y}$, so that $\mathbf{x} = \mathbf{y}$.

Nonnegative integers that are not in $\Gamma_k(a,b)$ have no representation by elements in $\mathcal{A}_k(a,b)$. Therefore, we must show that any $n = \mathbf{v}(\mathbf{x})$, $\mathbf{x} \notin X$ has at least two representations by elements in $\mathcal{A}_k(a,b)$. Note that $\mathbf{x} \notin X$ implies either $x_0 \ge b$ or $x_i \ge a$ for some $i \in \{1, \ldots, k\}$. Now

$$\mathbf{v}(x_0, x_1, x_2, \ldots, x_k) = \begin{cases} \mathbf{v}(x_0 - b, x_1 + a, x_2, \ldots, x_k), & \text{if } x_0 \ge b, \\ \mathbf{v}(x_0, \ldots, x_{j-2}, x_{j-1} + b, x_j - a, x_{j+1}, \ldots, x_k) & \\ & \text{if } x_j \ge a, j \in \{1, \ldots, k\}, \end{cases}$$

gives two representations of any such n by elements of $\mathcal{A}_k(a,b)$. □

Acknowledgement. The authors thank the referee for his/her comments.

References

[1] L. Comtet, *Advanced Combinatorics*, Dordrecht, Reidel, 1974.

[2] S. Chimni, S. Sankar and A. Tripathi, On integers that are uniquely representable by modified arithmetic progressions, *Notes in Number Theory and Discrete Mathematics* **22**, no.3 (2016), 36-44.

[3] L. E. Dickson, *History of the Theory of Numbers*, Vol. 2, Chelsea, New York, 1992.

[4] H. S. Wilf, *generatingfunctionology*, Academic Press, New York, 1990.