ALTERNATIVE FORMULATIONS OF THE GENERALIZED COSECANT AND SECANT NUMBERS

Victor Kowalenko
School of Mathematics and Statistics, The University of Melbourne, Victoria, Australia
vkowa@unimelb.edu.au

Received: 11/17/19, Accepted: 1/20/21, Published: 2/1/21

Abstract

Alternative formulations of the generalized cosecant and secant numbers, which represent the polynomial coefficients of the generating functions for arbitrary powers of the cosecant and secant functions, respectively, are presented via the partition method for a power series expansion. The new results have the advantage that expressions for the higher order coefficients are more easily computed with the aid of a computer program whose symbolic output can be introduced into Mathematica. In addition, expressions in terms of finite sums of products of the Riemann zeta function with even integer arguments are derived for the lowest order coefficients for both sets of numbers, which could not be evaluated previously.

1. Introduction

The generalized cosecant and secant numbers, denoted here by $c_{\rho,k}$ and $d_{\rho,k}$, respectively, represent the coefficients of the generating functions for the cosecant and secant functions raised to an arbirary power, ρ. That is, they are given by

$$\csc^{\rho} z = \frac{1}{\sin^{\rho} z} \equiv \sum_{k=0}^{\infty} c_{\rho,k} z^{2k-\rho}, \tag{1}$$

and

$$\sec^{\rho} z = \frac{1}{\cos^{\rho} z} \equiv \sum_{k=0}^{\infty} d_{\rho,k} z^{2k}. \tag{2}$$

The equivalence symbol has been introduced here because the right-hand sides of both equations can become divergent, specifically when $|z| \geq \pi$ in (1), while in the case of (2), when $|z| \geq \pi/2$. Inside the disks of absolute convergence the equivalence symbols may be replaced by an equals sign, whereas for z in the divergent regions, the above statements must be regularized in the manner described in [1]-[6].

Both sets of numbers not only appear in various applications, but they also possess many interesting properties of their own [7, 8]. Moreover, expressions for them are derived in these references, where it is found that they are, in fact, polynomials in powers of ρ of degree k with invariant coefficients that are functions of k or the order of z^2 in the generating functions. As a consequence, both sets of numbers can be expressed as

$$c_{\rho,k} = \sum_{i=1}^{k} C_{k,i}(k)\rho^i, \tag{3}$$

and

$$d_{\rho,k} = \sum_{i=1}^{k} D_{k,i}(k)\rho^i. \tag{4}$$

As explained in [7, 8], formulas or expressions for both sets of numbers can be obtained by using the discrete graphical method known as the partition method for a power series expansion, which relies on coding all the integer partitions summing to each order, k. In the case of the generalized cosecant numbers, one finds that

$$c_{\rho,k} = (-1)^k \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k=0 \\ \sum_{i=1}^{k} i\lambda_i = k}}^{k,\lfloor k/2\rfloor,\lfloor k/3\rfloor,\ldots,1} (-1)^{N_k}(\rho)_{N_k} \prod_{i=1}^{k} \Big(\frac{1}{(2i+1)!}\Big)^{\lambda_i} \frac{1}{\lambda_i!}, \tag{5}$$

where λ_i represents the multiplicity or the number of occurrences of each part i in the partitions, and the sum of the multiplicities or length of the partition is represented by N_k. That is, $N_k = \sum_{i=1}^{k} \lambda_i$. For partitions summing to k, the multiplicity of a part i can only range from zero to $\lfloor k/i \rfloor$, where $\lfloor x \rfloor$ denotes the floor function or the greatest integer less than or equal to x. When k is large, most of the multiplicities in the partitions vanish. Furthermore in the above result, $(\rho)_{N_k}$ denotes the Pochhammer notation for the quotient of gamma functions, namely, $\Gamma(\rho + N_k)/\Gamma(\rho)$. For $\rho = 1$, (5) reduces to the cosecant numbers, c_k, which were found in [7] to be given by

$$c_k = 2\big(1 - 2^{1-2k}\big)\frac{\zeta(2k)}{\pi^{2k}}, \tag{6}$$

with $\zeta(z)$ denoting the Riemann zeta function.

When the method is applied to yield an expression for the generalized secant numbers, the major difference occurs in assigning or coding a value to each part in the partitions. In (5) each part i has been assigned a value of $(-1)^{i+1}/(2i+1)!$, whereas for the generalized secant numbers, each part i is now assigned a value of $(-1)^{i+1}/(2i)!$. These assigned values are based on the coefficients in the power series expansions of the sine and cosine functions, which become the inner series in the partition method for a power series expansion. Therefore, via the partition method for a power series expansion, the generalized secant numbers are given by

$$d_{\rho,k} = (-1)^k \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k=0 \\ \sum_{i=1}^k i\lambda_i = k}}^{k,\lfloor k/2\rfloor,\lfloor k/3\rfloor,\ldots,1} (-1)^{N_k} (\rho)_{N_k} \prod_{i=1}^{k} \Big(\frac{1}{(2i)!}\Big)^{\lambda_i} \frac{1}{\lambda_i!}. \tag{7}$$

As in the case of the generalized cosecant numbers, the above result simplifies considerably when $\rho = 1$. In this instance, we obtain the secant numbers, which in terms of the Hurwitz zeta function, $\zeta(z,a)$, are given by

$$d_k = \frac{2}{(2\pi)^{2k+1}} \Big(\zeta(2k+1, 1/4) - \zeta(2k+1, 3/4)\Big). \tag{8}$$

Like the cosecant numbers, the secant numbers are rational and converge to zero as $k \to \infty$, although not as rapidly as their cosecant counterparts [7, 8].

In a recent work [9] formulas for the coefficients, $C_{k,i}$, in the generalized cosecant numbers were derived by expanding upon techniques sketched out in [8]. The three highest order coefficients, i.e., $C_{k,k}$ to $C_{k,k-2}$, were determined by applying (5) to the relevant partitions. For example, in order to determine $C_{k,k-1}$, one requires the contributions from two general partitions, the partition with k ones represented as $\{1_k\}$ and the partition with $k-2$ ones and a two, which is represented as $\{1_{k-2}, 2\}$. The ρ^{k-1} term in the contribution from the first partition is determined by expanding the Pochhammer factor $(\rho)_k$, while there is no need to do so for the contribution from the second partition. By combining the two contributions, one arrives at the value for $j = 1$ in Table 1 after a little algebra.

At the time it was found that the approach of isolating the appropriate power of ρ in the contributions from relevant partitions was awkward, although more recently, the approach has been refined. Consequently, another approach was adopted. From studying the three highest order coefficients, it was conjectured that each coefficient, $C_{k,k-j}$, possessed a polynomial in k of degree $j-1$ in the numerator, while the product of $(3!)^k$ and $(k-j-1)!$ appeared as a common factor in the denominator. Hence $C_{k,k-3}$ was conjectured to equal $(ak^2 + bk + c)/((3!)^{k+2}(k-4)!$. Then a set of three simultaneous equations was obtained by putting $k = 4$, 5 and 6 and setting them equal to $C_{4,1}$, $C_{5,2}$ and $C_{6,3}$, respectively, from the table of generalized cosecant numbers given as Table 2 in [9]. The equations were imported into Mathematica and solved by using the Solve routine. After further simplification, one arrives at the $j = 3$ result in Table 1.

The same procedure was applied to determine the $j = 4$ result in Table 1 except a third order polynomial was involved. In this instance a set of four simultaneous equations was obtained by putting k equal to 5, 6, 7, and 8 in the conjectured form for $C_{k,k-4}$ and then setting the resulting forms equal to the coefficients $C_{5,1}$,

j	$C_{k,k-j}$
0	$1/(3!)^k k!$
1	$1/5 \cdot (3!)^k (k-2)!$
2	$(21k+17)/175(3!)^{k+1}(k-3)!$
3	$(k^2+17k/7)/125(3!)^{k+1}(k-4)!$
4	$3(3k^3/625+102k^2/4375+289k/30625-2234/67375)/(3!)^{k+3}(k-5)!$
5	$(3k^4/31250+17k^3/21875+289k^2/306250-1117k/336875$ $-373248/109484375)/(3!)^{k+2}(k-6)!$
6	$(324k^5/78125+5508k^4/109375+93636k^3/765625-23388012k^2/58953125$ $-4719133656k/3831953125+103104/2234375)/((3!)^{k+6}(k-7)!)$
7	$(9k^6/2734375+153k^5/2734375+867k^4/3828125-180527k^3/294765625$ $-7641618k^2/1741796875-33159536k/19159765625$ $+421023744/65143203125)/(3!)^{k+4}(k-8)!$
8	$(27k^7/54687500+153k^6/13671875+2601k^5/38281250-189726k^4/1473828125$ $-5684318573k^3/2682367187500-1810590147k^2/670591796875$ $+218316311911k/25080133203125$ $+761597089236/95304506171875)/(3!)^{k+5}(k-9)!$
9	$(12k^8/68359375+2448k^7/478515625+20808k^6/478515625$ $-261248k^5/7369140625-6541965428k^4/3352958984375$ $-1329055472k^3/257919921875+8716556963888k^2/627003330078125$ $+19557983755584k/476522530859375$ $-3483186167808/476522530859375)/(3!)^{k+5}(k-10)!$
10	$(9k^9/6835937500+459k^8/9570312500+2601k^7/4785156250$ $+21387k^6/73691406250-10830689607k^5/335295898437500$ $-13912626361k^4/93882851562500+6432180898756k^3/21945116552734375$ $+183914974080577k^2/83391442900390625$ $+11275018983060292k/2710221894262695312$5 $-61312033048073664/17810029590869140625)/(3!)^{k+6}(k-11)!$

Table 1: Highest order coefficients of the generalized cosecant numbers

$C_{6,2}$, $C_{7,3}$ and $C_{8,4}$ in Table 2 of [9], respectively. Since then, the method has been applied to the $j=5$ and $j=6$ cases, the results of which also appear in Table 1. The last result produces a system of 6 simultaneous equations, which can be solved again via the Solve routine in Mathematica. However, for $j \geq 7$, this routine is unable to yield a solution for the coefficients, at least for some computing systems including the author's. The results for $j=7$ to 10 in Table 1 have been determined by applying an alternative formulation of the generalized cosecant numbers, which is presented in the following section.

2. Alternative Formulation of the Generalized Cosecant Numbers

An alternative formulation for the generalized cosecant numbers can be obtained first by introducing the infinite product formula for $\sin z$ (Equation (1.431.1) in [10]) into $z^\rho \csc^\rho z$. This gives

$$z^\rho \csc^\rho z = \prod_{j=1}^{\infty} \Big(1 - \frac{z^2}{j^2\pi^2}\Big)^{-\rho}. \tag{9}$$

Next, we exponentiate the product arriving at

$$z^\rho \csc^\rho z = \exp\Big(-\rho \sum_{j=1}^{\infty} \ln\Big(1 - \frac{z^2}{j^2\pi^2}\Big)\Big). \tag{10}$$

In order to apply the partition method for a power series expansion to the above result, we require an inner and outer power series. The inner series in the above result is obtained by replacing the logarithmic term by its Taylor series expansion, which according to [6] is given by

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, z^k \begin{cases} \equiv \ln(1+z), & \Re\, z \le -1 \quad, \\ = \ln(1+z), & \Re\, z > -1 \quad. \end{cases} \tag{11}$$

From this result we see that the power series is divergent for $\Re z \le -1$, in which case its regularized value is found to equal $\ln(1+z)$. Although the series is convergent for $\Re z > -1$, it is only absolutely convergent over the unit disk, namely, $|z| < 1$. Otherwise, it is conditionally convergent. Nevertheless, we can introduce the power series generally provided we use the equivalence symbol. Therefore, introducing (11) into (10) yields

$$z^\rho \csc^\rho z \equiv \exp\Bigg(\rho \sum_{j=1}^{\infty}\Big(\frac{z^2}{j^2\pi^2} + \frac{1}{2}\Big(\frac{z^2}{j^2\pi^2}\Big)^2 + \frac{1}{3}\Big(\frac{z^2}{j^2\pi^2}\Big)^3 + \cdots\Big)\Bigg). \tag{12}$$

The above result can be simplified by noting that the summations over j represent different values of the Riemann zeta function. Then we find that

$$z^\rho \csc^\rho z \equiv \exp\Bigg(\rho\Big(\frac{z^2\zeta(2)}{\pi^2} + \frac{\zeta(4)}{2}\Big(\frac{z^2}{\pi^2}\Big)^2 + \frac{\zeta(6)}{3}\Big(\frac{z^2}{\pi^2}\Big)^3 + \cdots\Big)\Bigg). \tag{13}$$

By replacing z^2 by z, we now have a formulation where we can apply the partition method for a power series expansion. In this instance, the inner power series whose coefficients are represented by p_k in Ch. 4 of [8] become $p_k = \rho\zeta(2k)/k\pi^{2k}$, while for the outer power series, the coefficients, q_k, are simply those for the power series expansion of the exponential function. That is, by expanding the exponential, we have $q_k = 1/k!$. Then the equivalent of (5) becomes

$$c_{\rho,k} = L_{P,k}\left[\prod_{i=1}^{k}\Big(\frac{\rho\zeta(2i)}{i\pi^{2i}}\Big)^{\lambda_i}\frac{1}{\lambda_i!}\right], \tag{14}$$

where the partition operator, $L_{P,k}\Big[\cdot\Big]$, is defined as

$$L_{P,k}\Big[\cdot\Big] \doteqdot \sum_{\substack{\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k=0 \\ \sum_{i=1}^{k} i\lambda_i=k}}^{k,\lfloor k/2\rfloor,\lfloor k/3\rfloor,\ldots,1}\Big(\cdot\Big). \tag{15}$$

Because this result does not have a Pochhammer term inside the partition operator, we do not require nearly all the partitions summing to k when evaluating the lowest order terms of the generalized cosecant numbers as in [9]. That is, each partition contributes only one power of ρ to the generalized cosecant numbers. Furthermore, from (14), we observe that the coefficients of $c_{\rho,k}$ are composed of finite sums over products of even integer values of the zeta function. Thus the coefficients of the generalized cosecant numbers possess a mathematical base.

Before studying the lowest order coefficients of the generalized cosecant numbers, let us examine the results from (14) for the highest order coefficients, which will enable us not only to verify the results in Table 1, but to extend them more easily beyond $C_{k,k-6}$, which was raised as a problem in the previous section. Another property of (14) is that the highest powers of ρ occur when the multiplicities are at their highest values. This means that the greater the length of a partition, N_k, the greater the power of ρ in $c_{\rho,k}$. Thus the coefficient of the highest power of ρ^k is determined by the contribution from the partition, $\{1_k\}$. Hence $C_{k,k} = (\zeta(2)/\pi^2)^k/k!$, which reduces to the $j = 0$ result in Table 1.

The coefficient of the second highest power of ρ is obtained by summing the contributions from partitions of length, $k-1$. In this instance, there is only one partition, namely, $\{1_{k-2}, 2\}$. Therefore, the coefficient is found to be

$$C_{k,k-1} = \Big(\frac{\zeta(2)}{\pi^2}\Big)^{k-2}\frac{1}{(k-2)!}\Big(\frac{\zeta(4)}{2\pi^4}\Big) = \frac{1}{5\cdot(3!)^k(k-2)!}. \tag{16}$$

So far, the results have been simple because only one partition has been involved in determining the coefficients of the two highest order terms in the generalized cosecant numbers. However, to determine $C_{k,k-2}$, we require the two partitions summing to k with a length of $k-2$. These are the partitions, $\{1_{k-3}, 3\}$ and $\{1_{k-4}, 2_2\}$. Then we arrive at

$$C_{k,k-2} = \Big(\frac{\zeta(2)}{\pi^2}\Big)^{k-3}\frac{1}{(k-3)!}\Big(\frac{\zeta(6)}{3\pi^6}\Big) + \Big(\frac{\zeta(2)}{\pi^2}\Big)^{k-4}\frac{1}{(k-4)!}\Big(\frac{\zeta(4)}{2\pi^4}\Big)^2\frac{1}{2!}. \tag{17}$$

	Partitions summing to k with length $k-5$	Partitions summing to 5
1	$\{1_{k-6}, 6\}$	{5}
2	$\{1_{k-7}, 2, 5\}$	{1,4}
3	$\{1_{k-7}, 3, 4\}$	{2,3}
4	$\{1_{k-8}, 2_2, 4\}$	$\{1_2, 3\}$
5	$\{1_{k-8}, 2, 3_2\}$	$\{1,2_2\}$
6	$\{1_{k-9}, 2_3, 3\}$	$\{1_3, 2\}$
7	$\{1_{k-10}, 2_5\}$	$\{1_5\}$

Table 2: Homology between partitions summing to k with length $k-5$ and standard integer partitions summing to 5

After a little algebra, one obtains the result for $C_{k,k-2}$ in Table 1.

To evaluate $C_{k,k-3}$, we require the contributions from the partitions summing to k of length, $k-3$. In this instance, there are three partitions, $\{1_{k-4}, 4\}$, $\{1_{k-5}, 2, 3\}$ and $\{1_{k-6}, 2_3\}$. For $C_{k,k-4}$, we need to evaluate the contributions from the five partitions summing to k of length $k-4$, which are $\{1_{k-5}, 5\}$, $\{1_{k-6}, 2, 4\}$, $\{1_{k-6}, 3_2\}$, $\{1_{k-7}, 2_2, 3\}$ and $\{1_{k-8}, 2_4\}$. For $C_{k,k-5}$, there are seven partitions summing to k of length $k-5$. These are listed in Table 2. In each instance, we require $P(j)$ partitions to evaluate $C_{k,k-j}$, where $P(j)$ denotes the partition function or the number of standard integer partitions summing to j. In fact, there is a homology between the partitions summing to k of length $k-j$ and those summing to j as depicted in Table 2, which displays the $j=5$ case. There it can be seen that whenever the length of the standard integer partition is ℓ, there will be $k-5-\ell$ ones in the corresponding partition summing to k of length, $k-5$. In addition, each part in a standard integer partition is incremented by unity to give the corresponding partition in the second column of Table 2. Therefore, we see that the three-part partition $\{1,2_2\}$ is homologous or corresponds to $\{1_{k-8}, 2, 3_2\}$, while the four-part partition $\{1_3, 2\}$ corresponds to $\{1_{k-9}, 2_3, 3\}$.

As a consequence of the homology between the classes of partitions, we do not need to create an entirely new program to generate the specific partitions that are required for evaluating $C_{k,k-\ell}$. Instead, all we need to do is modify an existing program, **numparts**, which is discussed extensively in [8]. This program generates the standard integer partitions summing to a value with a specified number of parts or length. For example, if we wish to know the number of partitions summing to 9 with 5 parts, then the command to run the program is ./numparts 9 5, which results in the following output:

1: 4(1) 1(5)
2: 3(1) 1(2) 1(4)
3: 3(1) 2(3)

4: 2(1) 2(2) 1(3)
5: 1(1) 4(2).
The first partition in the above output represents the partition with four ones and one five or $\{1_4, 5\}$, while the last partition is the partition with one one and four twos or $\{1, 2_4\}$. The reason why **numparts** has been chosen over other programs discussed in Chapter 3 of [8] is that in order to generate the partitions summing to k with $k - j$ parts, we have seen from Table 2 that they are dependent upon their lengths, whereas the other codes in Chapter 3 generate all the partitions summing to a specified value. That is, **numparts** possesses the capability of generating only those partitions with the same number of parts, which is required for generating the special class of partitions involved in the evaluation of $C_{k,k-\ell}$.

The modified version of **numparts** that generates the special partitions of length, $k - \ell$, summing to k appears as Program 1 in the appendix. In this program called **GNCP** the multiplicities of the partitions, as in the other codes discussed in [8], are stored in an array called *part*. That is, the multiplicity of a part i or λ_i is stored as *part[i]*. The tree-like structure of the partitions results from calling the bivariate recursive central partition or **brcp** for short, in the **main** function subprogram, while the results from parsing each partition are printed out by calling the **termgen** function subprogram. That is, **termgen** only prints out the specific class of partitions required by the user. Typical examples of interest to the user are: (1) distinct partitions, where each part only occurs once, (2) partitions with only odd or even parts, (3) perfect partitions and (4) those with a specific part in them. See Chapters 3 and 7 of [8] for more intricate examples.

The first major difference between the new code and **numparts** occurs in the main routine, which possesses an extra for loop. This is necessary so that all lengths of partitions summing to k be considered instead of specifying only one length in running **numparts**. For example, we need to specify 5 as the number of parts in the case above, but this is redundant in **GNCP**. The variable *tot* in the modified program represents not only the value of k to which the parts are summed, but also represents the maximum number of parts. Hence it appears in the upper limit of the extra for loop in **main**. In addition, the variable *numparts* gives the number of parts under consideration and is set equal to the variable *j*. It is incremented by unity in order to avoid the ones, which are handled separately in the first printf statement in **termgen** and set equal to $k - tot - j - 1$. Once the number of ones is printed out, the multiplicities of the remaining parts are printed out in another for loop in a similar manner to **numparts** except the values of the parts are incremented by two instead of unity. If we run the program for $\ell = 5$, then the following output is generated:
1: (k-6)(1) 1(6)
2: (k-7)(1) 1(2) 1(5)
3: (k-7)(1) 1(3) 1(4)

4: (k-8)(1) 2(2) 1(4)
5: (k-8)(1) 1(2) 2(3)
6: (k-9)(1) 3(2) 1(3)
7: (k-10)(1) 5(2).
Therefore, we obtain the seven partitions listed in the second column of Table 2.

By describing how the partitions summing to k of length, $k-\ell$, can be generated, we are now in a position to evaluate the coefficients, $C_{k,k-\ell}$. That is, we need to create a code that evaluates all the contributions from the above-mentioned partitions for each value of ℓ. Once again, all we need to do is modify an existing program. This is the program, **mathpm**, which computes the general coefficients, D_k, from the method for a power series expansion in symbolic form. It is discussed at length in Chapter 5 of [8]. In general, it is meant that the coefficients will be printed out in terms of the coefficients p_k and q_k of inner and outer power series mentioned earlier in this section. Consequently, we can apply the code to other situations such as the coefficients of generalized secant numbers in the following section.

The second modified program is called **gencoeff** and appears as the second program in the appendix. Structurally, it is identical to the first program with the main differences occurring inside **termgen**. In the first part of **termgen**, the program prints out LR[l_,*tot*], where the latter variable has been specified in the input statement. The l_ term represents symbolic input for Mathematica, indicating that the final result for each $C_{k,k-\ell}$ will be a function of ℓ. The variable *termcnt* ensures that at most only two contributions from the partitions appear on each line of the output. Next the number of parts for each partition or *num_parts* is determined. Then **termgen** prints out the contribution from the ones in each partition in powers of $p[1]$ divided by the factorial of their multiplicity. Finally, the code prints out $p[i]^{\lambda_i}/\lambda_i!$ for the other parts in each partition. Therefore, if we run the code by typing ./gencoeff 5, then the following output appears

```
LR[k_,5]:=((k-6)! (p[1]^(k-6)/(k-6)!) (p[6]^(1)/ 1!)) +((k-6)! (p[1]^(k-7)/(k-7)!) (p[2]^(1)/ 1!)
(p[5]^(1)/ 1!)) +
((k-6)! (p[1]^(k-8)/(k-8)!) (p[2]^(2)/ 2!) (p[4]^(1)/ 1!)) +((k-6)! (p[1]^ (k-9)/(k-9)!) (p[2]^(3)/
3!) (p[3]^(1)/ 1!)) +
((k-6)! (p[1]^(k-10)/(k-10)!) (p[2]^(5)/ 5!)) +((k-6)! (p[1]^(k-8)/(k-8)!) (p[2]^(1)/ 1!) (p[3]^(2)/
2!)) +
((k-6)! (p[1]^(k-7)/(k-7)!) (p[3]^(1)/ 1!) (p[4]^(1)/ 1!)).
```

Note that the above result has been multiplied by $(k-6)!$ so that it will yield polynomials when divided by the multiplicity of the ones in each partition. This means that we will need to divide by $(k-6)!$ to obtain the final value of $C_{k,k-5}$. More generally, to arrive at the final value of $C_{k,k-\ell}$, we need to divide the expression for LR by $(k-\ell-1)!$. In addition, the plus signs on the second and fourth lines and the last one at the end of the fifth line represent the ends of the lines when the above is imported into a Mathematica notebook.

The above result is now in a form that can be handled by Mathematica. To obtain $C_{k,k-5}$, we first type into a notebook:
p[k_] := Zeta[2 k]/(k Pi^(2 k)).
Next we replace the left-hand side of LR[k_,5] by Ckkminus5[k_] and enter the right-hand side of the above result for LR[k_,5] in the same notebook. By wrapping the expression in the FullSimplify routine and dividing by $(k-6)!$, we find that Mathematica prints out
In[3]:= Ckkminus5[k, 5]
Out[3]= (2^(-3 - k) 3^(-2 - k) (-746496 + 13 k (-55850 + 11 k (1445 + 7 k (170 + 21 k)))))/109484375 (k-6)!.
The above output can be simplified further by typing
In[4]:= Expand[(-746496 + 13 k (-55850 + 11 k (1445 + 7 k (170 + 21 k))))]
Out[4]= -746496 - 726050 k + 206635 k^2 + 170170 k^3 + 21021 k^4
This result can then replace the bracketed expression in the numerator of Ckkminus5[k,5] given above. Moreover, we can replace the factors outside the bracketed expression by $1/2 \cdot (3!)^{k+2}$ and divide throughout by 109484375, thereby obtaining the value for $C_{k,k-5}$ in Table 1. By carrying out the same procedure for $j = 7$ to 10, we arrive at the values displayed in Table 1 for these values of j, which is far more expedient than using the Solve routine in Mathematica described in the previous section.

Now we consider the issue of evaluating the lowest order terms in (14). The lowest order term in (14) is the linear power of ρ, which is obtained by considering 1-part partitions summing to k of which there is only one, {k}. The contribution from this partition yields $C_{k,1}$. According to (14), we obtain

$$C_{k,1} = \frac{\zeta(2k)}{k\pi^{2k}}. \tag{18}$$

As a check, if we examine $c_{\rho,11}$ in Table 2 of [9], then we see that the coefficient of the first order term equals $8\times 494848416153600/(243\times 25!)$ or 155366/147926426347074375. As expected, putting $k = 11$ in (18) yields the same value.

To calculate $C_{k,2}$, we need to determine all the contributions due to 2-part partitions, {j,k-j}, where j ranges from unity to $\lfloor k/2 \rfloor$. For all these partitions the multiplicity of the parts is equal to unity except when k is even and $j = k/2$. In this instance the multiplicity, $\lambda_{k/2}$, equals two. Hence we can sum over all values of j by setting the multiplicities to unity. However, for even values of k when $j = k/2$, we need to subtract half its contribution from the sum over j. Consequently, we arrive at

$$C_{k,2} = \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{\zeta(2j)\zeta(2k-2j)}{j(k-j)\pi^{2k}} - \big(1+(-1)^k\big)\frac{\zeta(k)^2}{k^2\pi^{2k}}. \tag{19}$$

The last term in (19) only contributes when k is even and $j = k/2$. When this occurs, it removes half the contribution from the $j = k/2$ term in the first sum.

If we put $k = 12$ in (19), then we obtain a value of 19311547193251/52337553053 005690470000000, which is identical to the coefficient of the second order term for $c_{\rho,12}$ in Table 2 of [9], where it is given as $5695207005856038912 \times 2/(2835 \times 27!)$.

The coefficient of the next highest order term in the generalized cosecant numbers, i.e., $C_{k,3}$, requires the contributions from all 3-part partitions summing to k. In this instance we fix the first value of the part to i_1, which will range from 1 to $\lfloor k/3 \rfloor$. Once i_1 is fixed, we need to sum the contributions from all 2-part contributions that sum to $k - i_1$. This means that another variable, i_2, is required in order to sum the contributions from the partitions $\{i_2, k\text{-}i_1\text{-}i_2\}$ with i_2 ranging from i_1 to $\lfloor (k - i_1)/2 \rfloor$. Nevertheless, there are some complications. The first is that at the lower end of summation over i_2, it is possible that $i_1 = i_2$, in which case the partition becomes $\{(i_1)_2, k\text{-}2i_1\}$ with a multiplicity of 2 for the part i_1. Half the contribution of this partition must be removed from the double summation over i_1 and i_2. In addition, when $k - i_1$ is even, we have the situation where the upper limit and i_2 can equal each other, in which case the multiplicity of the part $(k - i_1)/2$ equals 2. Thus we have to remove half this contribution from the sum over i_1 and i_2. Consequently, $C_{k,3}$ can be expressed as

$$C_{k,3} = \sum_{i_1=1}^{\lfloor k/3 \rfloor} \Big(\frac{\zeta(2i_1)}{i_1 \pi^{2i_1}} \Big) \sum_{i_2=i_1}^{\lfloor (k-i_1)/2 \rfloor} \Big(\frac{\zeta(2i_2)}{i_2 \pi^{2i_2}} \Big) \Big(\frac{\zeta(2k - 2i_1 - 2i_2)}{(k - i_1 - i_2)\pi^{2(k-i_1-i_2)}} \Big) - \frac{1}{2} \sum_{i_1=1}^{\lfloor k/3 \rfloor} \Big(\frac{\zeta(2i_1)}{i_1 \pi^{2i_1}} \Big)^2 \times \Big(\frac{\zeta(2k - 4i_1)}{(k - 2i_1)\pi^{2(k-2i_1)}} \Big) - \sum_{i_1=1}^{\lfloor k/3 \rfloor} \Big(\frac{\zeta(2i_1)}{i_1 \pi^{2i_1}} \Big) \Big(\frac{1 + (-1)^{k-i_1}}{(k - i_1)^2} \Big) \Big(\frac{\zeta(k - i_1)}{i_1 \pi^{k-i_1}} \Big)^2. \tag{20}$$

This, however, is not the end of the story. We also have the situation that when k is a multiple of 3, there is a partition where all three parts are equal to one another, namely, $\{(k/3)_3\}$. In this case the multiplicity of part, $k/3$, is three, which results in a value of $1/3!$ or $1/6$ appearing in the denominator of its contribution. If we put $i_1 = i_2 = k/3$ in the above result, then we find that there is no contribution from this partition whatsoever since all the relevant terms from the summations cancel each other. Hence we need to include the contribution from this partition specifically, but only when k is a multiple of 3. To ensure that there is only a contribution at multiples of three, we require the following identity:

$$\sum_{j=1}^{\ell} e^{2\pi ijk/\ell} = \begin{cases} \ell\,, & k \equiv 0 \mod \ell, \\ 0\,, & k, \quad \text{otherwise.} \end{cases} \tag{21}$$

In our case we put $\ell = 3$, which yields a factor of $(1 + 2(-1)^k \cos(\pi k/3))$. We divide this by ℓ or 3 to ensure that we only obtain unity when k is a multiple of 3. Therefore, to obtain the contribution when all parts are equal in a 3-part partition, we multiply this factor by the partition's contribution in (14), which is

$(3\zeta(2k/3))^3/(3!k^3\pi^{2k})$. Thus, we arrive at

$$C_{k,3} = \sum_{i_1=1}^{\lfloor k/3 \rfloor} \Big(\frac{\zeta(2i_1)}{i_1\pi^{2i_1}}\Big) \sum_{i_2=i_1}^{\lfloor (k-i_1)/2 \rfloor} \Big(\frac{\zeta(2i_2)}{i_2\pi^{2i_2}}\Big)\Big(\frac{\zeta(2k-2i_1-2i_2)}{(k-i_1-i_2)\pi^{2(k-i_1-i_2)}}\Big) - \frac{1}{2}\sum_{i_1=1}^{\lfloor k/3 \rfloor}\Big(\frac{\zeta(2i_1)}{i_1\pi^{2i_1}}\Big)^2$$
$$\times \Big(\frac{\zeta(2k-4i_1)}{(k-2i_1)\pi^{2(k-2i_1)}}\Big) - \sum_{i_1=1}^{\lfloor k/3 \rfloor}\Big(\frac{\zeta(2i_1)}{i_1\pi^{2i_1}}\Big)\Big(\frac{1+(-1)^{k-i_1}}{(k-i_1)^2}\Big)\Big(\frac{\zeta(k-i_1)}{\pi^{k-i_1}}\Big)^2$$
$$+ \frac{3}{2}\Big(1+2(-1)^k\cos(k\pi/3)\Big)\Big(\frac{\zeta(2k/3)}{k^3\pi^{2k/3}}\Big)^3. \tag{22}$$

If we type this result into a Mathematica notebook as Ck3[k_] and put $k = 13$, then we find that
In[10]:= Ck3[13]
Out[10]= 18909126568021/314025318318034142820000000,
which is indeed equal to $5576528334428209152 \times 232/(81 \times 30!)$, the coefficient of ρ^3 in $c_{\rho,13}$ according to Table 2 of [9]. For k, a multiple of 3 such as 12, where the last term in (22) does not vanish, we obtain
In[14]:= Ck3[12]
Out[14]:= 15649829911/2546065213888996800000.
This also agrees with the value of $9487372599204065280 \times 2/(2835 \times 27!)$ for the coefficient of ρ^3 in $c_{\rho,12}$ in the same table.

The above result for $C_{k,3}$ can be generalized by replacing $\rho\zeta(2i)/i\pi^{2i}$ by $p(i)$ in (14), which will be useful in the next section. Therefore, the third order coefficient can be expressed more generally as

$$C_{k,3}(k) = \sum_{i_1=1}^{\lfloor k/3 \rfloor} p(i_1) \sum_{i_2=i_1}^{\lfloor (k-i_1)/2 \rfloor} p(i_2)p(k-i_1-i_2) - \frac{1}{2}\sum_{i_1=1}^{\lfloor k/3 \rfloor} p(i_1)^2 p(k-2i_1)$$
$$- \sum_{i_1=1}^{\lfloor k/3 \rfloor}\Big(\frac{1+(-1)^{k-i_1}}{4}\Big)p(i_1)p\big((k-i_1)/2\big)^2 + \Big(\frac{1+2(-1)^k\cos(k\pi/3)}{3}\Big)\frac{p(k/3)^3}{3!}. \tag{23}$$

In fact, we shall adopt this approach as the order of the coefficients continues to increase such as in $C_{k,4}$. For this coefficient, the equivalent term of the first term on the right-hand side of (23) becomes

$$C_{k,4}(k) \le \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1}^{\lfloor (k-i_1)/3 \rfloor} \sum_{i_3=i_2}^{\lfloor (k-i_1-i_2)/2 \rfloor} p(i_1)p(i_2)p(i_3)p(k-i_1-i_2-i_3). \tag{24}$$

This term covers all the 4-part partitions summing to k. It would yield the correct value of the fourth order coefficient if all the multiplicities in the partitions were equal to unity. However, since this is not the case, it only represents an upper bound for the coefficient. To determine the coefficient exactly, we need to consider all the possible cases when at least one or more of i_1, i_2 and i_3 are equal to each another.

The first case to consider is when $i_1 = i_2$. Then the multiplicity is equal to 2. Consequently, we need to remove half the general term of its contribution from the right-hand side of (24). This yields

$$C_{k,4}(k) \le \sum_{i_1=1}^{\lfloor k/4\rfloor} \sum_{i_2=i_1}^{\lfloor (k-i_1)/3\rfloor} \sum_{i_3=i_2}^{\lfloor (k-i_1-i2)/2\rfloor} p(i_1)p(i_2)p(i_3)p(k-i_1-i_2-i_3) - \frac{1}{2}\sum_{i_1=1}^{\lfloor k/4\rfloor} \sum_{i_2=i_1}^{\lfloor (k-2i_1)/2\rfloor} p(i_1)^2 p(i_2)p(k-2i_1-i_2). \tag{25}$$

The second term on the right-hand side of the above result does not allow for the cases when either $i_2 = i_1$ or $i_2 = (k-2i_1)/2$. These terms need to correct (25). Consequently, the general fourth order coefficient becomes

$$\begin{aligned} C_{k,4}(k) \approx & \sum_{i_1=1}^{\lfloor k/4\rfloor} \sum_{i_2=i_1}^{\lfloor (k-i_1)/3\rfloor} \sum_{i_3=i_2}^{\lfloor (k-i_1-i_2)/2\rfloor} p(i_1)p(i_2)p(i_3)p(k-i_1-i_2-i_3) \\ & - \frac{1}{2}\sum_{i_1=1}^{\lfloor k/4\rfloor} \sum_{i_2=i_1}^{\lfloor (k-2i_1)/2\rfloor} p(i_1)^2 p(i_2)p(k-2i_1-i_2) - \frac{1}{3}\sum_{i_1=1}^{\lfloor k/4\rfloor} p(i_1)^3 p(k-3i_1) \\ & - \frac{1}{8}\sum_{i_1=1}^{\lfloor k/4\rfloor} \sum_{i_2=i_1+1}^{\lfloor (k-i_1-1)/3\rfloor} \Big(1+(-1)^{k-2i_1}\Big) p(i_1)^2 p\Big(\frac{k-2i_1}{2}\Big)^2. \end{aligned} \tag{26}$$

In addition, we have to remove half the term when $i_2 = i_3$ and $i_2 > i_1$, the last condition being necessary to stop double-counting due to the third and fourth terms on the right-hand side of the above approximation for $C_{k,4}(k)$. Hence the generalized coefficient can be expressed as

$$\begin{aligned} C_{k,4}(k) \approx & \sum_{i_1=1}^{\lfloor k/4\rfloor} \sum_{i_2=i_1}^{\lfloor (k-i_1)/3\rfloor} \sum_{i_3=i_2}^{\lfloor (k-i_1-i2)/2\rfloor} p(i_1)p(i_2)p(i_3)p(k-i_1-i_2-i_3) \\ & - \frac{1}{2}\sum_{i_1=1}^{\lfloor k/4\rfloor} \sum_{i_2=i_1}^{\lfloor (k-2i_1)/2\rfloor} p(i_1)^2 p(i_2)p(k-2i_1-i_2) - \frac{1}{3}\sum_{i_1=1}^{\lfloor k/4\rfloor} p(i_1)^3 p(k-3i_1) \\ & - \frac{1}{8}\sum_{i_1=1}^{\lfloor k/4\rfloor} \sum_{i_2=i_1+1}^{\lfloor (k-i_1-1)/3\rfloor} \Big(1+(-1)^{k-2i_1}\Big) p(i_1)^2 p\Big(\frac{k-2i_1}{2}\Big)^2 \\ & - \frac{1}{2}\sum_{i_1=1}^{\lfloor k/4\rfloor} \sum_{i_2=i_1+1}^{\lfloor (k-i_1-1)/3\rfloor} p(i_1)p(i_2)^2 p(k-i_1-2i_2). \end{aligned} \tag{27}$$

Furthermore, we need to remove half the term where $i_3 = i_4$, which only contributes when $k - i_1 - i_2$ is an even number. Then the generalized fourth order coefficient is given by

$$\begin{aligned}
C_{k,4}(k) \approx & \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1}^{\lfloor (k-i_1)/3 \rfloor} \sum_{i_3=i_2}^{\lfloor (k-i_1-i2)/2 \rfloor} p(i_1)p(i_2)p(i_3)p(k-i_1-i_2-i_3) \\
& - \frac{1}{2} \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1}^{\lfloor (k-2i_1)/2 \rfloor} p(i_1)^2 p(i_2) p(k-2i_1-i_2) - \frac{1}{3} \sum_{i_1=1}^{\lfloor k/4 \rfloor} p(i_1)^3 p(k-3i_1) \\
& - \frac{1}{8} \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1+1}^{\lfloor (k-i_1-1)/3 \rfloor} \Big(1+(-1)^{k-2i_1}\Big) p(i_1)^2 p\Big(\frac{k-2i_1}{2}\Big)^2 \\
& - \frac{1}{2} \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1+1}^{\lfloor (k-i_1-1)/3 \rfloor} p(i_1)p(i_2)^2 p(k-i_1-2i_2) \\
& - \frac{1}{4} \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1+1}^{\lfloor (k-i_1-1)/3 \rfloor} \Big(1+(-1)^{k-i_1-i_2}\Big) p(i_1)p(i_2) p\Big(\frac{k-i_1-i_2}{2}\Big)^2. \qquad (28)
\end{aligned}$$

We also need to remove from the first term on the right-hand side the contributions from the partitions where $i_2 = i_3 = i_4$. In this case the multiplicity yields a factor of $1/3!$, which means we have to remove $5/6$ of the term from the contribution due to the first term on the right-hand side. On top of that, the partitions only arise when $k - i_1$ is divisble by 3, which means in turn that we require (21) with $\ell = 3$ and $k = k - i_1$. Therefore, the generalized fourth order coefficient becomes

$$\begin{aligned}
C_{k,4}(k) = & \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1}^{\lfloor (k-i_1)/3 \rfloor} \sum_{i_3=i_2}^{\lfloor (k-i_1-i2)/2 \rfloor} p(i_1)p(i_2)p(i_3)p(k-i_1-i_2-i_3) \\
& - \frac{1}{2} \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1}^{\lfloor (k-2i_1)/2 \rfloor} p(i_1)^2 p(i_2) p(k-2i_1-i_2) - \frac{1}{3} \sum_{i_1=1}^{\lfloor k/4 \rfloor} p(i_1)^3 p(k-3i_1) \\
& - \frac{1}{8} \sum_{i_1=1}^{\lfloor k/4 \rfloor} \Big(1+(-1)^{k-2i_1}\Big) p(i_1)^2 p\Big(\frac{k-2i_1}{2}\Big)^2 \\
& - \frac{1}{2} \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1+1}^{\lfloor (k-i_1-1)/3 \rfloor} p(i_1)p(i_2)^2 p(k-i_1-2i_2) \\
& - \frac{1}{4} \sum_{i_1=1}^{\lfloor k/4 \rfloor} \sum_{i_2=i_1+1}^{\lfloor (k-i_1-1)/3 \rfloor} \Big(1+(-1)^{k-i_1-i_2}\Big) p(i_1)p(i_2) p\Big(\frac{k-i_1-i_2}{2}\Big)^2 \\
& - \frac{5}{18} \sum_{i_1=1}^{\lfloor k/4 \rfloor} p(i_1) p\Big(\frac{k-i_1}{3}\Big)^3 \Big(1+2(-1)^{k-i_1}\cos\big(\pi(k-i_1)/3\big)\Big) \\
& + \frac{A}{4 \cdot 4!} \Big(1+(-1)^k + 2\cos(\pi k/2)\Big) p(k/4)^4. \qquad (29)
\end{aligned}$$

The final term on the right-hand side of (29) only contributes when k is a multiple of 4 corresponding to the partition, $\{(k/4)_4\}$. The $\ell = 4$ case of (21) has been used to derive this term. It appears with a non-specific value of A because we need to consider contributions in the other sums due to $\{(k/4)_4\}$. For example, the first term on the right-hand side yields a contribution from this partition when $i_1 = k/4$, $i_2 = \lfloor (k - i_1)/3 \rfloor = k/4$, $i_3 = \lfloor (k - i_1 - i_2)/3 \rfloor = k/4$ and $k - i_1 - i_2 - i_3 = k/4$. That is, the first sum contributes $p(k/4)^4$. Similarly, we need to obtain $p(k/4)^4$-contributions from the second, third, fourth, and seventh terms on the right-hand side of (29). All these contributions must be summed to yield $p(k/4)/(4{\cdot}4!)$, which is the overall contribution to $C_{k,4}(k)$ due to $\{(k/4)_4\}$. Therefore, when k is a multiple of 4, we find that

$$\Big(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{3} - \frac{5}{6} + \frac{A}{24}\Big) = \frac{1}{24}. \tag{30}$$

In other words, $A = 23$.

We can verify the above result for the fourth order coefficient from the partition method of a power series expansion by applying it specifically to the results for the generalized cosecant numbers displayed in Table 2 of [9]. To achieve this, we set

$$p(i) = \frac{\zeta(2i)}{i\pi^{2i}}. \tag{31}$$

From the table, the fourth order coefficient equals

$$996352286992030556160 \times 2/(1215 \times 30!)$$

when $k = 14$. Introducing this value into Mathematica yields

In[19]:= 996352286992030556160 × 2/(1215 × 30!)

Out[19]= 279597781685533/45219645837796916566080000000.

Next we program (29) with $A = 23$ and its various nested sums as follows:

```
Ck4[k_] := Sum[p[i1] Sum[p[i2] Sum[p[i3] p[k-i1-i2-i3], {i3, i2,Floor[(k-i1-i2)/2]}],
{i2,i1,Floor[(k-i1)/3]}], {i1,1,Floor[k/4]}]-(1/2) Sum[p[i1]^2 Sum[p[i2] p[k-2 i1-i2],
{i2,i1,Floor[(k-2 i1)/2]}], {i1,1,Floor[k/4]}]-(1/2)Sum[p[i1] Sum[p[i2]^2 p[k-2i2-i1],
{i2,i1+1,Floor[(k-i1-1)/3]}], {i1,1,Floor[k/4]}]-(1/8) Sum[p[i1]^2 (1+(-1)^(k-2i1))
p[(k-2 i1)/2]^wedge 2, {i1,1,Floor[k/4]}]-(1/3)Sum[p[i1]^3 p[(k-3i1)], {i1,1,Floor[k/4]}]
-(1/4)Sum[p[i1] Sum[p[i2] p[(k-i1-i2)/2]^2 (1 + (-1)^(k-i1-i2)), {i2,i1+1,
Floor[(k-1-i1)/3]}], {i1,1,Floor[k/4]}]+23 (1 + (-1)^k+2Cos[Pi k/2]) p[k/4]^4/96
- (5/18) Sum[p[i1] p[(k-i1)/3]^3 (1+2 (-1)^(k-i1) Cos[Pi(k-i1)/3]), {i1,1,Floor[k/4]}].
```

Then we set

```
p[k_] := Zeta[2 k]/(k Pi^(2 k)).
```

By typing in the coefficient with k set equal to 14, we arrive at

In[3]:= Ck4[14]

Out[3]= 279597781685533/452196458377969165660800000000 .

For a multiple of 4 such as $k = 12$, we find from Table 2 in [9] that the fourth order coefficient in Mathematica reduces to

In[9]:= 9332354263294766080 × 2/(2835 × 27!)

Out[9]= 6096071712541/10082418247000427328000000,

while typing in $k = 12$ in (29) yields

In[3]:= Ck4[12]

Out[3]= 6096071712541/10082418247000427328000000 .

In principle, one can calculate the fifth or higher order coefficients by adopting the same approach used to obtain (29). However, it will become increasingly unwieldy adjusting the multiplicities when one or more partitions are equal to each other. Furthermore, as the order increases, there will be ever increasing groups of partitions where the partitions are equal to one another. Consequently, there will be far more finite sums in order to adjust the equivalent of the leading term on the right-hand side of (29). Nevertheless, we have seen as a result of (12) that the coefficients of the generalized cosecant numbers can be expressed as finite sums of products of the Riemann zeta function with even integer arguments.

3. Coefficients of the Generalized Secant Numbers

Before deriving the alternative formulation of the generalized secant numbers, we need to present a table of the generalized secant numbers using (6). This is necessary in order to verify the results derived in this section. Consequently, Table 3 presents the generalized secant numbers up to and including $k = 15$. They have been calculated by implementing (6) in Mathematica just as (5) was implemented in Section 2 of [9]. The table displays vastly different coefficients from Table 2 in [9] because, instead of $1/(2i + 1)!$ appearing in the denominator of the product inside the partition operator, we now have $1/(2i)!$ for each part i. As stated previosuly, this is solely due to the fact that the generalized cosecant numbers require the power series expansion for sine as their inner power series, while inner power series for the generalized secant numbers is the power series expansion for cosine.

To obtain the alternative form for the generalized secant numbers, we begin by introducing the infinite product as given by No. 1.431.3 in [10] into $\sec^{\rho} z$, which gives

$$\sec^{\rho} z = \prod_{j=1}^{\infty} \Big(1 - \frac{4z^2}{(2j+1)^2 \pi^2} \Big)^{-\rho}. \tag{32}$$

k	$d_{\rho,k}$
0	1
1	$\frac{1}{2!}\rho$
2	$\frac{1}{4!}\left(2\rho+3\rho^2\right)$
3	$\frac{1}{6!}\left(16\rho+30\rho^2+15\rho^3\right)$
4	$\frac{1}{8!}\left(272\rho+588\rho^2+420\rho^3+105\rho^4\right)$
5	$\frac{1}{10!}\left(7936\rho+18960\rho^2+16380\rho^3+6300\rho^4+945\rho^5\right)$
6	$\frac{1}{12!}\left(353792\rho+911328\rho^2+893640\rho^3+429660\rho^4+103950\rho^5+10395\rho^6\right)$
7	$\frac{1}{14!}\left(22368256\rho+61152000\rho^2+65825760\rho^3+36636600\rho^4+11351340\rho^5\right.$ $\left.+1891890\rho^6+135135\rho^7\right)$
8	$\frac{1}{16!}\left(1903757312\rho+5464904448\rho^2+6327135360\rho^3+3918554640\rho^4\right.$ $\left.+1427025600\rho^5+310269960\rho^6+37837800\rho^7+2027025\rho^8\right)$
9	$\frac{1}{18!}\left(209865342976\rho+627708979200\rho^2+771099175680\rho^3+518915779200\rho^4\right.$ $+212564111760\rho^5+54988053120\rho^6+8876747880\rho^7+827026200\rho^8$ $\left.+34459425\rho^9\right)$
10	$\frac{1}{20!}\left(29088885112832\rho+90133968949248\rho^2+116351757473280\rho^3\right.$ $+83750172011520\rho^4+37584782071200\rho^5+11033668966320\rho^6$ $\left.+2140527589200\rho^7+267129462600\rho^8+19641872250\rho^9+654729075\rho^{10}\right)$
11	$\frac{1}{22!}\left(4947139780649991\rho+15814534765170218\rho^2+21304894383760218\rho^3\right.$ $+16224432502183290\rho^4+7839772870777155\rho^5+2538741145580640\rho^6$ $+563078876920560\rho^7+85156682410800\rho^8+8469575314200\rho^9$ $\left.+504141387750\rho^{10}+13749310575\rho^{11}\right)$
12	$\frac{1}{24!}\left(1015423886506852352\rho+3334995367266582528\rho^2+4660452027922944000\rho^3\right.$ $+3722709536929152000\rho^4+1912883605551072000\rho^5+670567768280625600\rho^6$ $+165008446061659200\rho^7+28711168921234800\rho^8+3485201433714000\rho^9$ $\left.+282924146805300\rho^{10}+13914302301900\rho^{11}+316234143225\rho^{12}\right)$
13	$\frac{1}{26!}\left((246921480190207983616\rho+831075714033875681280\rho^2\right.$ $+1199816883168896778240\rho^3+999152056361851392000\rho^4$ $+541075658301443865600\rho^5+202638575299038278400\rho^6$ $+54234696715000401600\rho^7+10522205018757432000\rho^8$ $+1477579639013478000\rho^9+147292828652970000\rho^{10}$ $+9948726145858500\rho^{11}+411104386192500\rho^{12}$ $\left.7905853580625\rho^{13}\right)$
14	$\frac{1}{28!}\left(70251601603943959887872\rho+241739105025518063321088\rho^2\right.$ $+359285730111528382955520\rho^3+310374268041902641274880\rho^4$ $+175911626797846587648000\rho^5+69699511706049902476800\rho^6$ $+20007718263237296592000\rho^7+4239318795668748792000\rho^8$ $+666689627505922020000\rho^9+77249145233642382000\rho^{10}$ $+6441347964618567000\rho^{11}+367773983887810500\rho^{12}$ $\left.+12949788165063750\rho^{13}+213458046676875\rho^{14}\right)$
15	$\frac{1}{30!}\left(23119184187809597841473536\rho+81173430481947309385973760\rho^2\right.$ $+123843697685956391723335680\rho^3+110537942778566327962828800\rho^4$ $+65212319968676584133713920\rho^5+27132631482870606005913600\rho^6$ $+8267637211077679417382400\rho^7+1885316838557296448304000\rho^8$ $+324973012921416916056000\rho^9+42332829556495137660000\rho^{10}$ $+4120281213040990998000\rho^{11}+291779693682582105000\rho^{12}$ $\left.+14270666557900252500\rho^{12}+433319834754056250\rho^{13}+6190283353629375\rho^{15}\right)$

Table 3: Generalized secant numbers $d_{\rho,k}$

As in Sec. 2, we exponentiate the product, thereby obtaining

$$\sec^\rho z = \exp\Big(-\rho \sum_{j=1}^{\infty} \ln\Big(1 - \frac{4z^2}{(2j+1)^2\pi^2}\Big)\Big). \tag{33}$$

Next we replace the logarithms in (33) by using (11). Hence we find that

$$\sec^\rho z \equiv \exp\Bigg(\rho\Big(\frac{4z^2}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2} + \frac{1}{2}\Big(\frac{4z^2}{\pi^2}\Big)^2\sum_{j=1}^{\infty}\frac{1}{(2j+1)^4} + \frac{1}{3}\Big(\frac{4z^2}{\pi^2}\Big)^3\sum_{j=1}^{\infty}\frac{1}{(2j+1)^6} + \cdots\Big)\Bigg). \tag{34}$$

By noting that

$$\sum_{j=1}^{\infty}\frac{1}{(2j+1)^{2\ell}} = \Big(1 - 2^{-2\ell}\Big)\zeta(2\ell),$$

we can express (34) as

$$\sec^\rho z \equiv \exp\Bigg(\rho\,\Big(\frac{z^2}{\pi^2}\Big(2^2-1\Big)\zeta(2) + \frac{1}{2}\Big(\frac{z^2}{\pi^2}\Big)^2\Big(2^4-1\Big)\zeta(4) + \frac{1}{3}\Big(\frac{z^2}{\pi^2}\Big)^3\Big(2^6-1\Big)\zeta(6) + \cdots\Big)\Bigg). \tag{35}$$

If we compare (12) with (35), then we see that the powers of z^{2i} are now accompanied by an extra factor of $2^{2i} - 1$, which does not affect the rationality of the coefficients when they are multiplied by $\zeta(2i)/\pi^{2i}$. In other words, in applying the partition method for a power series expansion, we find that the coefficients of the inner power series become

$$p_k = \rho\Big(2^{2k} - 1\Big)\frac{\zeta(2k)}{k\pi^{2k}}, \tag{36}$$

while the coefficients of the outer power series remain the same as before. That is, $q_k = 1/k!$. Thus, analogous to (14), we see that the coefficients of the generating function for $\sec^\rho z$ are given by

$$d_{\rho,k} = L_{P,k}\left[\prod_{i=1}^{k}\Big(\frac{\rho(2^{2i}-1)\zeta(2i)}{i\pi^{2i}}\Big)^{\lambda_i}\frac{1}{\lambda_i!}\right], \tag{37}$$

We are now in a position to determine formulas for the highest order coefficients in the same manner as the previous section. First, we note that the highest order coefficient, $D_{k,k}$, is represented by the contribution from the partition with k ones or $\{1_k\}$. Thus we arrive at

$$D_{k,k} = \frac{((2^2-1)\zeta(2))^k}{\pi^{2k}k!} = \frac{1}{2^k k!}. \tag{38}$$

If we put $k = 8$ in the above result, then we obtain a value of $1/2^8 \cdot 8! = 1/10321920$, which is identical to the value of $2027025/16!$ for the coefficient of ρ^8 for $k = 8$ in Table 3.

The next highest coefficient, $D_{k,k-1}$, is also determined by the contribution from one partition, which is $\{1_{k-2},2\}$. Then we find that

$$D_{k,k-1} = \Big(\frac{(2^2-1)\zeta(2)}{\pi^2}\Big)^{k-2}\Big(\frac{(2^4-1)\zeta(4)}{2\pi^4}\Big)\frac{1}{(k-2)!} = \frac{1}{3\cdot 2^k(k-2)!}. \qquad (39)$$

For $k = 10$, the above result yields a value of $1/123863040$ for $D_{10,9}$, which is also identical to 19641872250/20! given for the coefficient of the second highest order term when $k = 10$ in Table 3.

The remaining higher order coefficients in the table can be determined more expeditiously by using the computer code **gencoeff** listed in the appendix. For example, if we wish to calculate $D_{k,k-3}$, then we run the code simply by typing ./gencoeff 3. This produces the following output:

```
LR[k_,3]:=((k-4)! (p[1]^(k-4)/(k-4)!) (p[4]^(1)/ 1!)) +((k-4)! (p[1]^(k-5)/(k-5)!) (p[2]^(1)/ 1!)
(p[3]^(1)/ 1!)) +
((k-4)! (p[1]^(k-6)/(k-6)!) (p[2]^(3)/ 3!)).
```

As in the case of LR[k_,5], the plus sign on the second line of the above output represents the end of line in a Mathematica notebook. Next we import the result into Mathematica and replace the reference to the reduced partition operator, LR[k_,3], by Dkkminus3[k_]. If we type
FullSimplify[Dkkminus3[k]],
then we obtain
$(2^{-1-k}(-32+7k(3+5k)))/2835$.
Finally, we expand the polynomial using the Expand routine, which gives the result for $D_{k,k-3}$ in Table 4.

The lowest order coefficients of the generalized secant numbers can also be evaluated by using the general results of the previous section. There it was found that the lowest order contribution or the coefficient of the first order term, in this case, $D_{k,1}(k)$, is determined from the partition summing to k with only one part, namely, {k}. Therefore, by using (37), we arrive at

$$D_{k,1}(k) = \big(2^{2k}-1\big)\frac{\zeta(2k)}{k\pi^{2k}}. \qquad (40)$$

If we put $k = 7$ in the above result, then we obtain a value of 10922/42567525, which is indeed equal to the coefficient of ρ in the $k = 7$ term of Table 3, given by 22368256/14!.

The second order coefficient of the generalized cosecant numbers, namely, (17), was found by determining the contributions due to all the 2-part partitions, {j,k-j}, where j ranges from unity to $\lfloor k/2\rfloor$. The multiplicity of the parts is equal to unity except when k is even and $j = k/2$, in which case it equals two. Then it was necessary to subtract half the contribution when $j = k/2$ from the sum over j. We can generalize (17) to the situation with an arbitrary inner series, p_k, by expressing it as

$$C_{k,2} = \sum_{j=1}^{\lfloor k/2\rfloor} p_j p_{k-j} - \Big(\frac{1+(-1)^k}{4}\Big)p_{k/2}^2. \qquad (41)$$

j	$D_{k,k-j}$
0	$1/2^k k!$
1	$1/3 \cdot 2^k(k-2)!$
2	$(5k+1)/45 \cdot 2^{k+1}(k-3)!$
3	$(35k^2+21k-32)/2835 \cdot 2^{k+1}(k-4)!$
4	$(175k^3+210k^2-619k-222)/42525 \cdot 2^{k+3}(k-5)!$
5	$(385k^4+770k^3-3289k^2-3146k+2688)/1403325 \cdot 2^{k+3}(k-6)!$
6	$(175175k^5+525525k^4/109375-2887885k^3-5234229k^2+8135894k$ $+3833328)/5746615875 \cdot 2^{k+4}(k-7)!$
7	$(25025k^6+105105k^5-695695k^4-2051049k^3+4370678k^3+4370678k^2$ $+5583024k-3737088)/17239847625 \cdot 2^{k+4}(k-8)!$
8	$(2127125k^7+11911900k^6-91040950k^5-399559160k^4+1037881637k^3$ $+2570031916k^2-2921125572k-1618011216)/4396161144375 \cdot 2^{k+7}(k-9)!$
9	$(282907625k^8+2036934900k^7-17449742310k^6-107550162720k^5$ $+318404757489k^4+1321571095740k^3-1923654995060k^2-3032116281360k$ $+1811862153216)/15786614669450625 \cdot 2^{k+7}(k-10)!$
10	$(15559919375k^9+140039274375k^8-1315035471750k^7-10923063401250k^6$ $+35202866673975k^5+225940696223319k^4-372503784321680k^3$ $-1203296864522940k^2+1149798167701200k$ $+705854043023616)/13023957102296765625 \cdot 2^{k+8}(k-11)!$

Table 4: Highest order coefficients of the generalized secant numbers

The last term only contributes when k is even. Putting $p_k = (2^{2k}-1)\zeta(2k)/(k\pi^{2k})$ in the above equation yields the second order coefficient of the generalized secant numbers. Thus we find that

$$D_{k,2} = \sum_{j=1}^{\lfloor k/2 \rfloor} \left(1-2^{2j}-2^{2k-2j}+2^{2k}\right) \frac{\zeta(2j)\zeta(2k-2j)}{j(k-j)\pi^{2k}} - \frac{(1+(-1)^k)}{2}\left(2^k-1\right)^2 \frac{\zeta(k)^2}{k^2\pi^{2k}}. \tag{42}$$

As a check, we shall consider both an even and an odd value of k since (42) is different in both cases. For $k = 12$, (42) yields a value of

$$1475014227059/274414240180080000,$$

which agrees with the value of 3334995367266582528/24! for the coefficient of the quadratic power in the $k = 12$ result of Table 3. On the other hand, Table 3 gives a value of 831075714033875681280/26! for $D_{13,2}$, which agrees with the value obtained from (42) of 1161282876869/563529243226950000.

To determine the third order coefficient, all that is required is to introduce (36) with $\rho = 1$ into (23). Then we arrive at

$$D_{k,3} = \tag{43}$$

$$\sum_{i_1=1}^{\lfloor k/3 \rfloor} \sum_{i_2=i_1}^{\lfloor (k-i_1)/2 \rfloor} \Big(2^{2i_1}-1\Big)\Big(1-2^{2i_2}-2^{2k-2i_1-2i_2}+2^{2k-2i_1}\Big) \frac{\zeta(2i_1)\zeta(2i_2)\zeta(2k-2i_1-2i_2)}{i_1 i_2 (k-i_1-i_2)\pi^{2k}}$$
$$-\frac{1}{2}\sum_{i_1=1}^{\lfloor k/3 \rfloor}\Big(2^{2k}-2^{2k+1-2i_1}+2^{2k-4i_1}+2^{2i_1+1}-2^{4i_1}-1\Big)\frac{\zeta(2i_1)^2\zeta(2k-4i_1)}{i_1^2(k-2i_1)\pi^{2k}}$$
$$-\sum_{i_1=1}^{\lfloor k/3 \rfloor}\Big(1+(-1)^{k-i_1}\Big)\Big(2^{2k}+2^{2i_1}-2^{2k-2i_1}-2^{k+i_1+1}+2^{k+1-i_1}-1\Big)\frac{\zeta(2i_1)\zeta(k-i_1)^2}{i_1(k-i_1)^2\pi^{2k}}$$
$$+\frac{3}{2}\big(1+2(-1)^k\cos(\pi k/3)\big)\Big(2^{2k/3}-1\Big)^3\frac{\zeta(2k/3)^3}{k^3\pi^{2k}}. \tag{44}$$

For $k = 14$, this result yields a value of 1338770493025181/113607495434553120000, which agrees with the value of the third order coefficient of the $k = 14$ result in Table 3 or 35928573011152838295552 0/28!. For a multiple of 3 such as $k = 15$, where the last term now contributes, $D_{15,3}$ equals 543018637847709181/1163056734511237566 000000, which agrees with 12384369768595639172333568 0/30! for the third order coefficient of the $k = 15$ result in Table 3.

The general fourth order coefficient for any inner power series is given by (29), bearing in mind that A is equal to 23. The fourth order coefficient of the generalized secant numbers is obtained by introducing (36) with $\rho = 1$ into (29). This yields

$$D_{k,4} = \tag{45}$$

$$\sum_{i_1=1}^{\lfloor k/4 \rfloor}\sum_{i_2=i_1}^{\lfloor (k-i_1)/3 \rfloor}\sum_{i_3=i_2}^{\lfloor (k-i_1-i_2)/2 \rfloor}\prod_{j=1}^{4}\Big(2^{2i_j}-1\Big)\frac{\zeta(2i_j)}{i_j\pi^{2k}}-\frac{1}{2}\sum_{i_1=1}^{\lfloor k/4 \rfloor}\sum_{i_2=i_1}^{\lfloor (k-2i_1)/2 \rfloor}\Big(2^{2i_1}-1\Big)^2\Big(2^{2i_2}-1\Big)$$
$$\times\Big(2^{2k-4i_1-2i_2}-1\Big)\frac{\zeta(2i_1)^2\zeta(2i_2)\zeta(2k-4i_1-2i_2)}{i_1^2 i_2(k-2i_1-i_2)\pi^{2k}}-\frac{1}{2}\sum_{i_1=1}^{\lfloor k/4 \rfloor}\sum_{i_2=i_1+1}^{\lfloor (k-1-i_1)/3 \rfloor}\Big(2^{2i_1}-1\Big)$$
$$\times\Big(2^{2i_2}-1\Big)^2\Big(2^{2k-2i_1-4i_2}-1\Big)\frac{\zeta(2i_1)\zeta(2i_2)^2\zeta(2k-2i_1-4i_2)}{i_1 i_2^2(k-i_1-2i_2)\pi^{2k}}-\frac{1}{2}\sum_{i_1=1}^{\lfloor k/4 \rfloor}\Big(1+(-1)^{k-2_1}\Big)$$
$$\times\Big(2^{2i_1}-1\Big)^2\Big(2^{k-2i_1}-1\Big)^2\frac{\zeta(2i_1)^2\zeta(k-2i_1)^2}{i_1^2(k-2i_1)^2\pi^{2k}}-\frac{1}{3}\sum_{i_1=1}^{\lfloor k/4 \rfloor}\Big(2^{2i_1}-1\Big)^3\Big(2^{2k-6i_1}-1\Big)$$
$$\times\frac{\zeta(2i_1)^3\zeta(2k-6i_1)}{i_1^3(k-3i_1)\pi^{2k}}-\sum_{i_1=1}^{\lfloor k/4 \rfloor}\sum_{i_2=i_1+1}^{\lfloor (k-1-i_1)/3 \rfloor}\Big(1+(-1)^{k-i_1-i_2}\Big)\Big(2^{2i_1}-1\Big)\Big(2^{2i_2}-1\Big)$$
$$\times\Big(2^{k-i_1-i_2}-1\Big)^2\frac{\zeta(2i_1)\zeta(2i_2)\zeta(k-i_1-i_2)^2}{i_1 i_2(k-i_1-i_2)^2\pi^{2k}}-\frac{15}{2}\sum_{i_1=1}^{\lfloor k/4 \rfloor}\Big(1+2(-1)^{k-i_1}\cos\big(\pi(k-i_1)/3\big)\Big)$$
$$\times\Big(2^{2i_1}-1\Big)\Big(2^{(2k-2i_1)/3}-1\Big)^3\frac{\zeta(2i_1)\zeta((2k-2i_1)/3)^3}{i_1(k-i_1)^3\pi^{2k}}+\frac{184}{3}\Big(1+(-1)^k+2\cos(k\pi/2)\Big)$$
$$\times\Big(2^{k/2}-1\Big)^4\frac{\zeta(k/2)^4}{k^4\pi^{2k}}, \tag{46}$$

where $i_4 = k - i_1 - i_2 - i_3$. For $k = 14$, the above result when programmed into Mathematica yields a value of 402266644283419/3951565058593152000000, which agrees with the value of 310374268041902641274880/28! for the fourth order coefficient of the penultimate result in Table 3. For the case of $k = 12$, where the last term in (45) contributes, $D_{k,4}$ equals 32012033/5335311421440, which also agrees with the fourth order coefficient of the corresponding result in Table 3 or 3722709536929152000/24!.

4. Conclusion

This paper has presented alternative formulations of the generalized cosecant and secant numbers obtained via the partition method for a power series expansion. These formulations indicate that the coefficients of both sets of numbers can be expressed as finite sums of products of the Riemann zeta function with even integer arguments. Consequently, both sets of coefficients are rational. In addition, the new formulations are far more expedient computationally than the previous formulations discussed in [7], [8] and [9]. With the aid of a special program in the appendix based on the bivariate recursive central partition or brcp algorithm [8], [12], the new formulations are able to evaluate the highest order coefficients of both sets of numbers in terms of the order k in their generating functions more efficiently. Therefore, formulas for the highest order coefficients ranging from k down to $k-10$ are presented in Tables 1 and 4. Moreover, the new formulations are able to provide expressions for the lowest order coefficients of both sets of numbers, which could not be determined previously. Nevertheless, although the new formulations are able to give the values of the coefficients very quickly when programmed in Mathematica, the forms become cumbersome and onerous as the order increases. Therefore, it would be of great benefit to derive more compact results such as those obtained for the highest order coefficients, but at present these remain elusive.

References

[1] V. Kowalenko, Towards a theory of divergent series and its importance to asymptotics, in *Recent Research Developments in Physics*, Vol. 2, Transworld Research Network, Trivandrum, India, 2001, 17-68.

[2] V. Kowalenko, Exactification of the asymptotics for Bessel and Hankel functions, *Appl. Math. and Comp.* **133** (2002), 487-518.

[3] V. Kowalenko, *The Stokes Phenomenon, Borel Summation and Mellin-Barnes Regularisation*, Bentham ebooks, http://www.bentham.org, 2009.

[4] V. Kowalenko, Properties and applications of the reciprocal logarithm numbers, *Acta Appl. Math.* **109** (2010), 413-437.

[5] V. Kowalenko, Generalizing the reciprocal logarithm numbers by adapting the partition method for a power series expansion, *Acta Appl. Math.* **106** (2009), No.3, 369-420.

[6] V. Kowalenko, Euler and divergent series, *Eur. J. Pure Appl. Math.* **4** (2011), 370-423.

[7] V. Kowalenko, Applications of the cosecant and related numbers, *Acta Appl. Math.* **114** (2011), No. 1-2, 15-134.

[8] V. Kowalenko, *The Partition Method for a Power Series Expansion: Theory and Applications*, Academic Press/Elsevier, Oxford, 2017.

[9] V. Kowalenko, Generalized cosecant numbers and the Hurwitz zeta function, *Integers* **19** (2019), A52.

[10] I.S. Gradshteyn, I.M. Ryzhik, A. Jeffrey (ed.), *Table of Integrals, Series, and Products*, Fifth Ed., Academic Press, Boston, 1994.

[11] V. Kowalenko, *The Stokes Phenomenon, Borel Summation and Mellin-Barnes Regularisation*, Bentham ebooks, http://www.bentham.org, 2009.

[12] V. Kowalenko, Developments from programming the partition method for a power series expansion, arXiv:1203.4967v1, (2012).

Appendix

Program 1 GNCP.cpp

```
/* This program determines all the partitions with k-l parts
   summing to k. In this code l is the sole input variable,
   which is set equal to the variable {\it tot}.*/

#include <stdio.h>
#include <memory.h>
#include <stdlib.h>

/* Global variables */

int tot,numparts,j,*part;
/* numparts is the number of parts excluding ones  */
long unsigned int term=1;

void termgen()
{
int freq,i,sumparts=0;
/* sumparts is the number of parts in a partition
   excluding ones */
for(i=0;i<tot;i++){
      sumparts= sumparts+part[i];
                  }
if(sumparts != numparts) goto end;
else
    printf("%ld: (k-%i)(1) ",term++,tot+j+1);
```

```
    for(i=0;i<tot;i++){
                freq=part[i];
                if(freq) printf("%i(%i) ",freq,i+2);
                      }
printf("\n");
end:        ;
}

void brcp(int p,int q)
{
part[p-1]++;
termgen();
part[p-1]--;
p -= q;
while(p >= q){
        part[q-1]++;
        brcp(p--,q);
        part[q++ -1]--;
             }
}

int main(int argc,char *argv[])
{
int i;
if(argc != 2) printf( "usage: numparts <#partitions> \n" );
else{
     tot=atoi(argv[1]);
     part=(int *) malloc(tot*sizeof(int));
     for (j=0;j<tot;j++){
        numparts=j+1;
        if(part == NULL) printf("unable to allocate array\n\n");
        else{
             for(i=0;i<tot;i++) part[i]=0;
             brcp(tot,1);
            }
        }
        free(part);
     }
printf("\n");
return(0);
}
```

Program 2 gencoeff.cpp

```
/* This program generates symbolic forms for the
generalized cosecant and secant numbers that can be
evaluated in Mathematica simply by changing the
coefficients of the inner power series, p[k].
It achieves this by summing the contributions
from the partitions generated by Program 1. */


#include <stdio.h>
#include <memory.h>
#include <stdlib.h>
#include <time.h>

int tot,*part;
long unsigned int termcnt=1;


void termgen(int p)
{
int f=0,i,num_parts=0,k;
/* num_parts is the total # of parts in the partition. */

if(p==tot) printf("LR[k_,%i]:=",p--);
else {
        if (termcnt>=1) printf("+");
     if (termcnt % 2 == 0) printf("\n");
     termcnt++;
     }
for (i=1;i<=tot; i++){
        f=part[i];
        if(f>0){
                num_parts += k = f;
                }
        }
f=0;
if( num_parts >= k ){
       printf("((k-%i)! (p[1]^(k-%i)/(k-%i)!)",tot+1,\
        tot+num_parts, tot+num_parts);
       for (i=1; i<=tot; i++){
              f=part[i];
              if(f>0){
                      printf(" (p[%i]^(%i)/ %i!)", i+1,f,f);
                                    }
                         }
                }
        printf(") ");
```

```
}
void brcp(int p1, int q)
{
part[p1]++;
termgen(p1);
part[p1]--;
p1 -= q;
while(p1 >= q){
        part[q]++;
        brcp(p1--,q);
        part[q++]--;
        }
}

int main( int argc, char *argv[] )
{
int i;
if(argc != 2)
  printf("execution: gen cos nos <sum of partitions>\n");
else{
        tot=atoi(argv[1]);
/* tot is the sum of the partitions required by the brcp
 function subprogram  */
       part=(int *) malloc((tot+1) *sizeof(int));
       if(part == NULL) printf("unable to allocate array\n\n");
       else{
                for (i=0;i<tot;i++) part[i]=0;
                brcp(tot,1);
                free(part);
             }
        }
printf("\n");
return(0);
}
```