INVARIANT MEASURES FOR 2-DIMENSIONAL MAPS AND ASSOCIATED RÉNYI MAPS

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Abstract

This paper considers 2-dimensional fractional linear maps and their invariant measures. In the first part maps are considered for which several invariant densities can be constructed. The second part deals with maps which have probably only one invariant density. One might conjecture that these maps are ergodic. However, a proof of the convergence of the whole family of these algorithms seems to be difficult. A new class of maps called associated Rényi maps is introduced. This map can be used to investigate the Rényi condition of the given map.

0. Introduction

Let $B = \{(x, y) : 0 \leq y \leq x \leq 1\}$. We work with a partition in two cylinders, namely

$$B(1) = \{(x, y) \in B : x + y < 1\} \text{ and } B(2) = \{(x, y) \in B : 1 \leq x + y\}.$$

We consider piecewise fractional linear maps $T : B \to B$ with the property $TB(1) = TB(2) = B$ and introduce the six matrices

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly these matrices form the symmetry group $\mathfrak{S}_3$ of the triangle B. As usual we identify a matrix with its associated fractional linear map. This means that the matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

is associated with the map

$$A(x,y) = (\frac{a_{10}+a_{11}x+a_{12}y}{a_{00}+a_{01}x+a_{02}y}, \frac{a_{20}+a_{21}x+a_{22}y}{a_{00}+a_{01}x+a_{02}y}).$$

We introduce the inverse branches of the map T (associated with $G_1 = G_2 = E$) as follows:

$$V_1(\beta,\gamma) = V(\beta,\gamma) = \begin{pmatrix} 2 & 2\beta & 2\gamma \\ 0 & 2+2\beta & -1-\beta+\gamma \\ 0 & 0 & 1+\beta+\gamma \end{pmatrix}$$

$$V_2(\mu,\nu) = V(\mu,\nu) = \begin{pmatrix} 2 & \mu & \nu \\ 1 & 1+\mu & \nu \\ 1 & -1 & 2+\mu+\nu \end{pmatrix}.$$

If we do not choose numerical values for the parameters, we suppress the lower index and we write $V(\beta,\gamma)$ for $V_1(\beta,\gamma)$ and $V(\mu,\nu)$ for $V_2(\mu,\nu)$. Then we find $V(\beta,\gamma)B = B(1)$ and $V(\mu,\nu)B = B(2)$. Note that $V(\beta,\gamma)$ has the fixed points $(0,0)$ and $(1,0)$ and maps the point $(1,1)$ onto the point $(\frac{1}{2},\frac{1}{2})$. The branch $V(\mu,\nu)$ has the fixed points $(1,0)$ and $(1,1)$ and maps $(0,0)$ onto $(\frac{1}{2},\frac{1}{2})$. If we replace the matrices $V(\beta,\gamma)$ and $V(\mu,\nu)$ by $V(\beta,\gamma)\circ G_1$ and $V(\mu,\nu)\circ G_2$ where G_1 and G_2 belong to $\mathfrak{S}_3$ we can obtain all 36 possible variations of the map T. The so-called 2-dimensional Mönkemeyer algorithm (see [2]) is the map with the pair $V(\beta,\gamma)\circ S_1, \beta = 0, \gamma = 1$ and $V(\mu,\nu)\circ D_2, \mu = -1, \nu = 0$.

Let T be a given map defined by $V(\beta,\gamma)\circ G_1$ and $V(\mu,\nu)\circ G_2$ where G_1 and G_2 belong to $\mathfrak{S}_3$. Then we introduce digits ϵ by the following definition.

$$\epsilon_1(x,y) = 1 \text{ if } (x,y) \in B(1) = V(\beta,\gamma)\circ G_1 B$$

$$\epsilon_1(x,y) = 2 \text{ if } (x,y) \in B(2) = V(\mu,\nu)\circ G_2 B$$

$$\epsilon_{s+1} = \epsilon_1(T^s(x,y)).$$

A *cylinder of rank s* is the set

$$B(\epsilon_1,...,\epsilon_s) = \{(x,y) \in B : T^j(x,y) \in B(\epsilon_{j+1}),\, 0 \le j < s\}.$$

We write

$$V(\epsilon_1,...,\epsilon_s)(x,y) = (\frac{A_1^{(s)}+B_1^{(s)}x+C_1^{(s)}y}{A_0^{(s)}+B_0^{(s)}x+C_0^{(s)}y}, \frac{A_2^{(s)}+B_2^{(s)}x+C_2^{(s)}y}{A_0^{(s)}+B_0^{(s)}x+C_0^{(s)}y}),$$

where the coefficients $A_j^{(s)}$, $B_j^{(s)}$ and $C_j^{(s)}$ depend on the block $(\epsilon_1,...,\epsilon_s)$.

A good device to find an invariant measure is to look for a symmetric matrix M with the property $M\circ T = T^*\circ M$. The map T^* belongs to a fibred system with piecewise fractional linear maps, but its branches are defined by the adjoined matrices $V(\beta,\gamma)^*$ and $V(\mu,\nu)^*$. If $B^{\#}$ is the set of definition of T^*, then

$$h(x,y) = \int_{B^{\#}} \frac{dudv}{(1+xu+yv)^3}$$

is an invariant density. If a symmetric matrix M with the property $M \circ T = T^* \circ M$ exists, then $B^{\#} = MB$ and we call T^* a *natural dual*. For the background of these ideas we refer to [6] and [4].

1. Several Invariant Densities

If we choose $G_1, G_2 \in \{E, S_2\}$ then these maps have the following property: all cylinders $B(\epsilon_1, ..., \epsilon_n)$ are triangles with the common vertex $(1,0)$. The other two vertices lie on the segment which joins the point $(0,0)$ with $(1,1)$. Therefore these maps are probably not ergodic (or conservative). We will discuss some maps with several invariant densities.

It turns out that it is more appropriate to use the map $\Phi(x,y) = \big(\frac{y}{1-x+y}, \frac{x-y}{1-x+y}\big)$ and study the conjugate map $S = \Phi \circ T \circ \Phi^{-1}$ on the stripe

$$\Phi B = A = [0,1] \times [0,\infty[.$$

We note that the inverse map is given as $\Phi^{-1}(x,y) = \big(\frac{x+y}{1+y}, \frac{x}{1+y}\big)$. Then the inverse branches are

$$W(\beta,\gamma)(x,y) = \Big(\frac{(1+\beta+\gamma)x}{2+(2\beta+2\gamma)x}, \frac{(2+2\beta)y}{2+(2\beta+2\gamma)x}\Big)$$

$$W(\mu,\nu)(x,y) = \Big(\frac{1+(1+\mu+\nu)x}{2+(\mu+\nu)x}, \frac{(2+\mu)y}{2+(\mu+\nu)x}\Big).$$

We note

$$\det W(\beta,\gamma) = 4(1+\beta)(1+\beta+\gamma) \text{ and } \det W(\mu,\nu) = (2+\mu)(2+\mu+\nu).$$

The map S_2 will be replaced by

$$R_2 = \Phi \circ S_2 \circ \Phi^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$W(\beta,\gamma) \circ R_2(x,y) = \Big(\frac{1+\beta+\gamma-(1+\beta+\gamma)x}{2+2\beta+2\gamma-(2\beta+2\gamma)x}, \frac{(2+2\beta)y}{2+2\beta+2\gamma-(2\beta+2\gamma)x}\Big)$$

$$W(\mu,\nu) \circ R_2(x,y) = \Big(\frac{2+\mu+\nu-(1+\mu+\nu)x}{2+\mu+\nu-(\mu+\nu)x}, \frac{(2+\mu)y}{2+\mu+\nu-(\mu+\nu)x}\Big).$$

There are four cases, but we will discuss only (a) and (b).

- (a) $W(\beta,\gamma), W(\mu,\nu)$
- (b) $W(\beta,\gamma)\circ R_2, W(\mu,\nu)$
- (c) $W(\beta,\gamma), W(\mu,\nu)\circ R_2$
- (d) $W(\beta,\gamma)\circ R_2, W(\mu,\nu)\circ R_2$.

1.1. Discussion of Case (a)

Since A lies in the upper half plane we have to assume $1+\beta>0$ and $2+\mu>0$. Since the two branches of the restriction of S to the interval $[0,1]$ are increasing we further suppose $1+\beta+\gamma>0$ and $2+\mu+\nu>0$. Furthermore the fixed points $\xi=(0,0)$ and $\eta=(1,0)$ should not be attractive. Hence $\beta+\gamma\le 1$ and $0\le 1+\mu+\nu$.

Theorem 1. *(1) If $\beta+\gamma=\mu+\nu=0$ then*

$$h(x,y)=\frac{1}{y}$$

is an invariant density.
(2) If $\beta+\gamma=\mu+\nu=0$ and $2\beta+\mu=0$ then

$$h(x,y)=1$$

is an invariant density.
(3) If

$$\frac{1+\beta+\gamma}{2(1+\beta)^2}+\frac{2+\mu+\nu}{(2+\mu)^2}=1,$$

then $h(x,y)=\frac{1}{y^3}$ is an invariant density.

Proof. The Kuzmin equation on A (with respect to Lebesgue measure) is given by

$$\phi(x,y)=\phi\Big(\frac{(1+\beta+\gamma)x}{2+(2\beta+2\gamma)x},\frac{(2+2\beta)y}{2+(2\beta+2\gamma)x}\Big)\frac{4(1+\beta)(1+\beta+\gamma)}{(2+(2\beta+2\gamma)x)^3}$$
$$+\phi\Big(\frac{1+(1+\mu+\nu)x}{2+(\mu+\nu)x},\frac{(2+\mu)y}{2+(\mu+\nu)x}\Big)\frac{(2+\mu)(2+\mu+\nu)}{(2+(\mu+\nu)x)^3}.$$

If $\phi(x,y)=\psi(y)$ then this equation reduces to

$$\psi(y)=\psi\Big(\frac{(2+2\beta)y}{2+(2\beta+2\gamma)x}\Big)\frac{4(1+\beta)(1+\beta+\gamma)}{(2+(2\beta+2\gamma)x)^3}$$
$$+\psi\Big(\frac{(2+\mu)y}{2+(\mu+\nu)x}\Big)\frac{(2+\mu)(2+\mu+\nu)}{(2+(\mu+\nu)x)^3},$$

and the claim of the theorem is easily verified. □

To find an invariant measure we look for a symmetric matrix

$$N = \begin{pmatrix} a & b & e \\ b & d & f \\ e & f & g \end{pmatrix}$$

with the property $N \circ S = S^* \circ N$. For a matrix L we denote the adjoined matrix as L^*. The map S^* of the *dual algorithm* is defined on the set $NA = A^\#$. The vertices of A have the projective coordinates $[1, 0, 0]$, $[1, 1, 0]$, and $[0, 0, 1]$. Therefore the set $A^\#$ has the projective coordinates $[a, b, e]$, $[a+b, b+d, e+f]$, and $[e, f, g]$.

Remark 1. Let $\Phi : B \to A$ and $S = \Phi \circ T \circ \Phi^{-1}$. If symmetric matrices M and N satisfy $M \circ T = T^* \circ M$ and $N \circ S = S^* \circ N$ then $M = \Phi^* \circ N \circ \Phi$. The map Φ induces a map $\Phi^\# : B^\# \to A^\#$ which satisfies $\Phi^\# \circ M = N \circ \Phi$.

The theory of dual algorithms (see [6], [5]) shows that

$$h(x, y) = \frac{1}{(a + bx + ey)(a + b + (b + d)x + (e + f)y)(e + fx + gy)}$$

is an invariant density for S.

The equation $N \circ W(\beta, \gamma) = W(\beta, \gamma)^* \circ N$ leads to the following six equations.

$$a(2\beta + 2\gamma) = b(1 - \beta - \gamma)$$

$$\beta e = 0$$

$$e(2\beta + 2\gamma) = f(1 + \beta - \gamma)$$

$$a(\mu + \nu) + b(-1 + \mu + \nu) = d$$

$$e\mu = f$$

$$e(\mu + \nu) = f(1 - \nu).$$

Since g does not appear in these equations we can always find non-trivial solutions! Using the first and the fourth equation we choose $a = 1 - \beta - \gamma$, $b = 2\beta + 2\gamma$, and then $d = (\mu + \nu)(1 + \beta + \gamma) - 2\beta - 2\gamma$.

If $\beta \neq 0$ then $e = 0$ and $f = 0$.

If $f = 0$ and $e \neq 0$ then $\mu = \nu = 0$.

If $e \neq 0$ and $f \neq 0$ then $\beta = 0$. Then we additionally get

$$2\gamma = \mu(1 - \gamma)$$

and

$$\nu + \nu\mu = 0.$$

If $\nu \neq 0$ then $\mu = -1$ and $\gamma = -\frac{1}{3}$. Otherwise we have $\nu = 0$.

Theorem 2. *(1) If we put $e = f = 0$, then we find that*

$$h(x,y) = \frac{1}{(1-\beta-\gamma+(2\beta+2\gamma)x)(1+(\mu+\nu)x)y}.$$

is an invariant density. (2) Suppose $\beta = \nu = 0$. If we assume $g = \mu^2$, we choose $a = 1$ and $e = \mu$. Then $b = \mu$, $d = \mu^2$, $f = \mu^2$ and

$$h(x,y) = \frac{1}{(1+\mu x+\mu y)^3}$$

is an invariant density.

Proof. The assertion (1) follows immediately from the general expression of an invariant density as given before. In case (2) note that $\beta = \nu = 0$ implies $2\gamma = \mu - \mu\gamma$. □

These and further considerations allow us to write down explicit expressions of invariant densities.

Example 1. If we take $\beta = 0$, $\gamma = \frac{1}{2}$, $\mu = 2$, and $\nu = 0$ then we find at least four different invariant densities,

$$h(x,y) = \frac{1}{(1+2x)^2 y},\ h(x,y) = \frac{1}{(1+2x)^3},$$

$$h(x,y) = \frac{1}{(1+2x+2y)^2(1+2x+y)},\ h(x,y) = \frac{1}{(1+2x+2y)^3}.$$

As noted before, the Kuzmin equation on A with respect to Lebesgue measure is given by

$$\phi(x,y) = \phi\Big(\frac{(1+\beta+\gamma)x}{2+(2\beta+2\gamma)x}, \frac{(2+2\beta)y}{2+(2\beta+2\gamma)x}\Big)\frac{4(1+\beta)(1+\beta+\gamma)}{(2+(2\beta+2\gamma)x)^3}$$

$$+\phi\Big(\frac{1+(1+\mu+\nu)x}{2+(\mu+\nu)x}, \frac{(2+\mu)y}{2+(\mu+\nu)x}\Big)\frac{(2+\mu)(2+\mu+\nu)}{(2+(\mu+\nu)x)^3}.$$

Therefore we conjecture the existence of an invariant density $h(x,y)$ which is dependent on the variable x only! For some values of the parameters it is easy to write down such densities.

Theorem 3. *Suppose that*

$$\beta+\gamma = \frac{\mu+\nu}{2+\mu+\nu}$$

and

$$\frac{2+\mu}{(2+\mu+\nu)^2} + \frac{(1+\beta)(1+\beta+\gamma)}{2} = 1,$$

then

$$h(x,y) = \frac{1}{(1+(\mu+\nu)x)^3}$$

is an invariant density.

Example 2. For $\beta = \frac{1}{8}$, $\gamma = \frac{5}{24}$, $\mu = \frac{1}{4}$, and $\nu = \frac{3}{4}$ we find two densities:

$$h(x,y) = \frac{1}{(1+x)^3},\ h(x,y) = \frac{1}{(1+x)^2 y}.$$

Remark 2. For some values of the parameters no invariant density $h(x,y) = g(x)$ which is bounded on the interval $[0,1]$ exists. Suppose

- there is a constant G such that $0 \le g(x) \le G$, $0 \le x \le 1$
- $\beta + \gamma \ge 0$ and $\mu + \nu > 0$ or $\beta + \gamma > 0$ and $\mu + \nu \ge 0$
- the inequality

$$\frac{(1+\beta)(1+\beta+\gamma)}{2} + \frac{(2+\mu)(2+\mu+\nu)}{8} \le 1$$

holds; then the Kuzmin equation gives for $0 < x$ the inequality

$$\psi(x) = \psi\Big(\frac{(1+\beta+\gamma)x}{2+(2\beta+2\gamma)x}\Big)\frac{4(1+\beta)(1+\beta+\gamma)}{(2+(2\beta+2\gamma)x)^3}$$

$$+\psi\Big(\frac{1+(1+\mu+\nu)x}{2+(\mu+\nu)x}\Big)\frac{(2+\mu)(2+\mu+\nu)}{(2+(\mu+\nu)x)^3} < G.$$

Note that $\psi(0) = \psi(\frac{1}{2}) = G$ is therefore also excluded.

Example 3. These conditions are satisfied by $\beta = \gamma = 0$, $\mu = -1$, and $\nu = 3$. The dual algorithm gives the density $h(x,y) = \frac{1}{(1+2x)y}$.

1.2. Discussion of Case (b)

We use again the standard device to find an invariant measure and look for a symmetric matrix

$$N = \begin{pmatrix} a & b & e \\ b & d & f \\ e & f & g \end{pmatrix}$$

with the property $N \circ S = S^* \circ N$. The six equations are given as follows.

$$a(-2\beta - 2\gamma) + b(-3 - 3\beta - 3\gamma) = d(1+\beta+\gamma)$$

$$e(2\gamma)+f(1+\beta+\gamma)=0$$
$$e(2\beta+2\gamma)+f(3+3\beta+\gamma)=0$$
$$a(\mu+\nu)+b(-1+\mu+\nu)=d$$
$$e\mu=f$$
$$e(\mu+\nu)=f(1-\nu).$$

The first and the fourth equation together imply

$$-a\frac{2\beta+2\gamma}{1+\beta+\gamma}-3b=d$$

and

$$a(\mu+\nu)+b(-1+\mu+\nu)=d.$$

Hence we get

$$a=2+\mu+\nu,\ b=-\frac{2\beta+2\gamma}{1+\beta+\gamma}-\mu-\nu,\ b+d=\frac{2(\mu+\nu)}{1+\beta+\gamma}.$$

If $e=0$ then $f=0$.

If $f=0$ and $\mu\neq 0$ or $\gamma\neq 0$ then $e=0$. If $e\neq 0$ and $f\neq 0$ we obtain the relations

$$\beta^2+\beta=\beta\gamma+2\gamma$$

and

$$\nu+\mu\nu=0.$$

The second and the fifth equation then show

$$\mu=-\frac{2\gamma}{1+\beta+\gamma}.$$

If $\mu=-1$ then we obtain $\gamma=1+\beta$. This eventually leads to $\beta=-1$ which is not an allowed value. Hence we obtain $\nu=0$. Now we get

$$a=2+\mu,\ b=-\frac{2\beta^2+3\beta}{\beta^2+2\beta+1}-\mu,\ b+d=\frac{\mu(\beta+2)}{\beta^2+2\beta+1}.$$

With the help of

$$h(x,y)=\frac{1}{(a+bx+ey)(a+b+(b+d)x+(e+f)y)(e+fx+gy)}$$

we can now write down several invariant densities.

It is possible to generalize Theorem 3. The Kuzmin equation reads

$$f\Big(\frac{1+\beta+\gamma-(1+\beta+2\gamma)x}{2+2\beta+2\gamma-(2\beta+2\gamma)x},\frac{(2+2\beta)y}{2+2\beta+2\gamma-(2\beta+\gamma)x}\Big)\frac{4(1+\beta)(1+\beta+\gamma)}{(2+2\beta+2\gamma-(2\beta+2\gamma)x)^3}$$
$$+f\Big(\frac{1+(1+\mu+\nu)x}{2+(\mu+\nu)x},\frac{(2+\mu)y}{2+(\mu+\nu)x}\Big)\frac{(2+\mu)(2+\mu+\nu)}{(2+(\mu+\nu)x)^3}=f(x,y).$$

Theorem 4. *If the conditions*

$$1+\beta+\gamma=\frac{2}{(1+\mu+\nu)(2+\mu+\nu)}$$

and

$$\frac{2+\mu}{(2+\mu+\nu)^2}+\frac{(1+\beta)(1+\mu+\nu)^2}{2+\mu+\nu}=1$$

hold, then

$$h(x,y)\frac{1}{(1+(\mu+\nu)x)^3}$$

is an invariant density.

Since the proof is an easy calculation, we omit it.

Example 4. Take $\beta=-\frac{29}{45}$, $\gamma=\frac{8}{45}$, $\mu=\nu=\frac{1}{4}$, then the map with branches

$$W(\beta,\gamma)\circ R_2(x,y)=\Big(\frac{12-12x}{24+21x},\frac{32y}{24+21x}\Big)$$

$$W(\mu,\nu)(x,y)=\Big(\frac{4+6x}{8+2x},\frac{9y}{8+2x}\Big)$$

has the density

$$h(x,y)=\frac{1}{(2+x)^3}.$$

2. Invariant Measures for Some Other Cases

The rotation D_1 will be replaced by the map

$$C_1=\Phi\circ D_1\circ\Phi^{-1}=\begin{pmatrix}0&1&1\\0&0&1\\1&-1&0\end{pmatrix}.$$

Theorem 5. *For $W(\beta,\gamma)\circ C_1$ and $W(\mu,\nu)$ a natural dual exists if*

$$\nu+\nu\mu=0.$$

Proof. To find a natural dual we try to find a matrix

$$N=\begin{pmatrix}a&b&e\\b&d&f\\e&f&g\end{pmatrix}.$$

The resulting system of linear equations has the deteminant

$$\Delta=2(\nu+\nu\mu)(1+\beta)(1+\beta+\gamma)(2+\mu+\nu).$$

Therefore a non-trivial solution exists only if $\nu+\nu\mu=0$. □

Example 5. Let $\beta = \gamma = \mu = \nu = 0$. Then we find

$$N = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then we obtain $NA = \{(0,1)\}$. If we replace Lebesgue measure by the Dirac measure concentrated in the point $(0,1)$ we get

$$h(x,y) = \frac{1}{(1+y)^3}.$$

Example 6. Let $\beta = \gamma = \nu = 0$ and $\mu = 1$. Then we have

$$N = \begin{pmatrix} 6 & -4 & 3 \\ -4 & 6 & 3 \\ 3 & 3 & 4 \end{pmatrix}$$

and

$$h(x,y) = \frac{1}{(6-4x+3y)(1+x+3y)(3+3x+4y)}.$$

Theorem 6. *For $W(\beta,\gamma) \circ C_1$ and $W(\mu,\nu) \circ C_1$ a natural dual exists if*

$$(i)\ \frac{2}{1+\beta+\gamma} = \frac{1}{2+\mu+\nu}$$

or

$$(ii)\ (2+\mu)^2(1+\beta+\gamma) - (2+\mu+\nu)(1+\beta)(2+2\mu-2\beta) = 0.$$

Note that (i) and (ii) cannot occur together. If (i) and (ii) are both satisfied for some values β and γ we obtain $(2+\mu)^2 = (1+\beta)(1+\mu+\beta)$. This leads to the quadratic equation

$$\phi(\mu) = \mu^2 + (3-\beta)\mu + 3 + \beta^2 = 0.$$

The value $\mu = \frac{\beta-3}{2}$ is the minimum of $\phi(\mu)$. We get $\phi'(\frac{\beta-3}{2}) = \frac{3(\beta+1)^2}{4} = 0$, but $\beta = -1$ is not an admissible parameter.

Proof. To find a natural dual we try to find a matrix

$$N = \begin{pmatrix} a & b & e \\ b & d & f \\ e & f & g \end{pmatrix}.$$

The resulting system of linear equations has the deteminant

$$\Delta = 2(3+2\mu+2\nu-\beta-\gamma)((2+\mu)^2(1+\beta+\gamma) - (2+\mu+\nu)(1+\beta)(2+2\mu-2\beta)).$$

If $a = 0$ we find case (i) and if $a \neq 0$ we find case (ii). □

Example 7. This example is case (i). If we put $J = \frac{2}{1+\beta+\gamma} = \frac{1}{2+\mu+\nu}$, then for case (i) we always find

$$N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & J & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Therefore, for all admissible values of β, γ, μ, and ν, we have the invariant density

$$h(x,y) = \frac{1}{x(1-x)y}.$$

Example 8. Here we consider case (ii). Let $\beta = \frac{1}{2}$, $\gamma = -\frac{3}{4}$, $\mu = \nu = 0$. Then we have

$$N = \begin{pmatrix} 39 & -26 & 12 \\ -26 & 32 & 14 \\ 12 & 14 & 13 \end{pmatrix}$$

and

$$h(x,y) = \frac{1}{(39-26x+12y)(13+6x+26y)(12+14x+13y)}.$$

We can also replace the reflection S_1 by the map

$$R_1 = \Phi \circ S_1 \circ \Phi^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and state the following theorem (as a further example).

Theorem 7. *For $W(\beta,\gamma)$ and $W(\mu,\nu) \circ R_1$ a natural dual exists if*

$$\beta(1+\beta-\gamma) = 0.$$

Proof. We note that

$$W(\mu,\nu) \circ R_1(x,y) = \left(\frac{1-x+(2+\mu+\nu)y}{2-2x+(2+\mu+\nu)y}, \frac{(2+\mu)x}{2-2x+(2+\mu+\nu)y}\right).$$

The use of a symmetric matrix N leads to the equations

$$a(2\beta+2\gamma) = b(1-\beta-\gamma)$$

$$\beta e = 0$$

$$e(2\beta+2\gamma) = f(1+\beta-\gamma)$$

$$-2a-3b+e(2+\mu) = d$$

$$(a+b)(2+\mu+\nu) = 2e+f$$

$$(b+d)(2+\mu+\nu) = -2e-f+g(2+\mu).$$

If $\beta(1+\beta-\gamma) = 0$, then we can find a non-trivial solution. □

Example 9. Let $\beta = \gamma = 0$, $\mu = \frac{1}{2}$, and $\nu = -\frac{1}{2}$. Then we find a natural dual and the invariant density

$$h(x,y) = \frac{1}{(1+y)(2+x+2y)(5+6y)}.$$

Remark 3. In the introduction the Mönkemeyer algorithm was mentioned. It is conjugate via the map $\Psi(x,y) = \left(\frac{1}{1+x-y}, \frac{x}{1+x-y}\right)$ to the Selmer algorithm restricted to its absorbing set. It is the algorithm with the pair $V(\beta,\gamma)S_1, \beta = 0, \gamma = 1$ and $V(\mu,\nu)D_2, \mu = -1, \nu = 0$ (see [2], [5]). Its invariant density can be calculated in the same way and is given as

$$h(x,y) = \frac{1}{x(1+x-y)}.$$

3. Associated Rényi Maps

The present study brought the idea of considering a new class of maps which we call *associated Rényi maps*. For illustration we first consider one-dimensional maps T on $[0,1]$. The partition is $0 < \frac{1}{2} < 1$ and there are two branches,

$$V(\alpha)(x) = \frac{A_1^{(1)}(\alpha) + B_1^{(1)}(\alpha)x}{A_0^{(1)}(\alpha) + B_0^{(1)}(\alpha)x},\ V(\beta)(x) = \frac{A_1^{(1]}(\beta) + B_1^{(1)}(\beta)x}{A_0^{(1)}(\beta) + B_0^{(1)}(\beta)x}.$$

The matrices $V(\alpha)$ and $V(\beta)$ provide us with recursion relations for the pair $(A_0^{(s)}, A_0^{(s)} + B_0^{(s)})$. Then we can deduce the relation

$$\frac{A_0^{(s+1)}}{A_0^{(s+1)} + B_0^{(s+1)}} = \frac{(A_0^{(s)} - A_1^{(s)})\frac{A_0^{(s)}}{A_0^{(s)}+B_0^{(s)}} + A_1^{(s)}}{(A_0^{(s)} - A_1^{(s)} + B_0^{(s)} - B_1^{(s)})\frac{A_0^{(s)}}{A_0^{(s)}+B_0^{(s)}} + A_1^{(s)} + B_1^{(s)}}.$$

This leads to new maps with inverse branches $G(\alpha)$ and $G(\beta)$ which control the growth of the fractions $\frac{A_0^{(s)}}{A_0^{(s)}+B_0^{(s)}}$ that is essential for the so-called Rényi condition. This condition menas that there is a constant $C \geq 1$ such that

$$\frac{1}{C} \leq \frac{A_0^{(s)} + B_0^{(s)}x}{A_0^{(s)} + B_0^{(s)}y} \leq C$$

for all $x, y \in [0,1]$, and all $s \geq 1$ (see [3]). Since $B = [0,1]$ all matrices of the associated maps are non-negative. We therefore call this map the *associated Rényi map*. Since $B = [0,1]$ its matrices are non-negative.

These maps are closely connected to the dual algorithm. If M is the symmetric matrix which satisfies

$$M \circ V(\alpha) = V^*(\alpha) \circ M,\ M \circ V(\beta) = V^*(\beta) \circ M,$$

then define

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ M$$

and the following theorem holds.

Theorem 8. *(i) The Rényi map and the dual algorithm are related by the equations*

$$G(\alpha) \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ V^*(\alpha),\ G(\beta) \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \circ V^*(\beta).$$

(ii) The associated Rényi map is isomorphic to the given map T, i.e.

$$\Gamma \circ V(\alpha) = G(\alpha) \circ \Gamma,\ \Gamma \circ V(\beta) = G(\beta) \circ \Gamma.$$

Note that for one-dimensional maps with two branches a suitable matrix M always exists.

Example 10. If we choose

$$V(\alpha) = \begin{pmatrix} \alpha & 2-\alpha \\ 0 & 1 \end{pmatrix},\ V(\beta) = \begin{pmatrix} 2 & \beta \\ 1 & 1+\beta \end{pmatrix},$$

we find

$$G(\alpha) = \begin{pmatrix} \alpha & 0 \\ 1 & 1 \end{pmatrix},\ G(\beta) = \begin{pmatrix} 1 & 1 \\ 0 & 2+\beta \end{pmatrix}.$$

Remark 4. If the domain $\Gamma[0,1]$ is bounded away from 0 and ∞, then the map T satisfies a Rényi condition, i.e., there is constant $C \geq 1$ such that

$$\frac{1}{C} \leq \frac{A_0^{(s)}}{A_0^{(s)} + B_0^{(s)}} \leq C.$$

Remark 5. We can apply these considerations to maps with three branches (see [4]). As an example we consider the partition $0 < \frac{1}{3} < \frac{2}{3} < 1$ and three increasing branches. Let

$$V(\alpha) = \frac{\alpha x}{3 + (3\alpha - 1)x},\ V(\beta) = \frac{2 + 2\beta x}{6 + (3\beta - 3)x},\ V(\gamma) = \frac{2 + \gamma x}{3 + (\gamma - 1)x}.$$

Then a matrix Γ exists if the condition

$$1 - 7\alpha + \beta + 3\alpha\beta - 2\alpha\gamma = 0$$

is satisfied and

$$\Gamma = \begin{pmatrix} 1-\alpha & 3\alpha - 1 \\ 2\alpha & \alpha(\gamma - 1) \end{pmatrix}.$$

Using the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ the associated Rényi map can also defined if there exists an *exceptional dual* (see [4]) but there exists no suitable map Γ.

Remark 6. If we apply this method to continued fractions $V(k) = \frac{1}{k+x}$, $k \geq 1$, then we find $G(k)(y) = \frac{k+y}{k-1+y}$, $\Gamma(x) = 1 + x$, and $\Gamma[0,1] = [1,2]$ as expected. Nakada's continued fraction on $B = [\frac{\sqrt{5}-3}{2}, \frac{\sqrt{5}-1}{2}]$ (see [1]) also has an exceptional dual on $[0,1]$. Hence its Rényi map is well defined on $[1,2]$.

4. The Question of Ergodicity

One may conjecture that the maps which were not considered in Section 1 are ergodic and conservative. Since Lebesgue measure is finite on $B = \{(x,y) : 0 \leq y \leq x \leq 1\}$, we return to maps on this domain. However, a general proof seems not to be obvious.

Here we study the connection with their associated Rényi maps. In a similar way these maps arise from the recurrence relations for the triple

$$(A_0^{(s)}, A_0^{(s)} + B_0^{(s)}, A_0^{(s)} + B_0^{(s)} + C_0^{(s)}).$$

Again, the matrices of the associated maps are non-negative.

We consider the case

$$V_{D1}(\beta,\gamma) = V(\beta,\gamma) \circ D_1 = \begin{pmatrix} 2+2\beta & 2\gamma & -2\beta-2\gamma \\ 2+2\beta & -1-\beta+\gamma & -1-\beta-\gamma \\ 0 & 1+\beta+\gamma & -1-\beta-\gamma \end{pmatrix}$$

$$V(\mu,\nu)_{D1} = V(\mu,\nu) \circ D_1 = \begin{pmatrix} 2+\mu & \nu & -\mu-\nu \\ 2+\mu & \nu & -1-\mu-\nu \\ 0 & 2+\mu+\nu & -1-\mu-\nu \end{pmatrix}.$$

The associated matrices are given by

$$G_{D1}(\beta,\gamma) = \begin{pmatrix} 0 & 2+2\beta & 0 \\ 1+\beta+\gamma & 0 & 1+\beta+\gamma \\ 2 & 0 & 0 \end{pmatrix}$$

$$G_{D1}(\mu,\nu) = \begin{pmatrix} 0 & 2+\mu & 0 \\ 0 & 0 & 2+\mu+\nu \\ 1 & 0 & 1 \end{pmatrix}.$$

In a similar way as before we define

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \circ M.$$

Theorem 9. *There exists a matrix* Γ *such that*

$$\Gamma \circ V_{D1}(\beta,\gamma) = G_{D1}(\beta,\gamma) \circ \Gamma$$

$$\Gamma \circ V_{D1}(\mu,\nu) = G_{D1}(\mu,\nu) \circ \Gamma,$$

if

$$(i)\ \frac{2}{1+\beta+\gamma} = \frac{1}{2+\mu+\nu},$$

or

$$(ii)\ (2+\mu)^2(1+\beta+\gamma) - (2+\mu+\nu)(1+\beta)(2+2\mu-2\beta) = 0.$$

Using the equality $M = \Phi^* \circ N \circ \Phi$ (see Remark 1) Theorem 9 is equivalent to Theorem 6. In case (i) the matrices M and Γ have a remarkable simple form, namely

$$M = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & J \end{pmatrix}, \Gamma = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & J \end{pmatrix},$$

where $J = \frac{2}{1+\beta+\gamma} = \frac{1}{2+\mu+\nu}$. However, in this case Γ maps the points $[1,0,0]$, $[1,1,0]$, $[1,1,1]$, and $[2,1,1]$ onto the points $[0,1,0]$, $[1,0,0]$, $[0,0,1]$, and $[0,1,J]$. Therefore a Rényi condition is not satisfied.

Example 11. If $\beta = \gamma = \mu = \nu = 0$, which means that T is piecewise linear, then we see that

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

works and maps B onto the point $(1,1)$ which is the fixed point of the associated Rényi map.

Example 12. If we take $\beta = \frac{1}{2}$, $\gamma = -\frac{3}{4}$, $\mu = \nu = 0$, then

$$\Gamma = \begin{pmatrix} 39 & -27 & 1 \\ 12 & 1 & 13 \\ 13 & 13 & -7 \end{pmatrix}.$$

Hence ΓB is bounded and a Rényi condition is satisfied.

Remark 7. We consider Selmer's algorithm on the triangle Δ with vertices $[1,1,0]$, $[1,1,1]$, and $[2,1,1]$. The inverse branches are given as

$$V(1)(x,y) = \big(\frac{1}{x+y}, \frac{y}{x+y}\big),\ V(2)(x,y) = \big(\frac{1}{x+y}, \frac{x}{x+y}\big)$$

and the associated maps by

$$G(1)(u,v) = \big(v, 1+u\big),\ G(2)(u,v) = \big(\frac{v}{u}, \frac{1+u}{u}\big).$$

Note that we use the triple

$$(A_0^{(s)} + B_0^{(s)}, A_0^{(s)} + B_0^{(s)} + C_0^{(s)}, 2A_0^{(s)} + B_0^{(s)} + C_0^{(s)})$$

which corresponds to the vertices of the domain. The connecting matrix is

$$\Gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and therefore $\Gamma\Delta$ is the stripe $1 \leq u \leq v \leq 1 + u$. The cylinder $\Gamma B(2)$ is the triangle with vertices $(1,1)$, $1,2)$, and $(2,2)$. This reflects the fact that the jump transformation which avoids the cylinder $B(1)$ satisfies a Rényi condition.

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