SOME CONGRUENCES FOR (S,T)-REGULAR BIPARTITIONS MODULO T

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Abstract

In this work, we study the function $B_{s,t}(n)$, which counts the number of (s,t)-regular bipartitions of n. Recently, many authors proved the existence of infinite families of congruences modulo 11 for $B_{3,11}(n)$, modulo 3 for $B_{3,s}(n)$ and modulo 5 for $B_{5,s}(n)$. Most recently, the first author proved the existence of infinite families of congruences modulo 11, 13 and 17 for $B_{5,11}(n)$, $B_{5,13}(n)$ and $B_{81,17}(n)$, respectively. In this paper, we establish the existence of infinite families of congruences modulo 5 for $B_{2,15}(n)$, modulo 11 for $B_{7,11}(n)$, modulo 11 for $B_{27,11}(n)$ and modulo 17 for $B_{243,17}(n)$.

1. Introduction

For a positive integer n, a *partition* of n is a non-increasing sequence of positive integers whose sum is n. The number of partitions of n is denoted by $p(n)$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{1}$$

For a non-zero integer k, we define the general partition function $p_k(n)$ as the coefficient of q^n in the expansion of $(q;q)_\infty^k$. If $k=-1$, then we have the usual partition function $p(n)$. The generating function for $p_k(n)$ is given by

$$\sum_{n=0}^{\infty} p_k(n)q^n = (q;q)_\infty^k, \tag{2}$$

where, as usual,

$$f_k := (q^k; q^k)_\infty = \prod_{m=1}^{\infty} (1 - q^{mk}).$$

In [18, 20], Ramanujan obtained the significant identities

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6} \tag{3}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{f_7^3}{f_1^4} + 49q\frac{f_7^7}{f_1^8}. \tag{4}$$

Ramanujan [18] gave a brief proof of (3). He did not prove (4) in [18], but he did give a sketch of his proof of (4) in his unpublished manuscript on the partition and τ-functions [19]. Note that (3) and (4) immediately yield the congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$.

Ramanujan's partition congruences motivated an investigation of many classes of partitions, such as ℓ-regular partitions. For a positive integer ℓ, a partition is said to be *ℓ-regular* if none of its parts is divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n. The generating function for $b_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}. \tag{5}$$

The arithmetic properties of ℓ-regular partitions have been extensively studied in recent years. Some of the references are [5, 6, 7, 9, 10, 13, 22, 23, 25].

A bipartition (λ, μ) of n is a pair of partitions (λ, μ) such that the sum of all the parts of λ and μ equals n. A *(s, t)-regular bipartition* of n is a bipartition (λ, μ) of n such that λ is an s-regular partition and μ is a t-regular partition. Let $B_{s,t}(n)$ denote the number of (s, t)-regular bipartitions of n. If $s = t$, then we denote $B_{s,t}(n)$ as $B_s(n)$ for simplicity. The generating function of $B_{s,t}(n)$ is given by

$$\sum_{n=0}^{\infty} B_{s,t}(n)q^n = \frac{f_s f_t}{f_1^2}. \tag{6}$$

Recently, Lin [16] proved the existence of infinite families of congruences modulo 3 for $B_7(n)$, by using Ramanujan's two modular equations of degree 7, and in [17] he proved the existence of infinite families of congruences modulo 3 for $B_{13}(n)$. For more related works, see [15, 21].

In a recent paper, Dou [8] proved that for $n \geq 0$ and $\alpha \geq 2$,

$$B_{3,11}\left(3^\alpha n + \frac{5 \cdot 3^{\alpha-1} - 1}{2}\right) \equiv 0 \pmod{11}.$$

Adiga and Ranganatha [1] proved the existence of infinite families of congruences modulo 3 for $B_{3,7}(n)$, and Xia and Yao [24] proved the existence of infinite families of congruences modulo 3 for $B_{3,s}(n)$, modulo 5 for $B_{5,s}(n)$ and modulo 7 for $B_{3,7}(n)$. Let us consider an example. For a positive integer s, non-negative integer n and a prime $p \geq 5$,

$$B_{3,s}\left(p^{2\alpha+1}n + \frac{(1+s)(p^{2\alpha+2}-1)}{24}\right) \equiv 0 \pmod{3}.$$

Most recently, the first author [14] proved the existence of infinite families of congruences modulo 11, 13 and 17 for $B_{5,11}(n)$, $B_{5,13}(n)$ and $B_{81,17}(n)$, respectively. For example, for all $n \geq 0$ and $m \geq 0$,

$$B_{5,13}\left(5^{6m+5}(5n+k) + \frac{5^{6m+5}-2}{3}\right) \equiv 0 \pmod{13}, \quad \text{where} \quad k \in \{1,5\}.$$

In this paper, we prove the existence of infinite families of congruences modulo 5, 11, 11 and 17 for $B_{2,15}(n)$, $B_{7,11}(n)$, $B_{27,11}(n)$ and $B_{243,17}(n)$, respectively. The following are the main results in this paper.

Theorem 1. *For all $n \geq 0$ and $m \geq 0$,*

$$B_{2,15}\left(3^{2m+1}n + \frac{7\cdot 3^{2m+1}-5}{8}\right) \equiv 2^m B_{2,15}(3n+2) \pmod{5}, \tag{7}$$

$$B_{2,15}\left(3^{2m+2}n + \frac{23\cdot 3^{2m+1}-5}{8}\right) \equiv 0 \pmod{5}, \tag{8}$$

$$B_{2,15}\left(3^{2(m+1)+1}n + \frac{13\cdot 3^{2(m+1)}-5}{8}\right) \equiv 0 \pmod{5}. \tag{9}$$

Theorem 2. *For all $n \geq 0$ and $m \geq 0$,*

$$B_{7,11}\left(7^{12m}n + \frac{2\cdot 7^{12m}-2}{3}\right) \equiv 3^m B_{7,11}(n) \pmod{11}, \tag{10}$$

$$B_{7,11}\left(7^{12m+11}(7n+k) + \frac{2\cdot 7^{12m+11}-2}{3}\right) \equiv 0 \pmod{11}, \tag{11}$$

where $k \in \{1,5,6\}$.

Theorem 3. *For all $n \geq 0$ and $m \geq 4$,*

$$B_{27,11}\left(3^m n + \frac{5\cdot 3^{m-1}-3}{2}\right) \equiv 0 \pmod{11}. \tag{12}$$

Theorem 4. *For all $n \geq 0$,*

$$B_{243,17}(81n+23) \equiv 0 \pmod{17}, \tag{13}$$

$$B_{243,17}(81n+77) \equiv 0 \pmod{17}. \tag{14}$$

2. The Identities

In this section, we prove some lemmas to prove our main results. For any prime p, by using binomial theorem, we have

$$f_p \equiv f_1^p \pmod{p}. \tag{15}$$

Let us recall the results due to Berndt [2], Hirschhorn [11], and Hirschhorn and Sellers [12].

Lemma 1 (Berndt [2, p. 49]). We have

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{16}$$

Lemma 2 (Hirschhorn [11, Eqs. (21.3.1), (22.1.4) and (39.2.8)]). We have

$$f_1^3 = a(q^3)f_3 - 3qf_9^3, \tag{17}$$

$$a(q) = a(q^3) + 6q\frac{f_9^3}{f_3}, \tag{18}$$

$$\frac{1}{f_1^3} = a^2(q^3)\frac{f_9^3}{f_3^{10}} + 3qa(q^3)\frac{f_9^6}{f_3^{11}} + 9q^2\frac{f_9^9}{f_3^{12}}, \tag{19}$$

where

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}.$$

Lemma 3 (Hirschhorn and Sellers [12]). We have

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{20}$$

Lemma 4. For $n \geq 0$, we have

$$\sum_{n=0}^{\infty} p_5(7n+3)q^n = 10f_1^4 f_7 + 49qf_7^5. \tag{21}$$

Proof. Setting $k = 5$ in (2), we have

$$\sum_{n=0}^{\infty} p_5(n)q^n = f_1^5. \tag{22}$$

In [2, p. 303, Entry 17(v)], we have

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - \frac{A(q^7)}{B(q^7)}q - q^2 + \frac{C(q^7)}{A(q^7)}q^5\right), \tag{23}$$

where A, B and C are defined by
$A = A(q) := \dfrac{f(-q^3,-q^4)}{f(-q^2)}$, $B = B(q) := \dfrac{f(-q^2,-q^5)}{f(-q^2)}$ and $C = C(q) := \dfrac{f(-q,-q^6)}{f(-q^2)}$.
Substituting (23) into (22), we have

$$\sum_{n=0}^{\infty} p_5(n)q^n = f_{49}^5 \left(\frac{B(q^7)}{C(q^7)} - \frac{A(q^7)}{B(q^7)}q - q^2 + \frac{C(q^7)}{A(q^7)}q^5 \right)^5. \tag{24}$$

If we extract those terms in which the power of q is congruent to 3 modulo 7, dividing by q^3 and replacing q^7 by q, then we have

$$\sum_{n=0}^{\infty} p_5(7n+3)q^n = f_7^5 \left(20 \left(\frac{AB^2}{C^3} + \frac{A^2Cq}{B^3} - \frac{BC^2q^2}{A^3} \right) \right.$$
$$\left. - 10 \left(\frac{A^3}{BC^2} - \frac{B^3q}{A^2C} - \frac{C^3q^2}{AB^2} \right) - 61q \right). \tag{25}$$

From [3, p. 174, Entry 31] and [4, Eq. 3.11 and Eq. 3.15] in the terms of A, B and C, we have

$$\frac{B^5}{AC^4} - \frac{A^5}{B^4C} - \frac{C^5q^3}{A^4B} = 3q, \tag{26}$$
$$\frac{AB^2}{C^3} + \frac{A^2Cq}{B^3} - \frac{BC^2q^2}{A^3} = \frac{f_1^4}{f_7^4} + 8q, \tag{27}$$
$$\frac{A^3}{BC^2} - \frac{B^3q}{A^2C} - \frac{C^3q^2}{AB^2} = \frac{f_1^4}{f_7^4} + 5q, \tag{28}$$
$$\frac{B^7}{C^7} - \frac{A^7q}{B^7} + \frac{C^7q^5}{A^7} = \frac{f_1^8}{f_7^8} + 14q\frac{f_1^4}{f_7^4} + 57q^2. \tag{29}$$

Substituting (27) and (28) into (25), and simplifying, completes the proof of Lemma 4. □

Lemma 5. For $n \geq 0$, we have

$$\sum_{n=0}^{\infty} p_9(7n+4)q^n = -90f_1^8f_7 - 882qf_1^4f_7^5 - 2401q^2f_7^9. \tag{30}$$

Proof. Setting $k = 9$ in (2), we have

$$\sum_{n=0}^{\infty} p_9(n)q^n = f_1^9. \tag{31}$$

Substituting (23) into (31), we have

$$\sum_{n=0}^{\infty} p_9(n)q^n = f_{49}^9 \left(\frac{B(q^7)}{C(q^7)} - \frac{A(q^7)}{B(q^7)}q - q^2 + \frac{C(q^7)}{A(q^7)}q^5 \right)^9. \tag{32}$$

If we extract those terms in which the power of q is congruent to 4 modulo 7, dividing by q^4 and replacing q^7 by q, then we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_9(7n+4)q^n = f_7^9 \Bigg(& \left(\frac{36B^7}{C^7} - \frac{252A^2B^4}{C^6} + \frac{126A^4B}{C^5}\right) \\ & + q\left(-\frac{36A^7}{B^7} + \frac{756A^5}{B^4C} - \frac{3780A^3}{BC^2} - \frac{756B^5}{AC^4} + \frac{4284AB^2}{C^3}\right) \\ & + q^2\left(-\frac{252A^4C^2}{B^6} + \frac{4284A^2C}{B^3} + \frac{3780B^3}{A^2C} - 9745\right) \\ & + q^3\left(\frac{126B^4C}{A^5} - \frac{4284BC^2}{A^3} - \frac{126AC^4}{B^5} + \frac{3780C^3}{AB^2}\right) \\ & + q^4\left(\frac{756C^5}{A^4B} - \frac{252B^2C^4}{A^6}\right) + \frac{36C^7}{A^7}q^5\Bigg). \end{aligned} \tag{33}$$

On rearranging the equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_9(7n+4)q^n = f_7^9 \Bigg(& 36\left(\frac{B^7}{C^7} - \frac{A^7q}{B^7} + \frac{C^7q^5}{A^7}\right) \\ & - 252\left(\frac{A^2B^4}{C^6} - \frac{A^4C^2q^2}{B^6} - \frac{B^2C^4q^4}{A^6}\right) + 126\left(\frac{A^4B}{C^5} + \frac{B^4Cq^3}{A^5} - \frac{AC^4q^3}{B^5}\right) \\ & - 756q\left(\frac{B^5}{AC^4} - \frac{A^5}{B^4C} - \frac{C^5q^3}{A^4B}\right) + 4284q\left(\frac{AB^2}{C^3} + \frac{A^2Cq}{B^3} - \frac{BC^2q^2}{A^3}\right) \\ & - 3780q\left(\frac{A^3}{BC^2} - \frac{B^3q}{A^2C} - \frac{C^3q^2}{AB^2}\right) - 9745q^2\Bigg). \end{aligned} \tag{34}$$

From the above equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_9(7n+4)q^n = {} & 36f_7^9\left(\frac{B^7}{C^7} - \frac{A^7q}{B^7} + \frac{C^7q^5}{A^7}\right) \\ & - 252f_7^9\left(\left(\frac{AB^2}{C^3} + \frac{A^2Cq}{B^3} - \frac{BC^2q^2}{A^3}\right)^2 - 2q\left(\frac{A^3}{BC^2} - \frac{B^3q}{A^2C} - \frac{C^3q^2}{AB^2}\right)\right) \\ & + 126f_7^9\Bigg(q\left(\frac{B^5}{AC^4} - \frac{A^5}{B^4C} - \frac{C^5q^3}{A^4B}\right) \\ & \quad + \left(\frac{AB^2}{C^3} + \frac{A^2Cq}{B^3} - \frac{BC^2q^2}{A^3}\right)\left(\frac{A^3}{BC^2} - \frac{B^3q}{A^2C} - \frac{C^3q^2}{AB^2}\right)\Bigg) \\ & - 756qf_7^9\left(\frac{B^5}{AC^4} - \frac{A^5}{B^4C} - \frac{C^5q^3}{A^4B}\right) + 4284qf_7^9\left(\frac{AB^2}{C^3} + \frac{A^2Cq}{B^3} - \frac{BC^2q^2}{A^3}\right) \\ & - 3780qf_7^9\left(\frac{A^3}{BC^2} - \frac{B^3q}{A^2C} - \frac{C^3q^2}{AB^2}\right) - 9367q^2f_7^9. \end{aligned} \tag{35}$$

Substituting (26), (27), (28) and (29) into (35), we have

$$\begin{aligned}\sum_{n=0}^{\infty} & p_9(7n+4)q^n \\ &= 36f_7^9\left(\frac{f_1^8}{f_7^8}+14q\frac{f_1^4}{f_7^4}+57q^2\right)-252f_7^9\left(\left(\frac{f_1^4}{f_7^4}+8q\right)^2-2q\left(\frac{f_1^4}{f_7^4}+5q\right)\right)\\ &\quad+126f_7^9\left((3q)+\left(\frac{f_1^4}{f_7^4}+5q\right)\left(\frac{f_1^4}{f_7^4}+8q\right)\right)-756qf_7^9(3q)\\ &\quad+4284qf_7^9\left(\frac{f_1^4}{f_7^4}+8q\right)-3780qf_7^9\left(\frac{f_1^4}{f_7^4}+5q\right)-9367q^2f_7^9. \qquad (36)\end{aligned}$$

Simplifying Equation (36) completes the proof. □

3. Congruence for $(2,15)$-regular Bipartition

Theorem 5. *For $n \geq 0$, we have*

$$B_{2,15}(9n+8) \equiv 0 \pmod 5, \qquad (37)$$

$$B_{2,15}(27n+14) \equiv 0 \pmod 5, \qquad (38)$$

$$B_{2,15}(27n+23) \equiv 2B_{2,15}(3n+2) \pmod 5. \qquad (39)$$

Proof. Setting $s=2$, and $t=15$ in (6), we have

$$\sum_{n=0}^{\infty} B_{2,15}(n)q^n = \frac{f_2f_{15}}{f_1^2}. \qquad (40)$$

Substituting (20) into (40), we have

$$\sum_{n=0}^{\infty} B_{2,15}(n)q^n = f_{15}\left(\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}\right). \qquad (41)$$

If we extract those terms in which the power of q is congruent to 2 modulo 3, dividing by q^2 and replacing q^3 by q, then we have

$$\sum_{n=0}^{\infty} B_{2,15}(3n+2)q^n \equiv 4\frac{f_2^2f_6^3}{f_1} \pmod 5. \qquad (42)$$

Substituting (16) into (42), we have

$$\sum_{n=0}^{\infty} B_{2,15}(3n+2)q^n \equiv 4f_6^3\left(\frac{f_6f_9^2}{f_3f_{18}}+q\frac{f_{18}^2}{f_9}\right) \pmod 5. \qquad (43)$$

If we extract those terms in which the power of q is congruent to 1 modulo 3, dividing by q and replacing q^3 by q, then we have

$$\sum_{n=0}^{\infty} B_{2,15}(9n+5)q^n \equiv 4\frac{f_6^2 f_2^3}{f_3} \pmod{5}. \tag{44}$$

Now replacing q by q^2 in (17), we have

$$f_2^3 = f_6 a(q^6) - 3q^2 f_{18}^3. \tag{45}$$

Substituting (45) into (44), extracting those terms in which the power of q is congruent to 2 modulo 3, dividing by q^2 and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} B_{2,15}(27n+23)q^n \equiv 3\frac{f_2^2 f_6^3}{f_1} \pmod{5}. \tag{46}$$

Substituting (45) into (44), extracting those terms in which the power of q is congruent to 1 modulo 3, dividing by q and replacing q^3 by q, we have (38). This completes the proof of Theorem 5, which follows from (43) and (46). □

Next, we give a proof of Theorem 1.

Proof of Theorem 1. By mathematical induction, Congruence (7) follows from (39). Employing (37) and (38) in (7), we obtain (8) and (9).

4. Congruence for $(7, 11)$-regular Bipartition

We now present a proof of Theorem 2.

Proof of Theorem 2. Setting $s = 7$, and $t = 11$ in (6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{7,11}(n)q^n &\equiv f_7 f_1^9 \pmod{11}\\ &\equiv f_7 \sum_{n=0}^{\infty} p_9(n)q^n \pmod{11}.\end{aligned} \tag{47}$$

It follows that

$$\sum_{n=0}^{\infty} B_{7,11}(7n+4)q^n \equiv f_1 \sum_{n=0}^{\infty} p_9(7n+4)q^n \pmod{11}. \tag{48}$$

Now from (30) and (48), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{7,11}(7n+4)q^n &\equiv f_1\left(9f_1^8 f_7 + 9q f_1^4 f_7^5 + 8q^2 f_7^9\right)\\ &\equiv 9f_7 f_1^9 + 9q f_7^5 f_1^5 + 8q^2 f_7^9 f_1 \pmod{11}.\end{aligned} \tag{49}$$

Substituting (22), (23) and (31) into (49), we have

$$\sum_{n=0}^{\infty} B_{7,11}(7n+4)q^n \equiv 9f_7 \sum_{n=0}^{\infty} p_9(n)q^n + 9f_7^5 \sum_{n=0}^{\infty} p_5(n)q^{n+1} + 8q^2 f_7^9 f_{49} \left(\frac{B(q^7)}{C(q^7)} - \frac{A(q^7)}{B(q^7)}q - q^2 + \frac{C(q^7)}{A(q^7)}q^5 \right) \pmod{11}.$$

It follows that

$$\sum_{n=0}^{\infty} B_{7,11}(49n+32)q^n \equiv 9f_1 \sum_{n=0}^{\infty} p_9(7n+4)q^n + 9f_1^5 \sum_{n=0}^{\infty} p_5(7n+3)q^n + 3f_1^9 f_7 \pmod{11}. \tag{50}$$

Substituting Lemma 4 and Lemma 5 into (50), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{7,11}(49n+32)q^n &\equiv 9f_1\left(9f_1^8 f_7 + 9q f_1^4 f_7^5 + 8q^2 f_7^9\right) + 9f_1^5\left(10 f_1^4 f_7 + 5q f_7^5\right) \\ &\quad + 3f_1^9 f_7 \\ &\equiv 9f_7 f_1^9 + 5q f_7^5 f_1^5 + 6q^2 f_7^9 f_1 \pmod{11}. \end{aligned} \tag{51}$$

Similarly, we find

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{7,11}(343n+228)q^n &\equiv 4f_7 f_1^9 + 7q f_7^5 f_1^5 + 6q^2 f_7^9 f_1 \pmod{11},\\
\sum_{n=0}^{\infty} B_{7,11}(2401n+1600)q^n &\equiv f_7 f_1^9 + 5q f_7^5 f_1^5 + 10q^2 f_7^9 f_1 \pmod{11},\\
\sum_{n=0}^{\infty} B_{7,11}(16807n+11204)q^n &\equiv 5f_7 f_1^9 + q f_7^5 f_1^5 + 8q^2 f_7^9 f_1 \pmod{11},\\
\sum_{n=0}^{\infty} B_{7,11}(117649n+78432)q^n &\equiv 3f_7 f_1^9 + 6q f_7^5 f_1^5 + 7q^2 f_7^9 f_1 \pmod{11},\\
\sum_{n=0}^{\infty} B_{7,11}(823543n+549028)q^n &\equiv 3f_7 f_1^9 + 2q f_7^5 f_1^5 + 2q^2 f_7^9 f_1 \pmod{11},\\
\sum_{n=0}^{\infty} B_{7,11}(5764801n+3843200)q^n &\equiv f_7 f_1^9 + 4q f_7^5 f_1^5 + 2q^2 f_7^9 f_1 \pmod{11},\\
\sum_{n=0}^{\infty} B_{7,11}(40353607n+26902404)q^n &\equiv 3f_7 f_1^9 + 7q f_7^5 f_1^5 + 8q^2 f_7^9 f_1 \pmod{11},\\
\sum_{n=0}^{\infty} B_{7,11}(282475249n+188316832)q^n &\equiv f_7 f_1^9 + 7q f_7^5 f_1^5 + 2q^2 f_7^9 f_1 \pmod{11},\\
\sum_{n=0}^{\infty} B_{7,11}(1977326743n+1318217828)q^n &\equiv 8q^2 f_7^9 f_1 \pmod{11}.
\end{aligned} \tag{52}$$

Substituting (23) into (52), we have

$$\sum_{n=0}^{\infty} B_{7,11}\left(7^{11}n+\frac{2(7^{11}-1)}{3}\right)q^n \equiv 8q^2 f_7^9 f_{49}\left(\frac{B(q^7)}{C(q^7)}-\frac{A(q^7)}{B(q^7)}q-q^2+\frac{C(q^7)}{A(q^7)}q^5\right). \tag{53}$$

There are no terms on the right of q^{7n+1}, q^{7n+5} and q^{7n+6}, so

$$B_{7,11}\left(7^{11}(7n+k)+\frac{2(7^{11}-1)}{3}\right) \equiv 0 \pmod{11}, \text{ where } k=1,5,6. \tag{54}$$

and

$$B_{7,11}\left(7^{12}n+\frac{2(7^{12}-1)}{3}\right) \equiv 3B_{7,11}(n) \pmod{11}. \tag{55}$$

By mathematical induction, Congruence (10) follows from (55). Employing (10) in (54), we obtain (11).

5. Congruence for $(27,11)$-regular Bipartition

We now present a proof of Theorem 3.

Proof of Theorem 3. Setting $s=27$, and $t=11$ in (6), we have

$$\sum_{n=0}^{\infty} B_{27,11}(n)q^n = \frac{f_{27}f_{11}}{f_1^2} \equiv f_{27}f_1^9 \pmod{11}. \tag{56}$$

Substituting (17) into (56), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{27,11}(n)q^n &\equiv f_{27}\left(a(q^3)f_3-3qf_9^3\right)^3 \pmod{11}\\ &\equiv a(q^3)^3 f_3^3 f_{27}+2qa(q^3)^2 f_3^2 f_9^3 f_{27}+5q^2 a(q^3)f_3 f_9^6 f_{27}\\ &\quad +6q^3 f_9^9 f_{27} \pmod{11}.\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} B_{27,11}(3n)q^n \equiv a(q)^3 f_1^3 f_9+6qf_3^9 f_9 \pmod{11}. \tag{57}$$

Substituting (17) and (18) into (57), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{27,11}(3n)q^n &\equiv f_9\left(a(q^3)+6q\frac{f_9^3}{f_3}\right)^3\left(a(q^3)f_3-3qf_9^3\right)+6qf_9 \pmod{11}\\ &\equiv a(q^3)^4 f_3 f_9+6qf_3^9 f_9+4qa(q^3)^3 f_9^4+10q^2\frac{a(q^3)^2 f_9^7}{f_3}\\ &\quad +2q^3\frac{a(q^3)f_9^{10}}{f_3^2}+q^4\frac{f_9^{13}}{f_3^3} \pmod{11}.\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} B_{27,11}(9n+3)q^n \equiv 6f_1^9 f_3 + 4a(q)^3 f_3^4 + q\frac{f_3^{13}}{f_1^3} \pmod{11}. \tag{58}$$

Substituting (17), (18) and (19) into (58), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{27,11}(9n+3)q^n &\equiv 6f_3\left(a(q^3)f_3 - 3qf_9^3\right)^3 + 4f_3^4\left(a(q^3) + 6q\frac{f_9^3}{f_3}\right)^3 \\ &\quad + qf_3^{13}\left(a^2(q^3)\frac{f_9^3}{f_3^{10}} + 3qa(q^3)\frac{f_9^6}{f_3^{11}} + 9q^2\frac{f_9^9}{f_3^{12}}\right) \\ &\equiv 10a(q^3)^3 f_3^4 + 8qa(q^3)^2 f_3^3 f_9^3 + 3q^2 a(q^3) f_3^2 f_9^6 \\ &\quad + 7q^3 f_3 f_9^9 \pmod{11}. \end{aligned} \tag{59}$$

It follows that

$$\sum_{n=0}^{\infty} B_{27,11}(27n+12)q^n \equiv 8a(q)^2 f_1^3 f_3^3 \pmod{11}. \tag{60}$$

Substituting (17) and (18) into (60), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{27,11}(27n+12)q^n &\equiv 8f_3^3\left(a(q^3) + 6q\frac{f_9^3}{f_3}\right)^2\left(a(q^3)f_3 - 3qf_9^3\right) \\ &\equiv 8a(q^3)^3 f_3^4 + 6qa(q^3)^2 f_3^3 f_9^3 + 5q^3 f_3 f_9^9. \end{aligned} \tag{61}$$

It follows that

$$B_{27,11}(81n+66) \equiv 0 \pmod{11} \tag{62}$$

and

$$B_{27,11}(81n+39) \equiv 9B_{27,11}(27n+12) \pmod{11}. \tag{63}$$

By (62) and (63), the proof is complete.

6. Congruence for $(243, 17)$-regular Bipartition

We now present a proof of Theorem 4.

Proof of Theorem 4. Setting $s = 243$, and $t = 17$ in (6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{243,17}(n)q^n &= \frac{f_{243}f_{17}}{f_1^2} \\ &\equiv f_{243}f_1^{15} \pmod{17}. \end{aligned} \tag{64}$$

Substituting (17) into (64), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{243,17}(n)q^n &\equiv f_{243}\left(a(q^3)f_3 - 3qf_9^3\right)^5 \pmod{17}\\ &\equiv a(q^3)^5 f_3^5 f_{243} + 2qa(q^3)^4 f_3^4 f_9^3 f_{243} + 5q^2 a(q^3)^3 f_3^3 f_9^6 f_{243}\\ &\quad +2q^3 a(q^3)^2 f_3^2 f_9^9 f_{243} + 14q^4 a(q^3) f_3 f_9^{12} f_{243}\\ &\quad +12q^5 f_9^{15} f_{243} \pmod{17}.\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} B_{243,17}(3n+2)q^n \equiv 5a(q)^3 f_1^3 f_3^6 f_{81} + 12qf_3^{15} f_{81} \pmod{17}. \tag{65}$$

Substituting (17) and (18) into (65), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{243,17}(3n+2)q^n &\equiv 5f_3^6 f_{81}\left(a(q^3) + 6q\frac{f_9^3}{f_3}\right)^3\left(a(q^3)f_3 - 3qf_9^3\right) + 12qf_3^{15} f_{81}\\ &\equiv 5a(q^3)^4 f_3^7 f_{81} + 12qf_3^{15} f_{81} + 7qa(q^3)^3 f_3^6 f_9^3 f_{81}\\ &\quad +15q^2 a(q^3)^2 f_3^5 f_9^6 f_{81} + 4q^3 a(q^3) f_3^4 f_9^9 f_{81}\\ &\quad +7q^4 f_3^3 f_9^{12} f_{81} \pmod{17}.\end{aligned}$$

If we extract those terms in which the power of q is congruent to 1 modulo 3, dividing by q and replacing q^3 by q, then we have

$$\sum_{n=0}^{\infty} B_{243,17}(9n+5)q^n \equiv 12f_1^{15} f_{27} + 7a(q)^3 f_1^6 f_3^3 f_{27} + 7qf_1^3 f_3^{12} f_{27} \pmod{17}. \tag{66}$$

Substituting (17) and (18) into (66), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{243,17}(9n+5)q^n &\equiv 12\left(a(q^3)f_3 - 3qf_9^3\right)^5 f_{27}\\ &\quad +7\left(a(q^3) + 6q\frac{f_9^3}{f_3}\right)^3\left(a(q^3)f_3 - 3qf_9^3\right)^2 f_3^3 f_{27}\\ &\quad +7q\left(a(q^3)f_3 - 3qf_9^3\right) f_3^{12} f_{27} \pmod{17}\\ &\equiv 2a(q^3)^5 f_3^5 f_{27} + 7qa(q^3) f_3^{13} f_{27} + 6qa(q^3)^4 f_3^4 f_9^3 f_{27}\\ &\quad +13q^2 f_3^{12} f_9^3 f_{27} + 4q^2 a(q^3)^3 f_3^3 f_9^6 f_{27} + 4q^3 a(q^3)^2 f_3^2 f_9^9 f_{27}\\ &\quad +8q^4 a(q^3) f_3 f_9^{12} f_{27} + 16q^5 f_9^{15} f_{27} \pmod{17}.\end{aligned}$$

If we extract those terms in which the power of q is congruent to 2 modulo 3, dividing by q and replacing q^3 by q, then we have

$$\sum_{n=0}^{\infty} B_{243,17}(27n+23)q^n \equiv 13f_1^{12} f_3^3 f_9 + 4a(q)^3 f_1^3 f_3^6 f_9 + 16qf_3^{15} f_9 \pmod{17}.$$

Substituting (17) and (18) into (67), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{243,17}(27n+23)q^n &\equiv 13\left(a(q^3)f_3 - 3qf_9^3\right)^4 f_3^3 f_9 \\ &\quad +4\left(a(q^3)+6q\frac{f_9^3}{f_3}\right)^3 \left(a(q^3)f_3 - 3qf_9^3\right) f_3^6 f_9 + 16qf_3^{15} f_9 \\ &\equiv 16qf_3^{15}f_9 + 6qa(q^3)^3 f_3^6 f_9^4 + 8q^4 f_3^3 f_9^{13} \pmod{17}. \qquad (67)\end{aligned}$$

This completes the proof of Theorem 4, follow from (67).

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