THE FROBENIUS PROBLEM FOR EXTENDED THABIT NUMERICAL SEMIGROUPS

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Abstract

The greatest integer that does not belong to a numerical semigroup S is called the Frobenius number of S, and finding the Frobenius number is called the Frobenius problem. In this paper, we extend the Frobenius problem for Thabit numerical semigroups to a more general class of semigroups.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. To begin, we introduce a numerical semigroup and submonoid generated by a nonempty subset.

Definition 1.1. A *numerical semigroup* is a subset S of $\mathbb{N}$ that is closed under addition and contains 0, such that $\mathbb{N}\backslash S$ is finite.

Definition 1.2. Given a nonempty subset A of a numerical semigroup $\mathbb{N}$, we denote by $\langle A \rangle$ *the submonoid of* $(\mathbb{N}, +)$ *generated by* A, that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n | n \in \mathbb{N}\backslash\{0\}, a_i \in A, \lambda_i \in \mathbb{N}$$

for all $i \in \{1, \cdots, n\}\}$.

In addition, we introduce several theorems and definitions directly related to the above definitions.

Theorem 1.3. ([23, 26]). Let $\langle A \rangle$ be the submonoid of $(\mathbb{N}, +)$ generated by a nonempty subset $A \subseteq \mathbb{N}$. Then $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

Definition 1.4. If S is a numerical semigroup and $S = \langle A \rangle$ then we say that A is a *system of generators of* S. Moreover, if $S \neq \langle X \rangle$ for all $X \subsetneq A$, we say that A is a *minimal system of generators of* S.

Theorem 1.5. ([26]). Every numerical semigroup admits a finite and unique minimal system of generators.

Definition 1.6. The cardinality of the minimal system of generators of S is called the *embedding dimension* of S and is denoted by $e(S)$.

Definition 1.7. The cardinality of $\mathbb{N}\backslash S$ is called the *genus of* S and is denoted by $g(S)$ for a numerical semigroup S.

Definition 1.8. An integer x is a *pseudo-Frobenius number* if $x \notin S$ and $x + s \in S$ for all $s \in S\backslash\{0\}$. The set of pseudo-Frobenius numbers of S is denoted by $PF(S)$. In addition, we call this set's cardinality the *type* of S and denote it by $t(S)$.

The greatest integer that does not belong to a numerical semigroup S is called the *Frobenius number* of S and is denoted by $F(S)$. In other words, the Frobenius number is the largest integer that cannot be expressed as a sum $\sum_{i=1}^{n} t_i a_i$, where $t_1, t_2, ..., t_n$ are nonnegative integers and $a_1, a_2, ..., a_n$ are given positive integers such that $\gcd(a_1, a_2, ..., a_n) = 1$. Finding the Frobenius number is called the Frobenius problem, the coin problem or the money changing problem. The Frobenius problem is not only interesting for pure mathematicians, but is also connected with graph theory in [10, 11] and the theory of computer science in [17], as introduced in [16]. There are explicit formulas for calculating the Frobenius number when only two relatively prime numbers are present [29]. For three relatively prime numbers, it was shown decades ago that there is a somewhat algorithmic method to obtain the Frobenius number using the Euclidean algorithm [21]. More recently, a semi-explicit formula [20] for the Frobenius number for three relatively prime numbers was presented. An improved semi-explicit formula was presented for this case in 2017 [31].

F. Curtis proved in [6] that the Frobenius number for three or more relatively prime numbers cannot be given by a finite set of polynomials and Ramírez-Alfonsín proved in [18] that the problem is NP-hard. Currently, only algorithmic methods of determining the general formula for the Frobenius number of a set that has three or more relatively prime numbers in [3, 4] exist. Some recent studies have reported that the running time for the fastest algorithm is $O(a_1)$, with the residue table in memory in [5] and $O(na_1)$ with no additional memory requirements in [4]. In addition, research on the limiting distribution and lower bound of the Frobenius number were presented in [30] and [1, 7], respectively. From an algebraic viewpoint, rather than finding the general formula for three or more relatively prime numbers, the formulae for special cases were found such as the Frobenius number of a set of integers in a geometric sequence in [15], Pythagorean triples in [8] and three consecutive squares or cubes in [12]. Recently, various methods of solving the Frobenius problem for numerical semigroups have been suggested in [2], [22], [26], [27], etc. In particular, a method of computing the Apéry set and obtaining the Frobenius number using the Apéry set is an efficient tool for solving the Frobenius problem of numerical semigroups as reported in [14, 26, 19]. Furthermore, in recent articles presenting the Frobenius problems for Fibonacci numerical semigroups in [13], Mersenne numerical semigroups in [25], Thabit numerical semigroups in [23] and repunit numerical semigroups in [24], this method is used to obtain the Frobenius number.

The Frobenius problem of the numerical semigroups $\langle\{3\cdot 2^{n+i}-1|i\in\{0,1,\cdots\}\}\rangle$ for $n\in\{0,1,\cdots\}$ was presented in [23]. In [23], the authors recall a Thabit number $3\cdot 2^n-1$ and Thabit numerical semigroups $T(n)=\langle\{3\cdot 2^{n+i}-1|i\in\{0,1,\cdots\}\}\rangle$ for a nonnegative integer n and they used the definition of the minimal system of generators for $T(n)$ as the smallest subset of $\langle\{3\cdot 2^{n+i}-1|i\in\mathbb{N}\}\rangle$ that equals $T(n)$. In [23], it is proven that the minimal system of generators for $T(n)$ is $\langle\{3\cdot 2^{n+i}-1|i\in\{0,1,\cdots,n+1\}\}\rangle$. The *embedding dimension* is the cardinality of the minimal system of generators. By the minimality of the system $\langle\{3\cdot 2^{n+i}-1|i\in\{0,1,\cdots,n+1\}\}\rangle$ for $T(n)$, the embedding dimension for $T(n)$ is $n+2$. For any set S and $x\in S\backslash\{0\}$, the Apéry set is defined by $Ap(S,x)=\{s\in S|s-x\notin S\}$. Let $s_i=3\cdot 2^{n+i}-1$ for each nonnegative integer i. Then, the Apéry set is defined by $Ap(T(n),s_0)=\{s\in T(n)|s-s_0\notin T(n)\}$ for s_0. In [23], $Ap(T(n),s_0)$ was described explicitly, leading to a solution to the Frobenius problem. Let $R(n)$ be the set of sequences $(t_1,\cdots,t_{n+1})\in\{0,1,2\}^{n+1}$ that satisfies the following conditions:

1. $t_{n+1}\in\{0,1\}$,
2. if $t_j=2$, then $t_i=0$ for all $i<j\leq n$,
3. if $t_n=2$, then $t_{n+1}=0$,
4. if $t_n=t_{n+1}=1$, $t_i=0$ for all $1\leq i<n$.

Then [23] concludes that $Ap(T(n),s_0)=\{t_1s_1+\cdots+t_{n+1}s_{n+1}|(t_1,\cdots,t_{n+1})\in R(n)\}$. The Frobenius number of the numerical semigroups can be represented by $F(S)=\max(Ap(S,x))-x$ in [26] and therefore the Frobenius number of Thabit numerical semigroups is $s_n+s_{n+1}-s_0=9\cdot 2^{2n}-3\cdot 2^n-1$. In addition, an extended result of [23] has been suggested in 2017 which dealt with the numerical semigroups for $n\in\{0,1,\cdots\}$ and $2\leq k\leq 2^n$ such that $S(k,n)=\langle\{(2^k-1)\cdot 2^{n+i}-1|i\in\{0,1,\cdots\}\}\rangle$ [9]. In other words, the coefficient 3 in Thabit numerical semigroups was extended.

The principal purpose of this paper is to extend the coefficients 3 and 1 in Thabit numerical semigroups in [23] to 2^k+1 and 2^k-1 for a positive integer k, and to define the minimal system of generators and the Apéry set. In addition, we discuss the Frobenius problem in these numerical semigroups. More precisely, we solve the Frobenius problem for extended Thabit numerical semigroups defined by $\{(2^k+1)\cdot 2^{n+i}-(2^k-1)|i\in\{0,1,\cdots\}\}$ for $n\in\{0,1,\cdots\}$, $k\in\{1,2,\cdots\}$. Obviously, the extended Thabit numerical semigroups are generalized forms of [23]. The paper in [23] is the case of $k=1$ in this paper. In addition, the extended Thabit numerical semigroups represent the numerical semigroups generated by the numbers (Thabit number) + $(2^k-2)\cdot$ (Mersenne number), where k is a positive integer.

The remainder of this paper is organized as follows. In Section 2, we compute the minimal system of generators and the embedding dimension for the extended Thabit numerical semigroups. In Section 3, we present a method of obtaining the Apéry set and Frobenius number for the extended Thabit numerical semigroups. In Section 4, we compute the pseudo-Frobenius numbers and type for the extended Thabit numerical semigroups. In Section 5, we summarize the results of the Frobenius problem for the extended Thabit numerical semigroups.

Some theorems and definitions essential to understanding this paper are provided below.

Definition 1.9. A positive integer x is an *extended Thabit number* if $x = (2^k+1)\cdot 2^n - (2^k-1)$ for some $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$.

Definition 1.10. A numerical semigroup S is called an *extended Thabit numerical semigroup* if there exists $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0\}$ such that $S = \big\langle\{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \mathbb{N}\}\big\rangle$. We denote by $GT(n,k)$ the extended Thabit numerical semigroup $\big\langle\{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \mathbb{N}\}\big\rangle$.

Now, when we consider this extended Thabit numerical semigroup, we usually assume that $k \in \mathbb{N}\backslash\{0,1\}$ because the case $k = 1$ was done previously [23].

2. The Embedding Dimension for $GT(n,k)$

If $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0,1\}$, then $GT(n,k)$ is a submonoid of $(\mathbb{N},+)$. Moreover we have $\{(2^k+1)\cdot 2^n - (2^k-1), (2^k+1)\cdot 2^{n+1} - (2^k-1)\} \subseteq GT(n,k)$ and $\gcd((2^k+1)\cdot 2^n - (2^k-1), (2^k+1)\cdot 2^{n+1} - (2^k-1)) = 1$. Hence $\gcd(GT(n,k)) = 1$ and $GT(n,k)$ is a numerical semigroup.

Lemma 2.1. Let A be a nonempty set of positive integers, $k \in \mathbb{N}\backslash\{0,1\}$ and $M = \big\langle A\big\rangle$. Then the following conditions are equivalent.

1. $2a + (2^k-1) \in M$ for all $a \in A$.
2. $2m + (2^k-1) \in M$ for all $m \in M\backslash\{0\}$.

The proof of the above lemma is similar to that of Lemma 1 in [23], and it is a special case of the Lemma 2 in [24].

Proposition 2.2. If $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0,1\}$, then $2t + (2^k-1) \in GT(n,k)$ for all $t \in GT(n,k)\backslash\{0\}$.

The proof of the above proposition is similar to that of Proposition 2 in [23].
We provide some results to determine the minimal system of generators of $GT(n,k)$.

Lemma 2.3. Let $n \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0,1\}$ and $S = \big\langle\{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0,1,\cdots,n+k\}\}\big\rangle$. Then $2s + (2^k-1) \in S$ for all $s \in S\backslash\{0\}$.

The proof of the above lemma is similar to that of Lemma 3 in [23].

A system of generators of $GT(n,k)$ is given in the next lemma; note that it is not a minimal system of generators in the general case.

Lemma 2.4. If $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0,1\}$, then $GT(n,k)$ $= \left\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0,1,\cdots,n+k\}\} \right\rangle$.

Proof. This proof is similar to the proof of Lemma 4 in [23]. □

The fact that $(2^k+1)\cdot 2^{2n+k} - (2^k-1)$ is in the minimal system of generators of $GT(n,k)$ if $k \leq n$ is shown in the next lemma.

Lemma 2.5. If $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0,1\}$, then $(2^k+1)\cdot 2^{2n+k} - (2^k-1) \notin \left\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0,1,\cdots,n+k-1\}\} \right\rangle$ for $k \leq n$.

Proof. This proof is similar to the proof of Lemma 5 in [23]. □

Now, we introduce a lemma leading to Theorem 2.7, the main result of this section.

Lemma 2.6. Let $n \in \mathbb{N}\backslash\{0\}$ and $k \in \mathbb{N}\backslash\{0,1\}$. There exists at least one tuple $(a_0, a_1, \cdots, a_{n+\alpha+\beta})$, and $a_0, a_1, \cdots, a_{n+\alpha+\beta} \in \mathbb{N}$ with $\alpha, \beta \in \mathbb{N}$, satisfying

1. $\sum_{j=1}^{n+\alpha+\beta} a_j \cdot (2^j - 1) = 2^{n+\alpha+\beta+1} + 2^k - 2 - (2^k+1)\cdot 2^n$ and

2. $a_0 = 1 + (2^k+1)\cdot 2^n - \sum_{j=1}^{n+\alpha+\beta} a_j$,

for each $0 \leq \beta \leq k-\alpha-1$ with fixed α, if and only if we have $\left\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0,1,\cdots,n+\alpha\}\} \right\rangle = GT(n,k)$.

Proof. First, let us verify whether

$$(2^k+1)\cdot 2^{2n+\alpha+\beta+1} - (2^k-1) \in \left\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0,1,\cdots,n+\alpha+\beta\}\} \right\rangle \tag{1}$$

for each $0 \leq \beta \leq k-\alpha-1$ with fixed α. In other words, we verify whether there exist $a_0, \cdots, a_{n+\alpha+\beta} \in \mathbb{N}$ satisfying the equation

$$\begin{aligned}(2^k+1)\cdot 2^{2n+\alpha+\beta+1} - (2^k-1) &= \sum_{j=0}^{n+\alpha+\beta} a_j\{(2^k+1)\cdot 2^{n+j} - (2^k-1)\} \\ &= (2^k+1)\cdot 2^n \sum_{j=0}^{n+\alpha+\beta} a_j 2^j - (2^k-1)\sum_{j=0}^{n+\alpha+\beta} a_j.\end{aligned}$$

Hence, if we find a tuple $(a_0, \cdots, a_{n+\alpha+\beta})$ satisfying the following system of equations

$$\sum_{j=0}^{n+\alpha+\beta} a_j 2^j = 2^{n+\alpha+\beta+1} + t(2^k - 1), \tag{2}$$

$$\sum_{j=0}^{n+\alpha+\beta} a_j = 1 + t(2^k + 1) \cdot 2^n, \tag{3}$$

we have (1). Let us observe the following facts.

- In this case, we may fix $t = 1$.
  - If $t < 0$, then $\sum_{j=0}^{n+\alpha+\beta} a_j < 0$, that is, we have a contradiction.
  - If $t = 0$, then $\sum_{j=0}^{n+\alpha+\beta} a_j = 1$ and, consequently, $\sum_{j=0}^{n+\alpha+\beta} a_j 2^j < 2^{n+\alpha+\beta+1}$, in contradiction with Equation (2).
  - If $t \geq 2$, then

$$\begin{aligned} 0 \leq \sum_{j=0}^{n+\alpha+\beta} a_j(2^j - 1) &= \sum_{j=0}^{n+\alpha+\beta} a_j 2^j - \sum_{j=0}^{n+\alpha+\beta} a_j \\ &= 2^{n+\alpha+\beta+1} + t(2^k - 1) - 1 - t(2^k + 1)2^n \\ &= 2^{n+\alpha+\beta+1} - t(2^{n+k} + 2^n - 2^k + 1) - 1 \\ &\leq 2^{n+\alpha+\beta+1} - 2(2^{n+k} + 2^n - 2^k + 1) - 1 \\ &\leq 2^{n+k} - 2^{n+k+1} - 2^{n+1} + 2^{k+1} - 3 = -2^{n+k} - 2^{n+1} + 2^{k+1} - 3 < 0. \end{aligned}$$

- The first condition of this lemma is immediately obtained by (2) – (3).
- The second condition is immediately obtained by (3).

Finally, if we verify the system of equations for each $0 \leq \beta \leq k - \alpha - 1$, this implies that

$$(2^k+1)\cdot 2^{2n+\alpha+\beta+1} - (2^k-1) \in \left\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0, 1, \cdots, n+\alpha+\beta\}\} \right\rangle$$

for each $0 \leq \beta \leq k - \alpha - 1$ and hence

$$(2^k + 1) \cdot 2^{2n+\alpha+\beta+1} - (2^k - 1) \in \left\langle \{(2^k + 1) \cdot 2^{n+i} - (2^k - 1) | i \in \{0, 1, \cdots, n + \alpha\}\} \right\rangle$$

for each $0 \leq \beta \leq k - \alpha - 1$. This leads to

$$\begin{aligned} &\left\langle \{(2^k + 1) \cdot 2^{n+i} - (2^k - 1) | i \in \{0, 1, \cdots, n + k\}\} \right\rangle \\ &= \left\langle \{(2^k + 1) \cdot 2^{n+i} - (2^k - 1) | i \in \{0, 1, \cdots, n + \alpha\}\} \right\rangle \end{aligned}$$

for some fixed $0 \leq \alpha \leq k - 1$, completing the proof by Lemma 2.4. □

We also suggest a conclusion for a minimal system of generators of $GT(n,k)$ in the following theorem.

Theorem 2.7. Let $n \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0,1\}$ and

$$\delta = \begin{cases} 1, & \text{if } n = 0, \\ k, & \text{if } n \neq 0, k \leq n, \\ k-1, & \text{if } n \neq 0, k > n. \end{cases}$$

Then $\{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0, \cdots, n+\delta\}\}$ is a minimal system of generators.

Proof. We classify the cases to obtain a system of generators as follows.

1. $n = 0$.
   Here, we can easily see that $(2^k+1)\cdot 2^0 - (2^k-1) = 2$ for all k and $(2^k+1)\cdot 2^i - (2^k-1)$ is an odd number for each $i \geq 1$ because $k \geq 1$. This implies that for the case of $n = 0$, a minimal system of generators of $GT(n,k)$ is $\{(2^k+1)\cdot 2^i - (2^k-1) | i \in \{0,1\}\}$.

2. $n \neq 0, k \leq n$.
   For this case, the proof is similar to the proof of Theorem 6 in [23].

3. $n \neq 0, k > n$.
   Let us verify whether

   $$(2^k+1)\cdot 2^{2n+k} - (2^k-1) \in \left\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0, 1, \cdots, n+k-1\}\} \right\rangle.$$

   Note that this is just the case of $\alpha = k-1, \beta = 0$ in Lemma 2.6. Because we have the tuple $(a_0, a_1, a_2, \cdots, a_{n+k-1}) = (2^{n+k} + 2^{n+1} - 2^k + 3, 2^k - 2^n - 2, 0, \cdots, 0)$ satisfying the conditions in Lemma 2.6, $\delta \leq k-1$ in this case. Note that the tuple $(a_0, a_1, a_2, \cdots, a_{n+k-1})$ is not unique in general. Because $2^k - 2^n - 2 \geq 0$ for any $k > n \geq 1$, the remaining step is to check the minimality of $\left\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0, 1, \cdots, n+k-1\}\} \right\rangle$.
   Let us assume that there exists $\alpha \leq k-2$ such that $\left\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \{0, 1, \cdots, n+\alpha\}\} \right\rangle = GT(n,k)$. First, we consider the case of $\alpha \leq k-2, \beta = 0$ in Lemma 2.6. Note that $2^{n+\alpha+1} + (2^k-2) - (2^k+1)\cdot 2^n$ is negative because $\alpha \leq k-2$. Hence, the first condition of Lemma 2.6 cannot be satisfied, which leads to a contradiction. This implies that the number of elements in the generator must be at least $n+k$ for $k > n$ to guarantee the minimality. This completes the proof.

$\square$

By Theorem 2.7, we can identify the embedding dimension of $GT(n,k)$ for all $n \in \mathbb{N}$, and $k \in \mathbb{N}\backslash\{0,1\}$.

Corollary 2.8. Let $n \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0,1\}$ and let $GT(n,k)$ be an extended Thabit numerical semigroup associated with n and k. Then $e(GT(n,k)) = n + \delta + 1$.

We propose some examples related to the extended Thabit numerical semigroups $GT(n,k)$ with various embedding dimensions.

Example 2.9. $GT(0,3) = \left\langle \{(2^3+1)\cdot 2^0-(2^3-1), (2^3+1)\cdot 2^1-(2^3-1)\} \right\rangle = \left\langle \{2,11\} \right\rangle$ is an extended Thabit numerical semigroup with embedding dimension $0+1+1=2$.

Example 2.10. $GT(3,2) = \left\langle \{(2^2+1)\cdot 2^3-(2^2-1), (2^2+1)\cdot 2^4-(2^2-1), (2^2+1)\cdot 2^5-(2^2-1), (2^2+1)\cdot 2^6-(2^2-1), (2^2+1)\cdot 2^7-(2^2-1), (2^2+1)\cdot 2^8-(2^2-1)\} \right\rangle = \left\langle \{37,77,157,317,637,1277\} \right\rangle$ is an extended Thabit numerical semigroup with embedding dimension $3+2+1=6$.

Example 2.11. $GT(1,3) = \left\langle \{(2^3+1)\cdot 2^1-(2^3-1), (2^3+1)\cdot 2^2-(2^3-1), (2^3+1)\cdot 2^3-(2^3-1), (2^3+1)\cdot 2^4-(2^3-1)\} \right\rangle = \left\langle \{11,29,65,137\} \right\rangle$ is an extended Thabit numerical semigroup with embedding dimension $1+(3-1)+1=4$.

3. The Apéry Set for $GT(n,k)$

Definition 3.1. Let S be a numerical semigroup and let $x \in S\backslash\{0\}$. Then, we have the Apéry set of x in S defined as $Ap(S,x) = \{s \in S | s-x \notin S\}$.

From the definition above, we have the following lemma.

Lemma 3.2. *([26]).* Let S be a numerical semigroup and let $x \in S\backslash\{0\}$. Then, $Ap(S,x)$ has cardinality equal to x. Moreover, $Ap(S,x) = \{w(0), w(1), \cdots, w(x-1)\}$, where $w(i)$ is the least element of S congruent with i modulo x for all $i \in \{0,\cdots,x-1\}$.

Example 3.3. Let $S = \langle\{7,11,13\}\rangle$. Then
$S = \{0,7,11,13,14,18,20,21,22,24,25,26,27,28,29,31,\rightarrow\}$ where the symbol $\rightarrow$ means that every integer greater than 31 belongs to the set.
Hence $Ap(S,7) = \{0,11,13,22,24,26,37\}$.

The relation among the Frobenius number, genus, and Apéry set of a numerical semigroup is provided in the following lemma.

Lemma 3.4. *([26, 28]).* Let S be a numerical semigroup and let $x \in S\backslash\{0\}$. Then,

1. $F(S) = \max(Ap(S,x)) - x$
2. $g(S) = \frac{1}{x}(\sum_{w \in Ap(S,x)} w) - \frac{x-1}{2}$

Hereinafter, we denote by s_i the elements $(2^k+1)\cdot 2^{n+i} - (2^k-1)$ for each $i \in \{0,1,\cdots,n+\delta\}$. Thus, with this notation, $\{s_0, s_1, \cdots, s_{n+\delta}\}$ is the minimal system of generators of $GT(n,k)$ for each $i \in \{0,1,\cdots,n+\delta\}$ (where δ is defined as in Section 2).

Lemma 3.5. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0,1\}$.

1. If $0 < i \le j < n+\delta$, then $s_i + 2s_j = 2s_{i-1} + s_{j+1}$.

2. If $0 < i \le n+\delta$, then

$$s_i + 2s_{n+\delta} = \begin{cases} 2s_{i-1} + s_1 + (2^k+1)s_0, & \text{if } \ n = 0, \\ \\ 2s_{i-1} + {s_0}^2 + s_1 + (2^{k-1}-1)s_2 \\ +s_n + (2^{k-1}-1)s_{n+1}, & \text{if } \ n \neq 0, k \le n, \\ \\ 2s_{i-1} + (2^{n+k} + 2^{n+1} - 2^k + 3)s_0 \\ +(2^k - 2^n - 2)s_1, & \text{if } \ n \neq 0, k > n. \end{cases}$$

Proof. (1) The proof is similar to that of (1) of Lemma 9 in [23].
(2) For convenience, if $0 < i \le n+\delta$, then we can consider three cases.

1. If $n = 0$, then

$$\begin{aligned} & s_i + 2s_{n+\delta} \\ & = s_i + 2s_1 \\ & = \{(2^k+1)\cdot 2^i - (2^k-1)\} + 2\{(2^k+1)\cdot 2^1 - (2^k-1)\} \\ & = 2\{(2^k+1)\cdot 2^{i-1} - (2^k-1)\} + \{(2^k+1)\cdot 2^2 - (2^k-1)\} \\ & = 2s_{i-1} \\ & \quad + \{(2^k+1)\cdot 2^1 - (2^k-1)\} + (2^k+1)\{(2^k+1)\cdot 2^0 - (2^k-1)\} \\ & = 2s_{i-1} + s_1 + (2^k+1)s_0. \end{aligned}$$

2. If $n \neq 0, k \le n$, then

$$\begin{aligned} & s_i + 2s_{n+\delta} \\ & = (2^k+1)\cdot 2^{n+i} - (2^k-1) + 2\{(2^k+1)\cdot 2^{2n+k} - (2^k-1)\} \\ & = (2^k+1)\cdot 2^{n+i} - 2(2^k-1) + (2^k+1)\cdot 2^{2n+k+1} - (2^k-1) \\ & = 2\{(2^k+1)\cdot 2^{n+i-1} - (2^k-1)\} + \{(2^k+1)\cdot 2^n - (2^k-1)\}^2 \\ & \quad + \{(2^k+1)\cdot 2^{n+1} - (2^k-1)\} \\ & \quad + (2^{k-1}-1)\{(2^k+1)\cdot 2^{n+2} - (2^k-1)\} \\ & \quad + \{(2^k+1)\cdot 2^{2n} - (2^k-1)\} \\ & \quad + (2^{k-1}-1)\{(2^k+1)\cdot 2^{2n+1} - (2^k-1)\} \\ & = 2s_{i-1} + {s_0}^2 + s_1 + (2^{k-1}-1)s_2 + s_n + (2^{k-1}-1)s_{n+1}. \end{aligned}$$

3. If $n \neq 0, k > n$, then

$$\begin{aligned}
&s_i + 2s_{n+\delta} \\
&= (2^k+1)\cdot 2^{n+i} - (2^k-1) + 2\{(2^k+1)\cdot 2^{2n+k-1} - (2^k-1)\} \\
&= (2^k+1)\cdot 2^{n+i} - 2(2^k-1) + (2^k+1)\cdot 2^{2n+k} - 2^k - 1 \\
&= 2\{(2^k+1)\cdot 2^{n+i-1} - (2^k-1)\} \\
&\quad + (2^{n+k} + 2^{n+1} - 2^k + 3)\{(2^k+1)\cdot 2^n - (2^k-1)\} \\
&\quad + (2^k - 2^n - 2)\{(2^k+1)\cdot 2^{n+1} - (2^k-1)\} \\
&= 2s_{i-1} + (2^{n+k} + 2^{n+1} - 2^k + 3)s_0 + (2^k - 2^n - 2)s_1.
\end{aligned}$$

□

By Lemma 3.5, we can consider the set of coefficients $(t_1, \cdots, t_{n+\delta})$ such that the expression $\sum_{j=1}^{n+\delta} t_j s_j$ represents all elements in $Ap(GT(n,k), s_0)$. We follow a step-by-step approach to establish the set of coefficients $(t_1, \cdots, t_{n+\delta})$. First, we obtain the set of coefficients $(t_1, \cdots, t_{n+\delta})$ such that $\sum_{j=1}^{n+\delta} t_j s_j$ contains all elements in $Ap(GT(n,k), s_0)$, but that might not be equal. We can obtain the set by the following lemma.

Lemma 3.6. Let $A(n,\delta)$ be the set of $(t_1, \cdots, t_{n+\delta}) \in \{0,1,2\}^{n+\delta}$ such that $t_{n+\delta} \in \{0,1\}$, and if $t_j = 2$ then $t_i = 0$ for all $i < j$. Then
$Ap(GT(n,k), s_0) \subseteq \{\sum_{j=1}^{n+\delta} t_j s_j | (t_1, \cdots, t_{n+\delta}) \in A(n,\delta)\}$ unless $n = 1$ and $k = 2$.

Proof. The proof is the same as that of Lemma 10 in [23], except for the case of $(n = 1,\ k = 2)$. We show that $2s_{n+\delta} \notin Ap(GT(n,k), s_0)$ unless $(n = 1,\ k = 2)$ in Corollary 3.10. □

Lemma 3.7. Let $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0,1\}$. If $x \in GT(n,k)$ and $x \not\equiv 0 \pmod{s_0}$, then $x - (2^k - 1) \in GT(n,k)$.

Proof. If $x \in GT(n,k)$, then there exist $a_0, \cdots, a_{n+\delta} \in \mathbb{N}$ such that $x = \sum_{j=0}^{n+\delta} a_j s_j$. On the other hand, if $x \not\equiv 0 \pmod{s_0}$ then there exists $i \in \{1, \cdots, n+\delta\}$ such that $a_i \neq 0$. Note that $s_i = (2^k+1)\cdot 2^{n+i} - (2^k-1)$ as defined in Section 2. Hence

$$\begin{aligned}
x - (2^k-1) &= \sum_{j=0, j\neq i}^{n+\delta} a_j s_j + a_i s_i - (2^k-1) \\
&= \sum_{j=0, j\neq i}^{n+\delta} a_j s_j + (a_i - 1)s_i + (2^k+1)\cdot 2^{n+i} - 2(2^k-1) \\
&= \sum_{j=0, j\neq i}^{n+\delta} a_j s_j + (a_i - 1)s_i + 2\{(2^k+1)\cdot 2^{n+i-1} - (2^k-1)\} \\
&= \sum_{j=0, j\neq i-1, i}^{n+\delta} a_j s_j + (a_{i-1} + 2)s_{i-1} + (a_i - 1)s_i \in GT(n,k).
\end{aligned}$$

□

Let us consider Lemma 12 in [23]. In that lemma, $x \in T(n)$ (and $x \not\equiv 0 \pmod{s_0}$) implies that $x-1 \in T(n)$. In general, $x \in GT(n,k)$ (and $x \not\equiv 0 \pmod{s_0}$) implies that $x-(2^k-1) \in GT(n,k)$ by Lemma 3.7. Note that $\gcd(2^k-1, s_0) = \gcd(2^k-1, (2^k+1)\cdot 2^n - (2^k-1)) = \gcd(2^k-1, (2^k+1)\cdot 2^n) = \gcd(2^k-1, 2^{n+1}) = 1$, for any $x > (2^k-1)\{(2^k+1)\cdot 2^n - 2^k\} = (2^k-1)(s_0-1)$. Hence, if $(x-\alpha(2^k-1)) \equiv (x-\beta(2^k-1)) \pmod{s_0}$, then $s_0|(\alpha-\beta)\cdot(2^k-1)$, which leads to $s_0|(\alpha-\beta)$ because $\gcd(2^k-1, s_0) = 1$. Therefore, the set $\{x-i(2^k-1) | i \in \{0,1,\cdots,(2^k+1)\cdot 2^n - 2^k\}\}$ is a complete system of residues modulo s_0. Hence, we obtain the following lemma.

Lemma 3.8. Let $n \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0,1\}$ and $w(i)$ be the least element of $GT(n,k)$ congruent with i modulo s_0 for all $i \in \{0,\cdots,s_0-1\}$. Then we have

$$w(0) < w(2^k-1) < \cdots < w((s_0-1)\cdot(2^k-1)) = w(s_0-(2^k-1)).$$

Proof. The proof is similar to that of Lemma 13 in [23]. To add an explanation, let us consider $w(s_0-(2^k-1))$. Note that $w(s_0-(i+1)(2^k-1)) \le w(s_0-(2^k-1)) - i(2^k-1) < w(s_0-(2^k-1))$ for any $i \in \{1,\cdots,s_0-1\}$ by the following facts:

1. $w(s_0-(2^k-1)) - i(2^k-1) \in GT(n,k)$ by Lemma 3.7,
2. $s_0-(2^k-1)-i(2^k-1) \not\equiv 0 \pmod{s_0}$ for any $i \in \{1,\cdots,s_0-2\}$ because $\gcd(2^k-1, s_0) = 1$.

□

Corollary 3.9. If $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0,1\}$ then

$$w(s_0-(2^k-1)) = \max(Ap(GT(n,k), s_0)).$$

From this corollary and the fact that $s_i \equiv (2^i-1)(2^k-1) \pmod{s_0}$, we can prove the following corollary previously announced in Lemma 3.6.

Corollary 3.10. If $n \in \mathbb{N}$ and $k \in \mathbb{N}\backslash\{0,1\}$ then $2s_{n+\delta} \notin Ap(GT(n,k), s_0)$ unless $n=1$ and $k=2$.

Proof. We already know from the proof of Lemma 3.5 that
$s_i + 2s_{n+\delta} \notin Ap(GT(n,k), s_0)$ for all $i \in \mathbb{N}$. Therefore, if $2s_{n+\delta} \in Ap(GT(n,k), s_0)$, it can be expected to be a maximal element. Let us consider the following three cases.

1. If $n=0$, then

$$2s_{n+\delta} - s_0 = (2^{k-1}+2)s_0 \in GT(0,k).$$

2. If $n \neq 0$ and $k \leq n$, then because $s_i \equiv (2^i - 1)(2^k - 1) \pmod{s_0}$, $2s_{n+k} \equiv (2^{n+k+1} - 2)(2^k - 1) \pmod{s_0}$. Note that $(2^{n+k} + 2^n - 2^k)(2^k - 1) \equiv -(2^k - 1) \pmod{s_0}$ and

$$2^{n+k} + 2^n - 2^k < 2^{n+k+1} - 2 < 2(2^{n+k} + 2^n - 2^k) + 1.$$

This implies that $(2^{n+k+1} - 2)(2^k - 1) \not\equiv -(2^k - 1) \pmod{s_0}$ and therefore $2s_{n+\delta} \notin Ap(GT(n,k), s_0)$ if $n \neq 0$, $k \leq n$.

3. If $n \neq 0$ and $k > n$, then $2s_{n+k-1} \equiv (2^{n+k} - 2)(2^k - 1) \pmod{s_0}$ and hence

$$2^{n+k} + 2^n - 2^k < 2^{n+k} - 2 < 2(2^{n+k} + 2^n - 2^k) + 1$$

unless $n = 1$ and $k = 2$. Hence, $2s_{n+\delta} \notin Ap(GT(n,k), s_0)$ if $n \neq 0$, $k > n$, and $(n,k) \neq (1,2)$. If $(n,k) = (1,2)$, $2^{n+k} + 2^n - 2^k = 2^{n+k} - 2$ is satisfied and $2s_{n+k-1} - s_0 = 2s_2 - s_0 = 67$ and it implies that $74 \in Ap(\langle 7, 17, 37 \rangle, 7)$.

$\square$

Example 3.11. As mentioned in Example 11 of [23], we know that $GT(1,1) = \langle s_0, s_1, s_2 \rangle = \langle 5, 11, 23 \rangle$ and $Ap(GT(1,1), s_0) = \{0, 11, 22, 23, 34\} = \{0, s_1, 2s_1, s_2, s_1 + s_2\}$. Note that $2s_2 \notin Ap(GT(1,1), s_0)$.

We check the following four cases to find the Frobenius number and the Apéry set for $GT(n,k)$.

1. $n = 0$.
2. $2 \leq k \leq n$.
3. $n \neq 0$ and $2 \neq k > n$.
4. $n = 1$ and $k = 2$.

Note that the case of $k = 1$ was previously solved in [23].

3.1. The Case of $n = 0$

We state the Frobenius number and the Apéry set for the case in the following lemma. The proof can be directly derived because $s_0 = 2 \in GT(0,k)$ and $s_i = (2^k + 1) \cdot 2^i - (2^k - 1)$ is an odd number for all $i \geq 1$ because $k \geq 1$.

Remark 3.12. If $n = 0$ and $k \in \mathbb{N} \backslash \{0\}$, note that $s_0 = 2$ and $s_i = (2^k + 1) \cdot 2^i - (2^k - 1)$ is an odd number which is greater than or equal to s_1. In other words, for all $i \geq 1, s_i = s_1 + 2t = s_1 + ts_0$ for some $t \in \mathbb{N}$.

By Remark 3.12, we obtain the Frobenius number and the Apéry set for $GT(0,k)$.

Lemma 3.13. We have $\max(Ap(GT(0,k), s_0)) = s_1$ and $F(GT(0,k)) = s_1 - s_0 = s_1 - 2$. Moreover, $Ap(GT(0,k), s_0) = Ap(GT(0,k), 2) = \{0, s_1\}$.

3.2. The Case of $2 \le k \le n$

Similar to Lemma 2.6, we have the following useful lemma in this case.

Lemma 3.14. There exists at least one tuple $(a_0, a_1, \cdots, a_{n+k})$, with $a_0, a_1, \cdots, a_{n+k} \in \mathbb{N}$, $(t_1, \cdots, t_{n+\delta}) \in A(n, \delta)$, and $t \in \mathbb{Z}$, satisfying the following two conditions

1. $\sum_{j=1}^{n+k} a_j \cdot (2^j - 1) = \sum_{j=1}^{n+k-1} t_j(2^j - 1) + 2^{n+k} - 1 + t(2^k - 1 - 2^{n+k} - 2^n)$,
2. $a_0 = \sum_{j=1}^{n+k-1} t_j + t(2^k + 1) \cdot 2^n - \sum_{j=1}^{n+k} a_j$,

if and only if $\sum_{j=1}^{n+k-1} t_j s_j + s_{n+k} \notin Ap(GT(n, k), s_0)$.

Proof. Because the idea in the proof is analogous to that of Lemma 2.6, we simply check the following equalities:

$$\begin{aligned}
&\sum_{j=1}^{n+k-1} t_j s_j + s_{n+k} - s_0 \\
&= (2^k + 1) \cdot 2^n \sum_{j=1}^{n+k-1} t_j 2^j - (2^k - 1) \sum_{j=1}^{n+k-1} t_j + (2^k + 1) \cdot 2^n 2^{n+k} - (2^k - 1) \\
&\quad - (2^k + 1) \cdot 2^n + (2^k - 1) \\
&= (2^k + 1) \cdot 2^n \left(\sum_{j=1}^{n+k-1} t_j 2^j + 2^{n+k} - 1 \right) - (2^k - 1) \sum_{j=1}^{n+k-1} t_j \\
&= \sum_{j=0}^{n+k} a_j s_j \\
&= (2^k + 1) \cdot 2^n \left(\sum_{j=0}^{n+k} a_j 2^j \right) - (2^k - 1) \sum_{j=0}^{n+k} a_j .
\end{aligned}$$

This leads to

$$\begin{aligned}
\sum_{j=0}^{n+k} a_j 2^j &= \sum_{j=1}^{n+k-1} t_j 2^j + 2^{n+k} - 1 + t(2^k - 1) \\
\sum_{j=0}^{n+k} a_j &= \sum_{j=1}^{n+k-1} t_j + t(2^k + 1) \cdot 2^n \\
\sum_{j=0}^{n+k} a_j (2^j - 1) &= \sum_{j=1}^{n+k-1} t_j (2^j - 1) + 2^{n+k} - 1 + t(2^k - 1 - 2^{n+k} - 2^n).
\end{aligned}$$

□

Now, let us consider whether $s_i + s_{n+k} \in Ap(GT(n,k), s_0)$.

Proposition 3.15. We have $s_i + s_{n+k} \notin Ap(GT(n,k), s_0)$ if one of the following holds:

1. $k = n$ and $i \geq 2$,
2. $i \geq n$.

Proof. By Lemma 3.14, we find only one tuple $(a_0, a_1, \cdots, a_{n+k})$ with $a_0, a_1, \cdots, a_{n+k} \in \mathbb{N}$ and $t \in \mathbb{Z}$ satisfying the conditions

1. $\sum_{j=1}^{n+k} a_j \cdot (2^j - 1) = (2^i - 1) + 2^{n+k} - 2 + t(2^k - 1 - 2^{n+k} - 2^n)$,
2. $a_0 = 1 + t(2^k + 1) \cdot 2^n - \sum_{j=1}^{n+k} a_j$.

We may assume that $t = 1$ (because we only need to find a single tuple). Then, we only have to check the following two conditions.

1. $\sum_{j=1}^{n+k} a_j \cdot (2^j - 1) = 2^i + 2^k - 2^n - 3$.
2. $a_0 = 1 + (2^k + 1) \cdot 2^n - \sum_{j=1}^{n+k} a_j$.

Note that if $2^i + 2^k - 2^n - 3 \geq 0$, then $(a_0, a_1, a_2, \cdots, a_{n+k}) = (2^{n+k} + 2^{n+1} - 2^k - 2^i + 4, 2^i + 2^k - 2^n - 3, 0, \cdots, 0)$ is a desired tuple. $\square$

We introduce the following lemma which improves the first case of Proposition 3.15.

Lemma 3.16. If $k = n \geq 2$, then $\max(Ap(GT(n,n), s_0)) = s_1 + s_{n+n}$ and $F(GT(n,n)) = s_1 + s_{n+n} - s_0 = (2^n + 1) \cdot 2^n(2^{2n} + 1) - (2^n - 1)$.

Proof. First, let us consider whether $2s_1 + s_{n+n} \in Ap(GT(n,n), s_0)$ because $2s_1 < s_2$. By Lemma 3.14, we only have to check the following two conditions.

1. $\sum_{j=1}^{n+n} a_j \cdot (2^j - 1) = 2^{2n} + 1 - t(2^{2n} + 1)$.
2. $a_0 = 2 + t(2^{2n} + 2^n) - \sum_{j=1}^{n+n} a_j$.

We may take $t = 1$ as before. Then, we have a desired tuple $(a_0, a_1, a_2, \cdots, a_{n+n}) = (2^{2n} + 2^n + 2, 0, \cdots, 0)$. Hence, $2s_1 + s_{n+n} \notin Ap(GT(n,n), s_0)$. Furthermore, let us consider whether $s_1 + s_{n+n} \in Ap(GT(n,n), s_0)$. By Lemma 3.14, we only have to check the following two conditions.

1. $\sum_{j=1}^{n+n} a_j \cdot (2^j - 1) = 2^{2n} - t(2^{2n} + 1)$.
2. $a_0 = 1 + t(2^{2n} + 2^n) - \sum_{j=1}^{n+n} a_j$.

To satisfy the first condition, $t \leq 0$, but this leads to $a_0 < 0$. $\square$

Furthermore, we introduce the following lemma, which improves the second case of Proposition 3.15.

Lemma 3.17. If $2 \leq k < n$, then $s_{n-1}+s_{n+k} \leq \max(Ap(GT(n,k),s_0)) < 2s_{n-1}+s_{n+k}$.

Proof. Let us consider whether $2s_{n-1}+s_{n+k} \in Ap(GT(n,k),s_0)$. By Lemma 3.14, we only have to check the following two conditions.

1. $\sum_{j=1}^{n+k} a_j \cdot (2^j-1) = 2^n + 2^{n+k} - 3 + t(2^k - 1 - 2^{n+k} - 2^n)$.

2. $a_0 = 2 + t(2^k+1) \cdot 2^n - \sum_{j=1}^{n+k} a_j$.

Let $t=1$. Then, we have a desired tuple $(a_0, a_1, a_2, \cdots, a_{n+k}) = (2^{n+k}+2^n-2^k+6, 2^k-4, 0, \cdots, 0)$. Therefore, $2s_{n-1}+s_{n+k} \notin Ap(GT(n,k),s_0)$ if $2 \leq k < n$.
Let us consider whether $s_{n-1}+s_{n+k} \in Ap(GT(n,k),s_0)$. By Lemma 3.14, we only have to check the following two conditions.

1. $\sum_{j=1}^{n+k} a_j \cdot (2^j-1) = 2^{n-1} + 2^{n+k} - 2 + t(2^k - 1 - 2^{n+k} - 2^n)$.

2. $a_0 = 1 + t(2^k+1) \cdot 2^n - \sum_{j=1}^{n+k} a_j$.

Because $k<n$, if $t \geq 1$, $2^{n-1}+2^{n+k}-2+t(2^k-1-2^{n+k}-2^n) < 0$, and the first condition cannot be satisfied. In addition, if $t \leq 0$, it leads to $a_0 < 0$. □

Next, let us consider whether $\sum_{j=1}^{n-2} t_j s_j + s_{n-1} + s_{n+k} \in Ap(GT(n,k),s_0)$, where $(t_1, \cdots, t_{n-2}, 1, 0, \cdots, 0, 1) \in A(n,\delta)$.

Proposition 3.18. Let $2 \leq k < n$ and let the coefficients $(t_1, \cdots, t_{n-2}, 1, 0, \cdots, 0, 1) \in A(n,\delta)$ be such that

$$\sum_{j=1}^{n-2} t_j(2^j-1) = 2^{n-1} - 2^k + 2.$$

Then

$$\max(Ap(GT(n,k),s_0)) \leq \sum_{j=1}^{n-2} t_j s_j + s_{n-1} + s_{n+k}.$$

Proof. By Lemma 3.14, we only have to check the following two conditions:

1. $\sum_{j=1}^{n+k} a_j \cdot (2^j-1) = \sum_{j=1}^{n-2} t_j(2^j-1) + 2^{n-1} + 2^{n+k} - 2 + t(2^k-1-2^{n+k}-2^n)$ and,

2. $a_0 = \sum_{j=1}^{n-2} t_j + 1 + t(2^k+1) \cdot 2^n - \sum_{j=1}^{n+k} a_j$.

Let $t=1$. Then, we only have to check the following two conditions.

1. $\sum_{j=1}^{n+k} a_j \cdot (2^j - 1) = \sum_{j=1}^{n-2} t_j(2^j - 1) - 2^{n-1} + 2^k - 3$.
2. $a_0 = \sum_{j=1}^{n-2} t_j + 1 + (2^k + 1) \cdot 2^n - \sum_{j=1}^{n+k} a_j$.

We can conclude that if

$$\sum_{j=1}^{n-2} t_j(2^j - 1) \geq 2^{n-1} - 2^k + 3,$$

then by letting $a_1 = \sum_{j=1}^{n-2} t_j(2^j - 1) - (2^{n-1} - 2^k + 3)$,

$$\begin{aligned} a_0 &= -\sum_{j=1}^{n-2} t_j(2^j - 2) + 2^{n+k} + 3 \cdot 2^{n-1} - 2^k + 4 \\ &\geq -2(2^{n-2} - 2) + 2^{n+k} + 3 \cdot 2^{n-1} - 2^k + 4 \\ &\geq 0. \end{aligned}$$

Therefore, if $\sum_{j=1}^{n-2} t_j(2^j - 1) \geq 2^{n-1} - 2^k + 3$, then $\sum_{j=1}^{n-2} t_j s_j + s_{n-1} + s_{n+k} \notin Ap(GT(n,k), s_0)$, and we obtain an upper bound of $\max(Ap(GT(n,k), s_0))$ for each $2 \leq k < n$ when $\sum_{j=1}^{n-2} t_j(2^j - 1) = 2^{n-1} - 2^k + 2$. □

Let $R_1(n,\delta)$ be the set of the sequences $(t_1, \cdots, t_{n+\delta}) \in A(n,\delta)$ such that if $t_{n+k} = 1$, the sequences satisfy the following conditions.

1. $t_n = \cdots = t_{n+k-1} = 0$.
2. If $t_{n-1} = 1$, $\sum_{j=1}^{n-2}(2^j - 1)t_j \leq 2^{n-1} - 2^k + 2$.

Note that $\delta = k$ by Theorem 2.7 in this case. Then, we obtain the following lemma.

Lemma 3.19. Let $2 \leq k < n$. Then

$$Ap(GT(n,k), s_0) = \{\sum_{j=1}^{n+\delta} t_j s_j | (t_1, \cdots, t_{n+\delta}) \in R_1(n,\delta)\}.$$

Proof. Note that $Ap(GT(n,k), s_0) \subseteq \{\sum_{j=1}^{n+\delta} t_j s_j | (t_1, \cdots, t_{n+\delta}) \in R_1(n,\delta)\}$ and therefore $\#\{\sum_{j=1}^{n+\delta} t_j s_j | (t_1, \cdots, t_{n+\delta}) \in R_1(n,\delta)\} \leq \#R_1(n,\delta)$. Then, $\#R_1(n,\delta)$ having the cardinality $s_0 = (2^k+1) \cdot 2^n - (2^k - 1)$ suffices for the proof. We classify the cases to obtain the cardinality as follows.

1. If $t_{n+\delta} = 0$, then it can be again classified into two cases, as follows.
   (a) Let $2 \notin \{t_1, \cdots, t_{n+\delta-1}\}$. Then, $\#\{(t_1, \cdots, t_{n+\delta}) \in R_1(n,\delta)\} = 2^{n+\delta-1}$ because $t_i \in \{0,1\}$ for all $1 \leq i \leq n + \delta - 1$.

(b) Let $2 \in \{t_1, \cdots, t_{n+\delta-1}\}$. If $t_i = 2$ for some $i \in \{1, \cdots, n+\delta-1\}$, then $t_j = 0$ for all $j < i$ and $t_j \in \{0, 1\}$ for all $i < j \leq n+\delta-1$. Hence, $\#R_1(n, \delta)$ in this case is $2^{n+\delta-i-1}$. Thus, we use summation to determine $\#R_1(n, \delta)$ for (1b): $\sum_{i=1}^{n+\delta-1} 2^{n+\delta-i-1} = 2^{n+\delta-1} - 1$.

Hence, $\#R_1(n, \delta) = 2^{n+\delta} - 1$ in case (1).

2. If $t_{n+\delta} = 1$, then it can be again classified into two cases as follows.

(a) Let $t_{n-1} = 1$. Then we can find the unique solution $(t_1, \cdots, t_{n-2})$ for each $0 \leq x \leq 2^{n-1} - 2$: $(t_1, \cdots, t_{n-2}) \in A(n, \delta)$ such that $\sum_{j=1}^{n-2}(2^j - 1)t_j = x$. Hence, the number of solutions $(t_1, \cdots, t_{n-2})$ satisfy the inequality $(t_1, \cdots, t_{n-2}) \in A(n, \delta)$ such that $\sum_{j=1}^{n-2}(2^j - 1)t_j \leq 2^{n-1} - 2^\delta + 2$ is $2^{n-1} - 2^\delta + 3$.

(b) Let $t_{n-1} = 0$. Then it can be again classified into two cases as follows.

i. If $2 \notin \{t_1, \cdots, t_{n-2}\}$ then $\#R_1(n, \delta) = 2^{n-2}$.

ii. If $2 \in \{t_1, \cdots, t_{n-2}\}$ then $\#R_1(n, \delta) = \sum_{i=1}^{n-2} 2^{n-2-i} = 2^{n-2} - 1$.

Note that we can verify that $\#R_1(n, \delta) = 2^{n-1} - 1$ in case (2b).

Hence, we can conclude that $\#R_1(n, \delta) = 2^{n+\delta} - 1 + 2^{n-1} - 2^\delta + 3 + 2^{n-1} - 1 = (2^\delta + 1) \cdot 2^n - (2^\delta - 1) = (2^k + 1) \cdot 2^n - (2^k - 1) = s_0$ because $\delta = k$ for $k \leq n$. $\square$

Lemma 3.19 determines the upper bound of $\max(Ap(GT(n, k), s_0))$, announced in Proposition 3.18, as the exact value for each n, k. In other words, we have the following result.

Lemma 3.20. Let $2 \leq k < n$ and let the coefficients $(t_1, \cdots, t_{n-2}, 1, 0, \cdots, 0, 1) \in A(n, \delta)$ be such that

$$\sum_{j=1}^{n-2}(2^j - 1)t_j = 2^{n-1} - 2^k + 2$$

as presented above. Then

$$\max(Ap(GT(n, k), s_0)) = \sum_{j=1}^{n-2} t_j s_j + s_{n-1} + s_{n+k}$$

and

$$F(GT(n, k)) = \max(Ap(GT(n, k), s_0)) - s_0 = \sum_{j=1}^{n-2} t_j s_j + s_{n-1} + s_{n+k} - s_0.$$

Example 3.21. $GT(5, 3) = \langle 281, 569, 1145, 2297, 4601, 9209, 18425, 36857, 73721 \rangle$ by Theorem 2.7. Because $2^{n-1} - 2^k + 2 = 10$, we conclude $\max(Ap(GT(5, 3), 281) = s_2 + s_3 + s_4 + s_8 = 81764$ and $F(GT(5, 3)) = \max(Ap(GT(5, 3), 281)) - 281 = 81764 - 281 = 81483$.

The following corollary can be obtained by applying Lemma 3.20 with a fixed $k = 2$.

Corollary 3.22. If $n > 2$, then $F(\langle\{5 \cdot 2^{n+i} - 3 | i \in \mathbb{N}\}\rangle) = 100 \cdot 2^{2n-2} - 5 \cdot 2^n - 9$.

Proof. Let us apply Lemma 3.20 by steps as follows.

1. Find $t_1, \cdots, t_{n-2}$ satisfying $\sum_{j=1}^{n-2}(2^j - 1)t_j = 2^{n-1} - 2$. Because $2 \cdot (2^{n-2} - 1) = 2^{n-1} - 2$, we have $(t_1, \cdots, t_{n-2}) = (0, \cdots, 0, 2)$.

2. Then we obtain $F(\langle\{5 \cdot 2^{n+i} - 3 | i \in \mathbb{N}, n \geq 2\}\rangle) = F(GT(n, 2)) = 2s_{n-2} + s_{n-1} + s_{n+2} - s_0 = 2 \cdot (5 \cdot 2^{2n-2} - 3) + (5 \cdot 2^{2n-1} - 3) + (5 \cdot 2^{2n+2} - 3) - (5 \cdot 2^n - 3) = 100 \cdot 2^{2n-2} - 5 \cdot 2^n - 9$.

□

Remark 3.23. Note that the proof of Corollary 3.22 is applicable only for $n > 2$ because $n - 2 \geq 1$. If $n = 2$ and $k = 2$, $F(GT(2, 2)) = 337$ using Lemma 3.16.

We introduce an interesting lemma that can be used to obtain the Frobenius number in a computationally more efficient manner in special cases.

Lemma 3.24. If $2 \leq k < n$, then $\sum_{j=k}^{n-1} s_j + s_{n+k} \in Ap(GT(n, k), s_0)$.

Proof. Let $k = n - 1 - \gamma$ such that $\gamma \geq 1$. Then, $2^{n-1} - 2^k + 3 = 2^{n-1-\gamma}(2^\gamma - 1) + 3$ and by $2^\gamma - 1 = \sum_{j=1}^{\gamma} 2^{j-1}$, we can establish the condition of $(t_1, \cdots, t_{n-2})$ such that $\sum_{j=1}^{n-2} t_j s_j + s_{n-1} + s_{n+k} \notin Ap(GT(n, k), s_0)$ for $2 \leq k < n$ by the inequality

$$\sum_{j=1}^{n-2}(2^j - 1)t_j \geq 2^{n-1-\gamma}(\sum_{j=1}^{\gamma} 2^{j-1}) + 3. \tag{4}$$

With the setting $t_{n-1-\gamma} = \cdots = t_{n-2} = 1$ and $t_i = 0$ for $i \neq n - 1 - \gamma, \cdots, n - 2$, the left-hand side of Equation (4) becomes smaller than the right-hand side of Equation (4) by $\gamma + 3$. Therefore, we can note that for $(t_1, \cdots, t_{n-2}, 1, 0, \cdots, 0, 1) \in A(n, \delta), \sum_{j=1}^{n-2} t_j s_j + s_{n-1} + s_{n+k} \in Ap(GT(n, k), s_0)$ if $t_k = \cdots = t_{n-2} = 1$ is satisfied. □

The reader may note that if $\gamma + 3$ in Lemma 3.24 is smaller than $2^k - 1$, then all of the coefficients of s_i for $k \leq i \leq n - 1$ are one in $\max(Ap(GT(n, k), s_0))$ because increasing a coefficient of s_i by one causes the value of $\sum_{j=1}^{n-2}(2^j - 1)t_j$ in (4) to increase by $2^i - 1$. Hence, by applying Lemma 3.20 and Lemma 3.24, we can obtain the following corollary, and for sufficiently small n, we can use the corollary to more easily calculate $\max(Ap(GT(n, k), s_0))$.

Corollary 3.25. Let $k = n - 1 - \gamma$, $2 \leq k < n$ and $\gamma + 3 = n + 2 - k \leq 2^k - 1$. In other words, $n \leq 2^k + k - 3$. In addition, let $(t_1, \cdots, t_{k-1}, 1, \cdots, 1(= t_{n-1}), 0, \cdots, 0, 1)$

be an element of $A(n,\delta)$ satisfying the equation

$$\sum_{j=1}^{k-1}(2^j-1)t_j = n+1-k. \tag{5}$$

Then

$$\max(Ap(GT(n,k),s_0)) = \sum_{j=1}^{k-1} t_j s_j + \sum_{j=k}^{n-1} s_j + s_{n+k}.$$

Example 3.26. Because $2 \le 3 < 5$ and $5 \le 2^3+3-3$, we can more easily obtain $\max(Ap(GT(5,3),281))$. Because $(2^2-1)\cdot 1 = 5+1-3 = 3$, we can conclude that $\max(Ap(GT(5,3),281)) = s_2+s_3+s_4+s_8 = 81764$.

Example 3.27. We can also consider $GT(7,3) = \langle\{1145, 2297, 4601, 9209, 18425, 36857, 73721, 147449, 294905, 589817, 1179641\}\rangle$. Because $7 \le 2^3+3-3$, we can easily notice that $2+1\cdot(2^2-1) = 5 = n+1-k$ leads to $\max(Ap(GT(7,3),1145)) = 2s_1+s_2+s_3+s_4+s_5+s_6+s_{10} = 1327048$ and $F(GT(7,3)) = \max(Ap(GT(7,3),1145)) - 1145 = 1325903$.

Example 3.28. Let $(n,k) = (3,2)$. Because $n-2 = 1$, the solutions of the inequality $\sum_{j=1}^{n-2}(2^j-1)t_j \le 2^{3-1}-2^2+2$ are $t_1 = 0, 1$ and 2. Hence, we can obtain

$$\begin{aligned}
&Ap(GT(3,2),37)\\
&= \{0, s_1, 2s_1, s_2, s_1+s_2,\\
&\quad 2s_1+s_2, 2s_2, s_3, s_1+s_3, 2s_1+s_3,\\
&\quad s_2+s_3, s_1+s_2+s_3, 2s_1+s_2+s_3, 2s_2+s_3, 2s_3,\\
&\quad s_4, s_1+s_4, 2s_1+s_4, s_2+s_4, s_1+s_2+s_4,\\
&\quad 2s_1+s_2+s_4, 2s_2+s_4, s_3+s_4, s_1+s_3+s_4, 2s_1+s_3+s_4,\\
&\quad s_2+s_3+s_4, s_1+s_2+s_3+s_4, 2s_1+s_2+s_3+s_4, 2s_2+s_3+s_4, 2s_3+s_4,\\
&\quad 2s_4, s_5, s_1+s_5, 2s_1+s_5, s_2+s_5,\\
&\quad s_1+s_2+s_5, 2s_1+s_2+s_5\}.
\end{aligned}$$

Note that $\#Ap(GT(3,2),37) = 37$ and $\max(Ap(GT(3,2),37)) = 2s_1+s_2+s_5$.

We already know from Lemma 3.16 that $\max(Ap(GT(n,k),s_0)) = s_1+s_{n+k}$ if $k = n$ and $F(GT(n,n)) = s_1+s_{n+n}-s_0 = (2^n+1)\cdot 2^n(2^{2n}+1)-(2^n-1)$. Then $R_2(n,\delta)$ can be defined as the set of sequences $(t_1,\cdots,t_{n+\delta}) \in A(n,\delta)$ that satisfy the condition that if $t_{2n} = 1$, then $t_1 = 1$ and $t_2 = \cdots = t_{2n-1} = 0$. Note that $\delta = k$ by Theorem 2.7 in this case. Then, we can obtain the following lemma.

Lemma 3.29. Let $k = n \ge 2$. Then

$$Ap(GT(n,k),s_0) = \{\sum_{j=0}^{n+\delta} t_j s_j | (t_1,\cdots,t_{n+\delta}) \in R_2(n,\delta)\}.$$

Proof. We can classify the cases as follows.

1. If $t_{2n} = 1$, then we can choose $t_1 \in \{0, 1\}$ such that there are two cases.
2. If $t_{2n} = 0$, then the situation can be again classified into two cases as follows.
   (a) Let $2 \notin \{t_1, \cdots, t_{2n-1}\}$. Then we can choose the coefficients $(t_1, \cdots, t_{2n-1})$, and each coefficient has the condition $t_i \in \{0, 1\}$ for all i. Hence, there are 2^{2n-1} cases.
   (b) Let $2 \in \{t_1, \cdots, t_{2n-1}\}$. Then, if $t_i = 2$, we can choose the coefficients $(t_{i+1}, \cdots, t_{2n-1})$, and each coefficient has the condition $t_j \in \{0, 1\}$ for all $i < j \leq 2n - 1$. Hence, there are 2^{2n-1-i} cases for each i. Then, we can express all of the cases by summation: $\sum_{i=1}^{2n-1} 2^{2n-1-i} = \sum_{i=0}^{2n-2} 2^i = 2^{2n-1} - 1$.

Therefore, we can obtain $2 + 2^{2n-1} + 2^{2n-1} - 1 = 2^{2n} + 1$ by cases (1) and (2). Note that $2^{2n} + 1 = (2^n + 1) \cdot 2^n - (2^n - 1) = s_0$, which completes the proof. □

Example 3.30. Let $(n, k) = (2, 2)$. Then we can obtain

$$\begin{aligned}
&Ap(GT(2, 2), 17) \\
&= \{0, s_1, 2s_1, s_2, s_1 + s_2, \\
&\quad 2s_1 + s_2, 2s_2, s_3, s_1 + s_3, 2s_1 + s_3, \\
&\quad s_2 + s_3, s_1 + s_2 + s_3, 2s_1 + s_2 + s_3, 2s_2 + s_3, 2s_3, \\
&\quad s_4, s_1 + s_4\}.
\end{aligned}$$

Note that $\#Ap(GT(2, 2), 17) = 17$ and $\max(Ap(GT(2, 2), 17)) = s_1 + s_4$.

3.3. The Case of $n \neq 0$ and $2 \neq k > n$

Similar to Lemmas 2.6 and 3.14, we have the following corollary, which is useful in this case.

Corollary 3.31. There exists at least one tuple $(a_0, a_1, \cdots, a_{n+k})$ with $a_0, a_1, \cdots, a_{n+k-1} \in \mathbb{N}$ and $t \in \mathbb{Z}$ satisfying

1. $\sum_{j=1}^{n+k-1} a_j \cdot (2^j - 1) = \sum_{j=1}^{n+k-1} t_j (2^j - 1) + t(2^k - 1 - 2^{n+k} - 2^n)$,
2. $a_0 = \sum_{j=1}^{n+k-1} t_j - 1 + t(2^k + 1) \cdot 2^n - \sum_{j=1}^{n+k-1} a_j$,

if and only if $\sum_{j=1}^{n+k-1} t_j s_j \notin Ap(GT(n, k), s_0)$.

Proof. The proof is similar to that of Lemma 3.14. □

Let $t = 1$. Then, we only have to check the following two conditions.

1. $\sum_{j=1}^{n+k-1} a_j \cdot (2^j - 1) = \sum_{j=1}^{n+k-1} t_j(2^j - 1) + 2^k - 1 - 2^{n+k} - 2^n$.

2. $a_0 = \sum_{j=1}^{n+k-1} t_j - 1 + (2^k + 1) \cdot 2^n - \sum_{j=1}^{n+k-1} a_j$.

We can conclude that if

$$\sum_{j=1}^{n+k-1} t_j(2^j - 1) + 2^k - 1 - 2^{n+k} - 2^n \geq 0,$$

then by letting $a_1 = \sum_{j=1}^{n+k-1} t_j(2^j - 1) + 2^k - 1 - 2^{n+k} - 2^n$,

$$\begin{aligned} a_0 &= -\sum_{j=1}^{n+k-1} t_j(2^j - 2) + 2^{n+k+1} + 2^{n+1} - 2^k \\ &\geq -2(2^{n+k-1} - 2) + 2^{n+k+1} + 2^{n+1} - 2^k \\ &\geq 0. \end{aligned}$$

Hence, $\sum_{j=1}^{n+k-1} t_j s_j \notin Ap(GT(n,k), s_0)$. Note that $2^{n+k} + 2^n - 2^k + 1 = (2^{n+k-1} - 1) + (2^{n+k-1} + 2^n - 2^k + 2) < 2(2^{n+k-1} - 1)$ if $n \neq 0$ and $2 \neq k > n$. Hence, we obtain the least upper bound of $\max(Ap(GT(n,k), s_0))$ for each $n \neq 0$ and $2 \neq k > n$.

Proposition 3.32. Let $n \neq 0$ and $2 \neq k > n$, and let the coefficients $(t_1, \cdots, t_{n+k-2}, 1) \in A(n, \delta)$ be such that

$$\sum_{j=1}^{n+k-2} (2^j - 1)t_j = 2^{n+k-1} + 2^n - 2^k + 1.$$

Then

$$\max(Ap(GT(n,k), s_0)) \leq \sum_{j=1}^{n+k-2} t_j s_j + s_{n+k-1}.$$

$R_3(n, \delta)$ can be defined by the set of the sequences $(t_1, \cdots, t_{n+\delta}) \in A(n, \delta)$ which satisfy the condition that if $t_k = t_{k+1} = \cdots = t_{n+k} = 1$,

$$\sum_{j=1}^{k-1} (2^j - 1)t_j \leq 2^n + n.$$

Note that $\delta = k - 1$ by Theorem 2.7 in this case. Then, we can obtain the following lemma.

Lemma 3.33. Let $n \neq 0$ and $2 \neq k > n$. Then

$$Ap(GT(n,k), s_0) = \{\sum_{j=1}^{n+\delta} t_j s_j | (t_1, \cdots, t_{n+\delta}) \in R_3(n, \delta)\}.$$

Proof. Note that $Ap(GT(n,k),s_0) \subseteq \{\sum_{j=1}^{n+\delta} t_j s_j | (t_1, \cdots, t_{n+\delta}) \in R_3(n,\delta)\}$ and $\#\{\sum_{j=1}^{n+\delta} t_j s_j | (t_1, \cdots, t_{n+\delta}) \in R_3(n,\delta)\} \leq \#R_3(n,\delta)$. Then, $\#R_3(n,\delta)$ having the cardinality $s_0 = (2^k+1)\cdot 2^n - (2^k-1)$ suffices for the proof. We can classify the cases to verify the cardinality as follows.

1. Let $(t_k, \cdots, t_{n+k-1}) \in \{0,1\}^n$ and assume there exists at least one $t_i \in \{t_k, \cdots, t_{n+k-1}\}$ such that $t_i = 0$. Then, there are $2^n - 1$ cases to choose $\{t_k, \cdots, t_{n+k-1}\}$, and to choose $\{t_1, \cdots, t_{k-1}\}$, the possibilities can be again classified into two cases as follows.

   (a) Let $2 \notin \{t_1, \cdots, t_{k-1}\}$. Then there are 2^{k-1} cases to choose $\{t_1, \cdots, t_{k-1}\}$.

   (b) Let $2 \in \{t_1, \cdots, t_{k-1}\}$. Then, if $t_i = 2$, there are 2^{k-i-1} cases to choose $\{t_{i+1}, \cdots, t_{k-1}\}$. Hence, $\sum_{i=1}^{k-1} 2^{k-i-1} = 2^{k-1} - 1$.

   Therefore, we can check that case (1) has $(2^{k-1} + 2^{k-1} - 1)(2^n - 1) = (2^k - 1)(2^n - 1) = 2^{n+k} - 2^n - 2^k + 1$ cases.

2. Let $t_k = t_{k+1} = \cdots = t_{n+k-1} = 1$. Then, we can obtain the unique solution $(t_1, \cdots, t_{k-1})$ for each $0 \leq x \leq 2^n + n$ such that $\sum_{j=1}^{k-1}(2^j - 1)t_j = x$. Hence, the number of solutions $(t_1, \cdots, t_{k-1})$ satisfy the inequality $\sum_{j=1}^{k-1}(2^j - 1)t_j \leq 2^n + n$ is $2^n + n + 1$.

3. Let $2 \in \{t_k, \cdots, t_{n+k-2}\}$. If $t_i = 2$, there are $2^{n+k-i-1} - 1$ cases to choose $\{t_{i+1}, \cdots, t_{n+k-1}\}$ because at least one $t_j \in \{t_{i+1}, \cdots, t_{n+k-1}\}$ must be 0. Then we can obtain the summation: $\sum_{i=k}^{n+k-2}(2^{n+k-i-1} - 1) = \sum_{i=1}^{n-1} 2^i - (n-1) = 2^n - 2 - (n-1) = 2^n - n - 1$.

You can verify easily that the equation

$$(2^{n+k} - 2^n - 2^k + 1) + (2^n + n + 1) + (2^n - n - 1) = 2^{n+k} + 2^n - (2^k - 1) = s_0$$

holds, and this completes the proof. □

Lemma 3.33 determines the least upper bound of $\max(Ap(GT(n,k),s_0))$, announced in Proposition 3.32, to be the exact value for each n,k. In other words,

Lemma 3.34. Let $n \neq 0$ and $2 \neq k > n$ and let the coefficients $(t_1, \cdots, t_{n+k-2}, 1) \in A(n,\delta)$ be such that

$$\sum_{j=1}^{n+k-2} (2^j - 1)t_j = 2^{n+k-1} + 2^n - 2^k + 1.$$

Then

$$\max(Ap(GT(n,k),s_0)) = \sum_{j=1}^{n+k-2} t_j s_j + s_{n+k-1}$$

and

$$F(GT(n,k)) = \max(Ap(GT(n,k),s_0)) - s_0 = \sum_{j=1}^{n+k-2} t_j s_j + s_{n+k-1} - s_0.$$

Note that

$$2^{n+k-1} + 2^n - 2^k + 1 = \sum_{j=k}^{n+k-2} (2^j - 1) + (2^n + n).$$

Further, we can easily deduce the fact that $2^n + n < 2^{n+1} - 1 \leq 2^k - 1$ for $n \neq 1$ and $2^n + n = 2^{n+1} - 1 < 2^k - 1$ for $n = 1$. Therefore, we can reduce the effort required to obtain $\max(Ap(GT(n,k),s_0))$ for $n \neq 0$ and $2 \neq k > n$.

Lemma 3.35. Let $n \neq 0$ and $2 \neq k > n$, and let the coefficients $(t_1, \cdots, t_{k-1}, 1, \cdots, 1) \in A(n,\delta)$ be such that

$$\sum_{j=1}^{k-1} (2^j - 1)t_j = 2^n + n.$$

Then

$$\max(Ap(GT(n,k),s_0)) = \sum_{j=1}^{k-1} t_j s_j + \sum_{j=k}^{n+k-1} s_j \text{ and}$$

$$F(GT(n,k)) = \max(Ap(GT(n,k),s_0)) - s_0 = \sum_{j=1}^{k-1} t_j s_j + \sum_{j=k}^{n+k-1} s_j - s_0.$$

Example 3.36. We have that $GT(2,3) = \left\langle \{29, 65, 137, 281, 569\} \right\rangle$ and $2^n + n = 2^2 + 2 = 6$. Thus, $t_2 = 2$ and $t_i = 0$ for all $i < k$ and $i \neq 2$. Then we can obtain $\max(Ap(GT(2,3),s_0)) = 2s_2 + s_3 + s_4 = 1124$ and $F(GT(2,3)) = \max(Ap(GT(2,3),s_0)) - s_0 = 1124 - 29 = 1095$ by Lemma 3.35.

Example 3.37. Let $(n,k) = (2,3)$. Because the solutions of the inequality $\sum_{j=1}^{k-1}(2^j - 1)t_j \leq 2^2 + 2$ are $(0,0,\cdots,0), (1,0,\cdots,0), (2,0,\cdots,0), (0,1,0,\cdots,0), (1,1,0,\cdots,0), (2,1,0,\cdots,0)$ and $(0,2,0,\cdots,0)$, we can obtain

$$\begin{aligned}
&Ap(GT(2,3),29)\\
&= \{0, s_1, 2s_1, s_2, s_1+s_2,\\
&\quad 2s_1+s_2, 2s_2, s_3, s_1+s_3, 2s_1+s_3,\\
&\quad s_2+s_3, s_1+s_2+s_3, 2s_1+s_2+s_3, 2s_2+s_3, 2s_3,\\
&\quad s_4, s_1+s_4, 2s_1+s_4, s_2+s_4, s_1+s_2+s_4,\\
&\quad 2s_1+s_2+s_4, 2s_2+s_4, s_3+s_4, s_1+s_3+s_4, 2s_1+s_3+s_4,\\
&\quad s_2+s_3+s_4, s_1+s_2+s_3+s_4, 2s_1+s_2+s_3+s_4, 2s_2+s_3+s_4\}.
\end{aligned}$$

Note that $\#Ap(GT(2,3),29) = 29$ and $\max(Ap(GT(2,3),29)) = 2s_2 + s_3 + s_4$.

3.4. The Case of $n = 1$ and $k = 2$

The Frobenius number and the Apéry set can be verified easily by the following lemma, which follows easily from Theorem 2.7.

Lemma 3.38. Let $n = 1$ and $k = 2$. Then we have
$GT(1,2) = \langle \{(2^2+1)\cdot 2^1 - (2^2-1), (2^2+1)\cdot 2^2 - (2^2-1), (2^2+1)\cdot 2^3 - (2^2-1)\} \rangle$ and $Ap(GT(1,2),7) = \{0, 17, 34, 37, 54, 71, 74\}$. Hence $\max(Ap(GT(1,2),7)) = 74$ and $F(GT(1,2)) = 74 - 7 = 67$.

4. Pseudo-Frobenius Numbers and Type

Because the Apéry set depends on both n and k, it is very difficult to obtain pseudo-Frobenius numbers and type of $GT(n,k)$ for arbitrary n and k. Hence, we show some results with fixed k as in [23, 24], or with fixed n.

First, let us recall the definition of pseudo-Frobenius numbers (see Definition 1.8). Afterwards, we introduce an order relation on the Apéry set, which, in turn, gives rise to the notion of maximal elements of an Apéry set with respect to that relation. Finally, we give a lemma relating the pseudo-Frobenius numbers to the Apéry set.

Definition 4.1. Let S be a numerical semigroup. Then the order relation $\leq_S$ is defined as $a \leq_S b$ if $b - a \in S$, and maximal elements in the Apéry set of S are defined as $maximals_{\leq_S}(\mathrm{Ap}(S,x)) = \{w \in \mathrm{Ap}(S,x) | w' - w \notin \mathrm{Ap}(S,x)\backslash\{0\}$ for all $w' \in \mathrm{Ap}(S,x)\}$.

Lemma 4.2. *([26])*. Let S be a numerical semigroup. Then, for any nonzero element x of S, we have

$$PF(S) = \{w - x | w \in maximals_{\leq S}(\mathrm{Ap}(S,x))\},$$

Let n be an integer greater than or equal to 3. Note that the maximal elements in $R_1(n,2)$ are

$$\Bigg\{\left\{2s_{n-2} + s_{n-1} + s_{n+2}, 2s_{n-3} + s_{n-2} + s_{n-1} + s_{n+2}, \cdots, 2s_1 + \sum_{k=2}^{n+2} s_k\right\}$$
$$\bigcup \left\{2s_2 + s_3 + \cdots + s_{n+1}, 2s_1 + s_2 + \cdots + s_{n+1}\right\}\Bigg\},$$

and we have

$$\left\{2s_{n-2}+s_{n-1}+s_{n+2}, 2s_{n-3}+s_{n-2}+s_{n-1}+s_{n+2}, \cdots, 2s_1+\sum_{k=2}^{n+2} s_k\right\}$$
$$=\left\{2s_{n-2}+s_{n-1}+s_{n+2}-3k \mid k\in\{0,1,\cdots,n-3\}\right\},$$

$$\left\{2s_2+s_3+\cdots+s_{n+1}, 2s_1+s_2+\cdots+s_{n+1}\right\}=\left\{2s_2+s_3+\cdots+s_{n+1}-3k \mid k\in\{0,1\}\right\}.$$

Hence, we state a theorem related to pseudo-Frobenius numbers of $GT(n,2)$ for $n\geq 3$.

Theorem 4.3. Let $n\in\mathbb{N}$, $n\geq 3$. Then

$$\begin{aligned}&maximals_{\leq GT(n,2)}(Ap(GT(n,2),s_0))\\&=\left\{2s_{n-2}+s_{n-1}+s_{n+2}-3k \mid k\in\{0,1,\cdots,n-3\}\right\}\\&\quad\bigcup\left\{2s_2+s_3+\cdots+s_{n+1}-3k \mid k\in\{0,1\}\right\}.\end{aligned}$$

Proof. Note that $2s_{n-2}+s_{n-1}+s_{n+2}-3i\in maximals_{\leq GT(n,2)}(Ap(GT(n,2),s_0))$ for $i\in\{0,1,\cdots,n-3\}$ because $3(n-3)<s_0=5\cdot 2^n-3$ for any $n\geq 2$. In addition, we know that

$$\begin{aligned}&(2s_{n-2}+s_{n-1}+s_{n+2})-(2s_2+s_3+\cdots+s_{n-1}+s_n+s_{n+1})\\&=s_{n+2}-s_{n+1}-s_n+s_{n-2}-s_{n-3}-\cdots-s_3-2s_2\\&=s_{n+2}-s_{n+1}-s_n+s_{n-2}-s_{n-3}-\cdots-s_4-2s_3+3\\&=\cdots\\&=s_{n+2}-s_{n+1}-s_n+s_{n-2}-2s_{n-3}+3(n-5)\\&=s_{n+2}-s_{n+1}-s_n+3(n-4)\\&=s_{n+1}-s_n+3(n-3)\\&=s_n+3(n-2).\end{aligned}$$

Hence, we only have to show that $s_n+3i\notin Ap(GT(n,2),s_0)$ for $1\leq i\leq n-1$. Note that $3i<s_0$ for any $1\leq i\leq n-1$ and hence $s_n<s_n+3i<s_0+s_n$ and if $t_j=0$ for any $j\leq n$, $\sum_{j=1}^{n-1} t_js_j<s_n$ for any $(t_1,\cdots,t_{n+2})\in R_1(n,2)$. This completes the proof of $s_n+3i\notin Ap(GT(n,2),s_0)$, which leads to $\{2s_2+s_3+\cdots+s_{n+1}, 2s_2+s_3+\cdots+s_{n+1}-3\}\subseteq maximals_{\leq GT(n,2)}(Ap(GT(n,2),s_0))$.
Finally, we exclude the other candidates for elements in

$maximals_{\leq GT(n,2)}(Ap(GT(n,2),s_0))$.

$$\begin{aligned}
&(2s_{n-2}+s_{n-1}+s_{n+2})-(2s_n+s_{n+1})=s_n,\\
&(2s_{n-3}+s_{n-2}+s_{n-1}+s_{n+2})-(2s_{n-1}+s_n+s_{n+1})\\
&\ =(2s_{n-2}+s_{n-1}+s_{n+2}-3)-(2s_n+s_{n+1}-3)=s_n,\\
&\vdots\\
&(2s_1+s_2+\cdots+s_{n-1}+s_{n+2})-(2s_3+s_4+\cdots+s_{n+1})=s_n.
\end{aligned}$$

□

In conclusion, we obtain pseudo-Frobenius numbers when $k=2$ and $n\geq 3$.

Theorem 4.4. Let $n\geq 3$. Then we have

$$\begin{aligned}
PF(GT(n,2)) = &\Big\{F(GT(n,2))-3i \mid i\in\{0,1,\cdots,n-3\}\Big\}\\
&\bigcup\Big\{2s_{n+1}-s_0-3(n-1), 2s_{n+1}-s_0-3n\Big\}.
\end{aligned}$$

Further, we obtain the type of $GT(n,2)$ when $n\geq 3$.

Corollary 4.5. Let $n\geq 3$. Then we have

$$t(GT(n,2))=n.$$

In the same manner, we obtain pseudo-Frobenius numbers and type when $k=n\geq 2$.

Theorem 4.6. Let $n\geq 2$. Then we have

$$\begin{aligned}
PF(GT(n,n)) = &\Big\{F(GT(n,n))\Big\}\\
&\bigcup\Big\{2s_{2n-1}-s_0-i(2^n-1) \mid i\in\{0,1,\cdots,2n-2\}\Big\},
\end{aligned}$$

Corollary 4.7. Let $n\geq 2$. Then we have

$$t(GT(n,n))=2n.$$

In addition, we obtain pseudo-Frobenius numbers and type when $k\geq 4$ and $n=2$.

Theorem 4.8. Let $k\geq 4$. Then we have

$$\begin{aligned}
PF(GT(2,k)) = &\Big\{F(GT(k,2))\Big\}\\
&\bigcup\Big\{2s_{k-1}+s_{k+1}-s_0-i(2^k-1) \mid i\in\{2,3,\cdots,k-2\}\Big\}\\
&\bigcup\Big\{2s_k-s_0-6(2^k-1)\Big\}.
\end{aligned}$$

Corollary 4.9. Let $k \geq 4$. Then we have

$$t(GT(2,k)) = k-1.$$

5. Conclusion

We summarize all the previous results and three principal theorems.

The first principal result, already stated in Theorem 2.7, concerns the minimal system of generators of $GT(n,k)$.

By Lemmas 3.13, 3.16, 3.20, 3.35, and 3.38, we obtain $\max(Ap(GT(n,k),s_0))$ and $F(GT(n,k))$ for the general case stated in Theorem 5.1 as follows. Note that the case for $n \neq 0$ and $k = 1$ is in [23].

Finally, in Theorems 4.4, 4.6, and 4.8, we have pseudo-Frobenius numbers of $GT(n,2), GT(2,k)$, and $GT(n,n)$. In addition, we have the type of $GT(n,2)$, $GT(2,k)$, and $GT(n,n)$ in Corollaries 4.5, 4.7, and 4.9.

For convenience, we restate the theorems that we have described for the largest part: the Frobenius numbers and the Apéry set.

Theorem 5.1. Let $n \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0,1\}$ and let $GT(n,k) = \left\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \mathbb{N}\} \right\rangle$ be the extended Thabit numerical semigroup associated with n,k. Then, the maximal element of the Apéry set $Ap(GT(n,k),s_0)$ is

$$\begin{cases} s_1 & \text{if } \ n=0, \\ s_n + s_{n+1} & \text{if } \ n \neq 0 \ \ \textit{and} \ \ k=1, \\ \sum_{j=1}^{n-2} t_j s_j + s_{n-1} + s_{n+k} & \\ \text{where} \sum_{j=1}^{n-2}(2^j-1)t_j = 2^{n-1} - 2^k + 2 & \text{if } \ 2 \leq k < n, \\ s_1 + s_{2n} & \text{if } \ n = k \geq 2, \\ \sum_{j=1}^{k-1} t_j s_j + \sum_{i=k}^{n+k-1} s_i & \\ \text{where} \sum_{j=1}^{k-1}(2^j-1)t_j = 2^n + n & \text{if } \ n \neq 0 \text{ and } 2 \neq k > n, \\ 74 & \text{if } \ n=1 \ \ \textit{and} \ \ k=2. \end{cases}$$

Moreover, the Frobenius number $F(GT(n,k))$ is

$$\begin{cases} s_1 - 2 & \text{if } \ n=0, \\ s_n + s_{n+1} - s_0 & \text{if } \ n \neq 0 \ \ \text{and} \ \ k=1, \\ \sum_{j=1}^{n-2} t_j s_j + s_{n-1} + s_{n+k} - s_0 & \\ \text{where} \sum_{j=1}^{n-2}(2^j-1)t_j = 2^{n-1} - 2^k + 2 & \text{if } \ 2 \leq k < n, \\ s_1 + s_{2n} - s_0 & \text{if } \ n = k \geq 2, \\ \sum_{j=1}^{k-1} t_j s_j + \sum_{i=k}^{n+k-1} s_i - s_0 & \\ \text{where} \sum_{j=1}^{k-1}(2^j-1)t_j = 2^n + n & \text{if } \ n \neq 0 \text{ and } 2 \neq k > n, \\ 67 & \text{if } \ n=1 \ \ \text{and} \ \ k=2. \end{cases}$$

Note that $(t_1, \cdots, t_{n+\delta}) \in A(n,\delta)$, as stated in Lemma 3.6.

Note the following statements.

1. In Theorem 5.1, for fixed n, k, $\sum_{j=1}^{n-2}(2^j-1)t_j = 2^{n-1}-2^k+2$ and $\sum_{j=1}^{k-1}(2^j-1)t_j = 2^n+n$ determine all t_j, and this determines $\sum_{j=1}^{n-2} t_j s_j$ and $\sum_{j=1}^{k-1} t_j s_j$. Hence, we obtain $\max(Ap(GT(n,k), s_0))$ and $F(GT(n,k))$ for associated cases.

2. For sufficiently small n in the case of $2 \leq k < n$, we can simplify the calculation for obtaining $\max(Ap(GT(n,k), s_0))$ and $F(GT(n,k))$ using Corollary 3.25.

By Lemmas 3.6, 3.13, 3.19, 3.29, 3.33, and 3.38, we obtain the Apéry set for $GT(n,k)$ in explicit form stated in Theorem 5.2 as follows. Note that the case for $n \neq 0$ and $k = 1$ is in [23].

Theorem 5.2. Let $n \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0\}$, and let $GT(n,k) = \big\langle \{(2^k+1)\cdot 2^{n+i} - (2^k-1) | i \in \mathbb{N}\}$ *big* $>$ be an extended Thabit numerical semigroup associated with n, k. In addition, let $A(n,\delta)$ be the set of sequences $(t_1, \cdots, t_{n+\delta}) \in \{0,1,2\}^{n+\delta}$ that satisfy the following conditions:

1. $t_{n+\delta} = 0$ or 1.

2. If $t_j = 2$, then $t_i = 0$ for all $i < j$.

Moreover, let $R(n), R_1(n,\delta), R_2(n,\delta), R_3(n,\delta) \subseteq A(n,\delta)$ be the set of sequences defined by the following additional statements.

1. Each sequence of $R(n)$ satisfies the following conditions

   (a) $t_{n+1} \in \{0,1\}$,

   (b) If $t_n = 2$, then $t_{n+1} = 0$,

   (c) If $t_n = t_{n+1} = 1$, $t_i = 0$ for all $1 \leq i < n$.

2. If $t_{n+\delta} = 1$, then each sequence of $R_1(n,\delta)$ satisfies the following conditions

   (a) $t_n = \cdots = t_{n+\delta-1} = 0$,

   (b) $\sum_{j=1}^{n-2}(2^j-1)t_j \leq 2^{n-1} - 2^k + 2$.

3. If $t_{2n} = 1$, then each sequence of $R_2(n,\delta)$ satisfies the following conditions

   (a) $t_1 = 1$,

   (b) $t_2 = \cdots = t_{2n-1} = 0$.

4. If $t_{\delta+1} = t_{\delta+2} = \cdots = t_{n+\delta} = 1$, then each sequence of $R_3(n,\delta)$ satisfies the following inequality $\sum_{j=1}^{\delta}(2^j-1)t_j \leq 2^n + n$.

Then, we have the explicit form of the Apéry set for extended Thabit numerical semigroups. Namely, $Ap(GT(n,k),s_0)$ is

$$\begin{cases} \{0, s_1\}, & \text{if } n=0, \\ \{\sum_{j=1}^{n+1} t_j s_j | (t_1,\cdots,t_{n+1}) \in R(n)\}, & \text{if } n \neq 0 \ \ and \ \ k=1, \\ \{\sum_{j=1}^{n+\delta} t_j s_j | (t_1,\cdots,t_{n+\delta}) \in R_1(n,\delta)\}, & \text{if } 2 \leq k < n, \\ \{\sum_{j=1}^{n+\delta} t_j s_j | (t_1,\cdots,t_{n+\delta}) \in R_2(n,\delta)\}, & \text{if } n = k \geq 2, \\ \{\sum_{j=1}^{n+\delta} t_j s_j | (t_1,\cdots,t_{n+\delta}) \in R_3(n,\delta)\}, & \text{if } n \neq 0 \text{ and } 2 \neq k > n, \\ \{0,17,34,37,54,71,74\}, & \text{if } n=1 \ \ and \ \ k=2. \end{cases}$$

Hence, we can obtain $g(GT(n,k))$ using the formula $g(S) = \frac{1}{x}(\sum_{w \in Ap(S,x)} w) - \frac{x-1}{2}$ in Lemma 3.4.

Finally, we suggest some open problems related to this paper.

Open problem 5.3. The Frobenius problem for a more extended Thabit numerical semigroup such as $\langle\{(2k_1+1)\cdot 2^{n+i} - (2k_2-1)|i \in \mathbb{N}\}\rangle$ for any $n \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{N}\backslash\{0\}$ is still open. We have tried to solve this problem, but doing so requires far more extensive calculation. We hope that there are more efficient methods of solving it.

Open problem 5.4. Let A be the minimal system of generators of S and B a proper subset of A. In other words, $\langle A\rangle = S$ and $\langle B\rangle$ is a proper subset of S, but $F(B) = F(A)$ might be satisfied in this condition. For example, let $S = \langle\{(2^3+1)\cdot 2^{2+i} - (2^3-1)|i \in \mathbb{N}\}\rangle$. Then, the minimal system of generators of S is $A = \langle\{(2^3+1)\cdot 2^{2+i} - (2^3-1)|i \in \{0,1,2,3,4\}\}\rangle$, because $4 = 2+3-1$. Letting $B = \langle\{(2^3+1)\cdot 2^{2+i} - (2^3-1)|i \in \{0,1,2,3\}\}\rangle$, we can easily observe $F(B) = F(A) = 1095$. Finding the condition of the proper subset B of the minimal system of generators A such that $F(B) = F(A)$ for a general numerical semigroup A is a further interesting problem.

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