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EXACT FORMULAS FOR THE GENERALIZED SUM-OF-DIVISORS FUNCTIONS

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Abstract

We prove new exact formulas for the generalized sum-of-divisors functions, $\sigma_\alpha(x) := \sum_{d|x} d^\alpha$, for fixed $\alpha \in \mathbb{R}$ and any integers $x \geq 1$. The formulas for $\sigma_\alpha(x)$ involve finite sums over the primes $p \leq x$ with terms involving the r-order harmonic number sequences, $H_n^{(r)} := \sum_{k=1}^n k^{-r}$, and the Ramanujan sums, $c_d(x) := \sum_{r|(d,x)} r\mu(d/r)$. We focus on the computational aspects of the resulting exact expressions with emphasis on obtaining new, and more precise asymptotic properties satisfied by the scaled summatory functions $\sum_{n \leq x} \sigma_\alpha(n) n^{-\beta}$ for integers $\alpha > 1$ and $2 \leq \beta \leq \alpha$.

1. Introduction

We begin our search for interesting formulas for the *generalized sum-of-divisors* functions, $\sigma_\alpha(x)$ for $\alpha \in \mathbb{R}$, by expanding the partial sums of the Lambert series which generate these functions in the form of [6, §17.10] [13, §27.7]

$$L_\alpha(q) := \sum_{n \geq 1} \frac{n^\alpha q^n}{1-q^n} = \sum_{m \geq 1} \sigma_\alpha(m) q^m, \alpha \in \mathbb{R}, |q| < 1. \tag{1}$$

We find new expansions of the partial sums of Lambert series generating functions in Equation (1) which generate our special arithmetic functions as

$$\sigma_\alpha(x) = [q^x] \left(\sum_{n=1}^{x} \frac{n^\alpha q^n}{1-q^n} \right) = \sum_{d|x} d^\alpha, x \geq 1, \alpha \in \mathbb{Z}^+ \cup \{0\}. \tag{2}$$

The identity in Equation (2) is proved by evaluating the series coefficients of the Lambert series expansions in Equation (1) considering expansions of the truncated generating function series, each scaled by a factor of q^n.

The formulas we arrive at to express $\sigma_\alpha(x)$ when $x \geq 1$ involving sums over prime powers $p^k \leq x$ follow from repeated use of properties of the sequence of *cyclotomic* *polynomials*, $\Phi_n(q)$. For $n \geq 1$ and any indeterminate $q \in \mathbb{C}$, these polynomials are defined by (see [4, §3] and [10, §13.2])

$$\Phi_n(q) := \prod_{\substack{1 \leq k \leq n \\ (k,n)=1}} \left(q - e^{2\pi\imath\frac{k}{n}}\right). \tag{3}$$

For each integer $n \geq 1$ we have an initial insight provided by the factorizations

$$q^n - 1 = \prod_{d|n} \Phi_d(q). \tag{4}$$

Equivalently we have that

$$\Phi_n(q) = \prod_{d|n} (q^d - 1)^{\mu(n/d)}, \tag{5}$$

where $\mu(n)$ denotes the *Möbius function*.

If $n = p^m r$ with p prime and $(p, r) = 1$, we have the identity that $\Phi_n(q) = \Phi_{pr}(q^{p^{m-1}})$. In later results stated and proved within the article, we use the next few known expansions of the cyclotomic polynomials which reduce the order n of the polynomials by exponentiation of the indeterminate q when n contains a factor of a prime power. A short list summarizing these transformation properties is given as follows for p an odd prime, integers $r, k \geq 1$, and where $p \nmid r$:

$$\Phi_{2p}(q) = \Phi_p(-q), \Phi_{p^k}(q) = \Phi_p\left(q^{p^{k-1}}\right), \Phi_{p^k r}(q) = \Phi_{pr}\left(q^{p^{k-1}}\right), \Phi_{2^k}(q) = q^{2^{k-1}} + 1. \tag{6}$$

The next definitions expand our Lambert series generating functions further by factoring its terms by the cyclotomic polynomials[1].

Definition 1.1 (Notation and logatithmic derivatives)**.** For $n \geq 2$ and an indeterminate q, we define the following rational functions of q related to the logarithmic derivatives of the cyclotomic polynomials:

$$\Pi_n(q) := \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{(1-q^n)} = \frac{(n-1) - nq - q^n}{(1-q)(1-q^n)} \tag{7}$$

$$\widetilde{\Phi}_n(q) := \frac{1}{q} \cdot \frac{d}{dw}\left[\log \Phi_n(w)\right]\Big|_{w \to \frac{1}{q}}.$$

For any natural number $n \geq 2$ and prime p, we use $\nu_p(n)$ to denote the largest power of p dividing n. If $p \nmid n$, then $\nu_p(n) = 0$ and if $n = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k}$ denotes the

[1] *Iverson's convention* compactly specifies boolean-valued conditions and is equivalent to the *Kronecker delta function*, $\delta_{i,j}$, as $[n = k]_\delta \equiv \delta_{n,k}$. Similarly, $[\texttt{cond}]_\delta \equiv \delta_{\texttt{cond},\texttt{True}} \in \{0, 1\}$, which is 1 if and only if the boolean-valued predicate `cond` is true, and 0 otherwise.

prime factorization of n then $\nu_{p_i}(n) = \gamma_i$. That is, $\nu_p(n)$ is the *valuation function* indicating the exact non-negative exponent of the prime p dividing any $n \geq 2$ so that $\nu_p(n) = \alpha$ if and only if $p^\alpha || n$. In the notation that follows, we consider sums indexed by p to be summed over only the primes. Finally, we define the function $\widetilde{\chi}_{\mathrm{PP}}(n) \mapsto \{0, 1\}$ to denote the indicator function of the positive natural numbers n which are not of the form $n = p^k, 2p^k$ for any primes p and exponents $k \geq 1$.

n	**Series Term Expansions** $\left(\frac{nq^n}{1-q^n} + n - \frac{1}{1-q}\right)$	**Formula Expansions**	**Reduced-Index Formula**
2	$\frac{1}{1+q}$	$\widetilde{\Phi}_2(q)$	--
3	$\frac{2+q}{1+q+q^2}$	$\widetilde{\Phi}_3(q)$	--
4	$\frac{1}{1+q} + \frac{2}{1+q^2}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_4(q)$	$\widetilde{\Phi}_2(q)+2\widetilde{\Phi}_2\left(q^2\right)$
5	$\frac{4+3q+2q^2+q^3}{1+q+q^2+q^3+q^4}$	$\widetilde{\Phi}_5(q)$	--
6	$\frac{1}{1+q} + \frac{2-q}{1-q+q^2} + \frac{2+q}{1+q+q^2}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_3(q)+\widetilde{\Phi}_6(q)$	--
7	$\frac{6+5q+4q^2+3q^3+2q^4+q^5}{1+q+q^2+q^3+q^4+q^5+q^6}$	$\widetilde{\Phi}_7(q)$	--
8	$\frac{1}{1+q} + \frac{2}{1+q^2} + \frac{4}{1+q^4}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_4(q)+\widetilde{\Phi}_8(q)$	$\widetilde{\Phi}_2(q)+2\widetilde{\Phi}_2\left(q^2\right)+4\widetilde{\Phi}_2\left(q^4\right)$
9	$\frac{2+q}{1+q+q^2} + \frac{3\left(2+q^3\right)}{1+q^3+q^6}$	$\widetilde{\Phi}_3(q)+\widetilde{\Phi}_9(q)$	$\widetilde{\Phi}_3(q)+3\widetilde{\Phi}_3\left(q^2\right)$
10	$\frac{1}{1+q} + \frac{4-3q+2q^2-q^3}{1-q+q^2-q^3+q^4} + \frac{4+3q+2q^2+q^3}{1+q+q^2+q^3+q^4}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_5(q)+\widetilde{\Phi}_{10}(q)$	--
11	$\frac{10+9q+8q^2+7q^3+6q^4+5q^5+4q^6+3q^7+2q^8+q^9}{1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}}$	$\widetilde{\Phi}_{11}(q)$	--
12	$\frac{1}{1+q} + \frac{2}{1+q^2} + \frac{2-q}{1-q+q^2} + \frac{2+q}{1+q+q^2} - \frac{2\left(-2+q^2\right)}{1-q^2+q^4}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_3(q)+\widetilde{\Phi}_4(q)$	$\widetilde{\Phi}_2(q)+2\widetilde{\Phi}_2\left(q^2\right)+\widetilde{\Phi}_3(q)$
		$+\widetilde{\Phi}_6(q)+\widetilde{\Phi}_{12}(q)$	$+\widetilde{\Phi}_6(q)+2\widetilde{\Phi}_6(q)$
13	$\frac{12+11q+10q^2+9q^3+8q^4+7q^5+6q^6+5q^7+4q^8+3q^9+2q^{10}+q^{11}}{1+q+q^2+q^3+q^4+q^5+q^6+q^7+q^8+q^9+q^{10}+q^{11}+q^{12}}$	$\widetilde{\Phi}_{13}(q)$	--
14	$\frac{1}{1+q} + \frac{6-5q+4q^2-3q^3+2q^4-q^5}{1-q+q^2-q^3+q^4-q^5+q^6} + \frac{6+5q+4q^2+3q^3+2q^4+q^5}{1+q+q^2+q^3+q^4+q^5+q^6}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_7(q)+\widetilde{\Phi}_{14}(q)$	--
15	$\frac{2+q}{1+q+q^2} + \frac{4+3q+2q^2+q^3}{1+q+q^2+q^3+q^4} + \frac{8-7q+5q^3-4q^4+3q^5-q^7}{1-q+q^3-q^4+q^5-q^7+q^8}$	$\widetilde{\Phi}_3(q)+\widetilde{\Phi}_5(q)+\widetilde{\Phi}_{15}(q)$	--
16	$\frac{1}{1+q} + \frac{2}{1+q^2} + \frac{4}{1+q^4} + \frac{8}{1+q^8}$	$\widetilde{\Phi}_2(q)+\widetilde{\Phi}_4(q)+\widetilde{\Phi}_8(q)+\widetilde{\Phi}_{16}(q)$	$\widetilde{\Phi}_2(q)+2\widetilde{\Phi}_2\left(q^2\right)+4\widetilde{\Phi}_2\left(q^4\right)+8\widetilde{\Phi}_2\left(q^8\right)$

Table 1: Expansions of Lambert series terms by cyclotomic polynomial primitives. The double dashes (--) in the rightmost column of the table indicate that the entry is the same as the previous column to distinguish between the cases where we apply our special case reduction formulas.

To provide intuition behind the factorizations of the terms in our Lambert series generating functions defined above, the listings in Table 1 provide the first several expansions of the right-hand-side of the next equations. These expanded and factored terms are generated with *Mathematica* according to applications of the identities in Equation (6). The components highlighted by the examples in the table form the key terms of our new exact formula expansions. Notably, we see that we may write the expansions of the individual Lambert series terms as follows:

$$\frac{q^n}{1-q^n} = -1 + \frac{1}{n(1-q)} + \frac{1}{n}\sum_{\substack{d|n \\ d>1}} \widetilde{\Phi}_d(q), n \geq 2. \tag{8}$$

We can reduce the index orders of the cyclotomic polynomials, $\Phi_n(q)$, and their logarithmic derivatives, denoted by $\widetilde{\Phi}_d(q)$, in terms of lower-indexed cyclotomic polynomials with q transformed into powers of q raised to powers of primes, as q^{p^k} (see [4, cf. §3] and [10, cf. §13.2]).

Remark 1.2 (Experimental intuition for the new formulas)**.** We begin by observing that the start of the formulas proved in Section 2 (stated precisely below) were initially recognized by experimentally factoring the exact polynomial expansions of the Lambert series terms $\frac{q^n}{1-q^n}$. Namely, the computer algebra routines employed by default in *Mathematica* are able to produce the already suggestive semi-factored output reproduced in Table 1 (cf. Section 1.2). The third and fourth columns of Table 1 naturally suggest by computation the exact forms of the (logarithmic derivative) polynomial expansions we are looking for to expand our Lambert series terms. The observation of these trends in the polynomial factorizations of $1-q^n$ into products of irreducible polynomials led to the intuition motivating our new results proved rigorously in the next sections of this article.

1.1. Statements of Key Results and Characterizations

Definition 1.3 (Notation for component divisor sums)**.** For fixed q and any $n \geq 1$, we define the component sums, $\widetilde{S}_{i,n}(q)$ for $i = 0, 1, 2$, as follows:

$$\begin{aligned}
\widetilde{S}_{0,n}(q) &= \sum_{\substack{d|n \\ d>1 \\ d\neq p^k, 2p^k}} \widetilde{\Phi}_d(q) \\
\widetilde{S}_{1,n}(q) &= \sum_{p|n} \Pi_{p^{\nu_p(n)}}(q) \\
\widetilde{S}_{2,n}(q) &= \sum_{\substack{2p|n \\ p>2}} \Pi_{p^{\nu_p(n)}}(-q).
\end{aligned}$$

We use the following notation for the generalized α-order harmonic number sequences to state our next few results:

$$H_n^{(\alpha)} := \sum_{k=1}^{n} k^{-\alpha}, n \geq 0, \alpha \in \mathbb{R}.$$

The generalized harmonic number sequences correspond to the partial sums of the Riemann zeta function $\zeta(\alpha)$ when $\alpha > 1$ and are related to generalized Bernoulli numbers and polynomials when $\alpha \leq 0$ is integer-valued.

Proposition 1.4 (Series coefficients of the component sums)**.** *For any fixed $\alpha \in \mathbb{R}$ and integers $x \geq 1$, we have the following component sum expressions:*

$$\widehat{S}_0^{(\alpha)}(x) := [q^x] \sum_{n=1}^{x} \widetilde{S}_{0,n}(q) n^{\alpha-1} =: \tau_\alpha(x) \tag{i}$$

$$\begin{aligned}\widehat{S}_1^{(\alpha)}(x) &:= [q^x] \sum_{n=1}^{x} \widetilde{S}_{1,n}(q) n^{\alpha-1} \\ &= \sum_{p \leq x} \sum_{k=1}^{\nu_p(x)+1} p^{\alpha k-1} H^{(1-\alpha)}_{\left\lfloor \frac{x}{p^k} \right\rfloor} \left(p \left\lfloor \frac{x}{p^k} \right\rfloor - p \left\lfloor \frac{x}{p^k} - \frac{1}{p} \right\rfloor - 1 \right)\end{aligned} \tag{ii}$$

$$\begin{aligned}\widehat{S}_2^{(\alpha)}(x) &:= [q^x] \sum_{n=1}^{x} \widetilde{S}_{2,n}(q) n^{\alpha-1} \\ &= \sum_{3 \leq p \leq x} \sum_{k=1}^{\nu_p(x)+1} \frac{p^{\alpha k-1}}{2^{1-\alpha}} H^{(1-\alpha)}_{\left\lfloor \frac{x}{2p^k} \right\rfloor} (-1)^{\left\lfloor \frac{x}{p^{k-1}} \right\rfloor} \left(p \left\lfloor \frac{x}{p^k} \right\rfloor - p \left\lfloor \frac{x}{p^k} - \frac{1}{p} \right\rfloor - 1 \right).\end{aligned} \tag{iii}$$

The precise form of the expansions in (i) of the previous proposition, denoted by $\tau_\alpha(x)$, and its particular natural connection to the Ramanujan sums, $c_q(n)$, is explored by the results stated in Proposition 2.3 of the next section.

Theorem 1.5 (Exact formulas for the generalized sum-of-divisors functions)**.** *For any fixed $\alpha \in \mathbb{R}$ and natural numbers $x \geq 1$, the following formula holds:*

$$\sigma_\alpha(x) = H_x^{(1-\alpha)} + \widehat{S}_0^{(\alpha)}(x) + \widehat{S}_1^{(\alpha)}(x) + \widehat{S}_2^{(\alpha)}(x).$$

While our new exact sum formulas in Theorem 1.5 are deeply tied to the distribution of the prime numbers $p \leq x$ for any large x, we observe that the expansions of sums for the divisor function special case given in the references (see [7, §9; p. 141] and [2]) are of a much more distinctive character than our new exact finite sum formulas proved by the theorem. We also recall the following prime product formula expansion that classically characterizes these multiplicative functions for comparision (cf. [13, §27.3] and [6, §16.7]):

$$\sigma_\alpha(n) = \prod_{p^\gamma || n} \left(\frac{p^{(\gamma+1)\alpha} - 1}{p^\alpha - 1} \right), n \geq 1, \alpha \in \mathbb{R}.$$

We next have a few remarks about symmetry in the identity from the theorem in the context of negative-order divisor functions of the form

$$\sigma_{-\alpha}(x) = \sum_{d|x} d^{-\alpha} = \sum_{d|x} \left(\frac{x}{d} \right)^{-\alpha} = \frac{\sigma_\alpha(x)}{x^\alpha}, x \geq 1, \alpha \geq 0. \tag{9}$$

For integers $\alpha \in \mathbb{Z}^+ \cup \{0\}$, we can express sums corresponding to the so-called *negative-order* harmonic numbers, $H_n^{(-\alpha)}$, in terms of the *generalized Bernoulli numbers* (polynomials) by *Faulhaber's formula* in the following forms:

$$\begin{aligned}\sum_{m=1}^{n} m^{\alpha} &= \frac{1}{\alpha+1}\left(B_{\alpha+1}(n+1) - B_{\alpha+1}\right) \\ &= \frac{1}{(\alpha+1)} \sum_{j=0}^{\alpha} \binom{\alpha+1}{j} B_j \cdot n^{\alpha+1-j}. \end{aligned} \tag{10}$$

Since the convolution formula in equation (9) above proves that $\sigma_{-\beta}(n) = \sigma_{\beta}(n) \cdot n^{-\beta}$ whenever $\beta > 0$, we also expand the right-hand-side of the theorem in the symmetric form of

$$\sigma_{\alpha}(x) = x^{\alpha}\left(H_x^{(\alpha+1)} + \tau_{-\alpha}(x) + \widehat{S}_1^{(-\alpha)}(x) + \widehat{S}_2^{(-\alpha)}(x)\right), \alpha > 0.$$

In particular, we are able to restate Proposition 1.4 and Theorem 1.5 together in the following alternate form where $c_d(x)$ denotes a *Ramanujan sum* (see Proposition 2.3):

Theorem 1.6 (Symmetric Forms of the Exact Formulas)**.** *For any fixed $\alpha \in \mathbb{R}$ and integers $x \geq 1$, we have the following formulas:*

$$\widehat{S}_0^{(-\alpha)}(x) = \sum_{d=1}^{x} H_{\left\lfloor \frac{x}{d} \right\rfloor}^{(\alpha+1)} \cdot \frac{c_d(x)}{d^{\alpha+1}} \cdot \chi_{\mathrm{PP}}(d) \tag{i}$$

$$\widehat{S}_1^{(-\alpha)}(x) = \sum_{p \leq x}\left[\sum_{k=1}^{\nu_p(x)} \frac{(p-1)}{p^{\alpha k+1}} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(\alpha+1)} - \frac{1}{p^{\alpha \cdot \nu_p(x)+\alpha+1}} H_{\left\lfloor \frac{x}{p^{\nu_p(x)+1}} \right\rfloor}^{(\alpha+1)}\right] \tag{ii}$$

$$\widehat{S}_2^{(-\alpha)}(x) = \frac{(-1)^x}{2^{\alpha+1}} \sum_{p \leq x}\left[\sum_{k=1}^{\nu_p(x)} \frac{(p-1)}{p^{\alpha k+1}} H_{\left\lfloor \frac{x}{2p^k} \right\rfloor}^{(\alpha+1)} - \frac{1}{p^{\alpha \cdot \nu_p(x)+\alpha+1}} H_{\left\lfloor \frac{x}{2p^{\nu_p(x)+1}} \right\rfloor}^{(\alpha+1)}\right]. \tag{iii}$$

The generalized sum-of-divisors functions are then expanded in the following form:

$$\sigma_{\alpha}(x) = x^{\alpha}\left(H_x^{(\alpha+1)} + \widehat{S}_0^{(-\alpha)}(x) + \widehat{S}_1^{(-\alpha)}(x) + \widehat{S}_2^{(-\alpha)}(x)\right). \tag{11}$$

We notice that the symmetric form of the identity given in Theorem 1.6 provides a curious, and necessarily deep, relation between the Bernoulli numbers and the partial sums of the Riemann zeta function involving the nested sums over the primes. It also leads to a direct proof of the known asymptotic results for the summatory functions [13, §27.11]

$$\sum_{n \leq x} \sigma_{\alpha}(n) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O\left(x^{\max(1,\alpha)}\right), \ \alpha > 0, \alpha \neq 1. \tag{12}$$

We will explore this direct proof based on Theorem 1.6 in more detail as an application given in Section 3. The new results in the two theorems also identify a method by which we can sum the left-hand-side of Equation (12) using a heuristic on the distribution of $\nu_p(n)$ for $p \leq n \leq x$.

1.2. Supplementary computational data reference

A *Mathematica* notebook containing definitions that can be used to computationally verify the formulas proved in this manuscript is made freely available to readers online at the following link:
https://github.com/maxieds/ManuscriptComputationalData/blob/master/sod-formulas-updated.nb.
The functional definitions provided in this notebook are intended to be of use to readers for experimental mathematics based on the contents of this article. Without this computationally driven means motivating our experimental work with new polynomial expansions involving the generating functions of these special classical functions, we would most likely never have noticed these subtle new formulas for the often studied classical sum-of-divisors functions.

2. Proofs of Our New Results

2.1. Motivating the Proof of the New Formulas

Example 2.1. We first revisit a computational example of the rational functions defined by the logarithmic derivatives in Definition 1.1 that is illustrated in the computations in Table 1. We will make use of the next variant of the identity in Equation (4) in the proof below which is again obtained by Möbius inversion:

$$\Phi_n(q) = \prod_{d|n} (q^d - 1)^{\mu(n/d)}. \tag{13}$$

In the case of our modified rational cyclotomic polynomial functions, $\widetilde{\Phi}_n(q)$, when $n := 15$, we use this product to expand the definition of the function as

$$\begin{aligned}
\widetilde{\Phi}_{15}(q) &= \frac{1}{q} \cdot \frac{d}{dq} \left[\log \left(\frac{(1-q^3)(1-q^5)}{(1-q)(1-q^{15})} \right) \right] \Bigg|_{q \rightarrow 1/q} \\
&= \frac{3}{1-q^3} + \frac{5}{1-q^5} - \frac{1}{1-q} - \frac{15}{1-q^{15}} \\
&= \frac{8 - 7q + 5q^3 - 4q^4 + 3q^5 - q^7}{1 - q + q^3 - q^4 + q^5 - q^7 + q^8}.
\end{aligned}$$

The procedure for transforming the difficult-looking terms involving the cyclotomic polynomials when the Lambert series terms, $q^n \cdot (1-q^n)^{-1}$, are expanded in partial fractions as in Table 1 is essentially the same as this example for the cases we will encounter here. More generally, we have the next lemma for any $n \geq 1$.

Lemma 2.2 (Key characterizations of the divisor sums, $\tau_\alpha(x)$)**.** *For integers $n \geq 1$ and any indeterminate q that is not a d^{th} root of unity for any of the divisors d of n, we have the following expansion:*

$$\widetilde{\Phi}_n(q) = \sum_{d|n} \frac{d \cdot \mu(n/d)}{(1-q^d)}.$$

In particular, we have that

$$\widetilde{S}_{0,n}(q) = \sum_{d|n} \sum_{r|d} \frac{r \cdot \widetilde{\chi}_{\mathrm{PP}}(d) \cdot \mu(d/r)}{(1-q^r)}.$$

Proof. The proof is essentially the same as the example given above. Since we can refer to this illustrative example, we only need to sketch the details to the remainder of the proof. In particular, we notice that since we have the known identity for the cyclotomic polynomials given by Equation (13), we can take logarithmic derivatives to obtain that

$$\frac{1}{x} \cdot \frac{d}{dq}\left[\log\left(1-q^d\right)^{\pm 1}\right]\Bigg|_{q \to 1/q} = \mp \frac{d}{q^d\left(1-\frac{1}{q^d}\right)} = \pm \frac{d}{1-q^d},$$

where for $d|n$ and $\frac{n}{d}$ not squarefree, we have that the resulting terms $\log(1) = 0$ vanish. This observation applied iteratively leads us to conclude the result. □

Proposition 2.3 (Connections to Ramanujan sums)**.** *Let the following notation denote a shorthand for the divisor sum terms in Theorem 1.5:*

$$\tau_\alpha(x) := \widehat{S}_0^{(\alpha)}(x) = [q^x] \sum_{n=1}^{x} \widetilde{S}_{0,n}(q) n^{\alpha-1}, x \geq 1, \alpha \in \mathbb{R}.$$

We have the following characterizations of the function $\tau_\alpha(x)$ expanded in terms of Ramanujan's sum, $c_q(n)$, where $\mu(n)$ denotes the Möbius function and $\varphi(n)$ is Euler's totient function:

$$\tau_{\alpha+1}(x) = \sum_{\substack{d=1 \\ d \neq p^k}}^{x} H_{\lfloor \frac{x}{d} \rfloor}^{(-\alpha)} \cdot d^\alpha \cdot c_d(x), x \geq 1, \alpha \in \mathbb{R}$$

$$\tau_{\alpha+1}(x) = \sum_{\substack{d=1 \\ d \neq p^k}}^{x} H_{\lfloor \frac{x}{d} \rfloor}^{(-\alpha)} \cdot d^\alpha \cdot \mu\left(\frac{d}{(d,x)}\right) \frac{\varphi(d)}{\varphi\left(\frac{d}{(d,x)}\right)}, x \geq 1, \alpha \in \mathbb{R},$$

$$\tau_{-\alpha}(x) = \sum_{d \leq x} H_{\lfloor \frac{x}{d} \rfloor}^{(\alpha+1)} \cdot \frac{c_d(x)}{d^{\alpha+1}} - \sum_{\substack{p^k | x \\ p^k \leq \frac{x}{p}}} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(\alpha+1)} \cdot \frac{(p-1)}{p^{\alpha k+1}} + \sum_{\substack{p^k \leq x \\ p^k > \frac{x}{p}}} H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(\alpha+1)} \cdot p^{-k}, x \geq 1, \alpha \in \mathbb{R}.$$

Proof. First, we observe that the contribution of the first (zero-indexed) sums in Theorem 1.5 correspond to computing the coefficients

$$\begin{aligned}\tau_{\alpha+1}(x) &= [q^x]\left(\sum_{k=1}^{x}\sum_{\substack{d\mid k\\ d\neq p^s}}\sum_{r\mid d}\frac{r\cdot\mu(d/r)}{(1-q^r)}k^{\alpha}\right)\\ &= \sum_{k=1}^{x}\sum_{r\mid x}\sum_{\substack{d\mid k\\ d\neq p^s}} r\cdot\mu(d/r)\cdot[r\mid d]_{\delta}\cdot k^{\alpha}\\ &= \sum_{k=1}^{x}\sum_{\substack{d\mid k\\ d\neq p^s}}\sum_{r\mid(d,x)} r\cdot\mu(d/r)\cdot k^{\alpha}.\end{aligned}$$

Since we can easily prove the identity that [1, cf. §2.14; §3.10; §3.12]

$$\sum_{k=1}^{x}\sum_{d\mid k} f(d)g(k/d) = \sum_{d=1}^{x} f(d)\left(\sum_{k=1}^{\lfloor \frac{x}{d}\rfloor} g(k)\right), x\geq 1,$$

for any fixed arithmetic functions f and g, we can also expand the right-hand-side of the previous equation for $\tau_{\alpha+1}(x)$ as follows:

$$\tau_{\alpha+1}(x) = \sum_{\substack{d=1\\ d\neq p^k}}^{x}\left(\sum_{r\mid(d,x)} r\mu(d/r)\right) H_{\lfloor \frac{x}{d}\rfloor}^{(-\alpha)}\cdot d^{\alpha}. \tag{14}$$

Thus the identities stated in the proposition follow by expanding out Ramanujan's sum in the form of the divisor sums (see [13, §27.10], [12, §A.7] and [6, cf. §5.6])

$$c_q(n) = \sum_{d\mid(q,n)} d\cdot\mu(q/d),\ \text{ for } n,q\in\mathbb{Z}^+.$$

The last identity stated in this proposition follows from the first by re-writing the formula for $c_d(x)$ at prime powers p^d given in cases by the formulas

$$c_{p^d}(n) = \begin{cases} 0, & \text{if } p^{d-1}\nmid n;\\ -p^{d-1}, & \text{if } p^{d-1}\mid n \wedge p^d\nmid n;\\ p^d - p^{d-1}, & \text{if } p^d\mid n\wedge p^{d+1}\mid n.\end{cases}$$ □

The last set of identities in the proof above implies that with $k\approx\log_p(x)$ on the last terms, we have that

$$\tau_{\alpha+1}(x) = \sum_{\substack{p^k\leq x\\ p^k\leq\frac{x}{p}}} H_{\lfloor \frac{x}{p^k}\rfloor}^{(\alpha+1)}\cdot\frac{(p-1)}{p^{\alpha k+1}} + O\left(\frac{1}{x}\right).$$

Remark 2.4. Ramanujan's sum satisfies the upper bound that $|c_q(n)| \leq (n,q)$ for all $n,q \geq 1$. This known bound can be used to obtain asymptotic estimates in the form of upper bounds for these sums when q is not prime or a prime power. For positive integers $n,q \geq 1$, Ramanujan's sum also has the following representation as a finite degree *exponential sum*:

$$c_q(n) = \sum_{d|q} \left[\sum_{k=1}^{d} \exp\left(2\pi\imath \cdot \frac{kn}{d}\right)\right] \mu\left(\frac{q}{d}\right), \text{ for } n,q \in \mathbb{Z}^+.$$

This formula for $c_q(n)$ is related to periodic exponential sums (modulo k) of the more general form

$$s_k(n) = \sum_{d|(n,k)} f(d) g\left(\frac{k}{d}\right).$$

The functions $s_k(n)$ are periodic in n (modulo k) with a finite Fourier series expansion

$$s_k(n) = \sum_{m=1}^{k} a_k(m) e^{2\pi\imath n/k},$$

and coefficients given by the auxiliary divisor sums [13, §27.10]

$$a_k(m) = \sum_{d|(m,k)} g(d) f\left(\frac{k}{d}\right) \frac{d}{k}.$$

It turns out that the terms in the formulas for $\sigma_\alpha(x)$ represented by the sums $\tau_\alpha(x)$ from Theorem 1.5 and Theorem 1.6 provide detailed insight into the error estimates for the summatory functions over the generalized sum-of-divisors functions. We use these resulting estimates in proving the main applications in Section 3.

2.2. Proofs of the Key New Theorems and Results

Proof of Theorem 1.5. We begin with the divisor product formula from Equation (4) involving the cyclotomic polynomials when $n \geq 1$ and q is fixed. Then by logarithmic differentiation we can see that

$$\begin{aligned}\frac{q^n}{1-q^n} &= -1 + \frac{1}{n(1-q)} + \frac{1}{n}\sum_{\substack{d|n \\ d>1}} \widetilde{\Phi}_d(q) \qquad (15)\\ &= -1 + \frac{1}{n(1-q)} + \frac{1}{n}\left(\widetilde{S}_{0,n}(q) + \widetilde{S}_{1,n}(q) + \widetilde{S}_{2,n}(q)\right).\end{aligned}$$

The last equation is obtained from the first expansion in Equation (15) above by identifying the next two sums as

$$\Pi_n(q) = \sum_{\substack{d|n \\ d>1}} \widetilde{\Phi}_n(1/q) = \sum_{j=0}^{n-2} \frac{(n-1-j)q^j(1-q)}{1-q^n}.$$

Here we are applying the known expansions of the cyclotomic polynomials which condense the order n of the polynomials by exponentiation of the indeterminate q when n contains a factor of a prime power given by Equation (6). Finally, we complete the proof by summing the right-hand-side of Equation (15) over $n \leq x$ times the weight n^{α} to obtain the x^{th} partial sum of the Lambert series generating function for $\sigma_{\alpha}(x)$ (cf. [6, §17.10] and [13, §27.7]). Since each term in the summation contains a power of q^n, replacing the infinite Lambert series generating function sums by this partial summation expression forms an $(x+1)$-order accurate generating series for the terms in the infinite series. □

Proof of Proposition 1.4. The identity in (i) follows from Lemma 2.2. Since $\Phi_{2p}(q) = \Phi_p(-q)$ for any prime $p > 2$, we are essentially in the same case with the two component sums in (ii) and (iii). We outline the proof of our expansion for the first sum, $\widetilde{S}_{1,n}(q)$, and note the small changes necessary along the way to adapt the proof to the other sum cases. By the properties of the cyclotomic polynomials expanded in Equation (6), we may factor the denominators of $\Pi_{p^{\nu_p(n)}}(q)$ into smaller irreducible factors of the same polynomial, $\Phi_p(q)$, with inputs varying as prime-power powers of the series variable q. More precisely, we define the functions

$$Q_{p,k}^{(n)}(q) := \frac{\sum\limits_{j=0}^{p-2} (p-1-j) q^{p^{k-1}j}}{\sum\limits_{i=0}^{p-1} q^{p^{k-1}i}},$$

and use them to expand the sums

$$\widetilde{S}_{1,n}(q) = \sum_{p \leq n} \sum_{k=1}^{\nu_p(n)} Q_{p,k}^{(n)}(q) \cdot p^{k-1}.$$

In evaluating the coefficients of powers of q in the sum $\sum_{n \leq x} Q_{p,k}^{(n)}(q) p^{k-1} n^{\alpha-1}$, these terms have a repeat coefficient, every p^k terms, so that we can form the coefficient sums for these terms as follows:

$$\sum_{i=i}^{\left\lfloor \frac{x}{p^k} \right\rfloor} \left(ip^k\right)^{\alpha-1} \cdot p^{k-1} = p^{k\alpha-1} \cdot H_{\left\lfloor \frac{x}{p^k} \right\rfloor}^{(1-\alpha)}.$$

We can also compute the inner sums in the previous equations exactly for any fixed t as

$$\sum_{j=0}^{p-2} (p-1-j) t^j = \frac{(p-1) - pt - t^p}{(1-t)^2},$$

where the corresponding paired denominator sums in these terms are given by $1 + t + t^2 + \cdots + t^{p-1} = (1-t^p) \cdot (1-t)^{-1}$. We now assemble the full sum over $n \leq x$ as

$$\sum_{n \leq x} \widetilde{S}_{1,n}(q) \cdot n^{\alpha-1} = \sum_{p \leq x} \sum_{k=1}^{\nu_p(x)} p^{k\alpha-1} H^{(1-\alpha)}_{\left\lfloor \frac{x}{p^k} \right\rfloor} \frac{(p-1) - pq^{p^{k-1}} + q^{p^k}}{(1-q^{p^{k-1}})(1-q^{p^k})}.$$

The analogous result for the second sum cases is obtained similarly with the exception of sign changes on the coefficients of the powers of q in the last expansion.

We compute the series coefficients of one of the three cases in the last equation to show our method of obtaining the full formula with the remaining two sum cases. The right-most term in these expansions leads to the double sum

$$\begin{aligned} C_{3,x,p} &:= [q^x] \frac{q^{p^k}}{(1 \mp q^{p^{k-1}})(1 \mp q^{p^k})} \\ &= [q^x] \sum_{n,j \geq 0} (\pm 1)^{n+j} q^{p^{k-1}(n+p+jp)}. \end{aligned}$$

We must have that $p^{k-1}|x$ in order to have a non-zero coefficient and for $n := x/p^{k-1} - jp - p$ with $0 \leq j \leq x/p^k - 1$. We can then compute these coefficients explicitly as

$$C_{3,x,p} := (\pm 1)^{\lfloor x/p^{k-1} \rfloor} \times \sum_{j=0}^{\lfloor x/p^k - 1 \rfloor} 1 = (\pm 1)^{\lfloor x/p^{k-1} \rfloor} \left\lfloor \frac{x}{p^k} - 1 \right\rfloor + 1 = (\pm 1)^{\lfloor x/p^{k-1} \rfloor} \left\lfloor \frac{x}{p^k} \right\rfloor.$$

With minimal simplifications we have arrived at the claimed result in the proposition. The other two cases of the series coefficient computations follow similarly. □

Proof of Theorem 1.6. Since we first have that Equation (9) holds for any $\alpha \geq 0$, we can see that the formula in Equation (11) follows immediately from Theorem 1.5. It remains to prove the subformulas in tagged equations (i)–(iii) of the theorem. The first formula for $S_0^{(-\alpha)}(x)$ corresponds to the formulas we derived in Proposition 2.3 of the previous subsection for these cases of negative-order α. The second two formulas follow from Proposition 1.4 by expanding the cases of the floor function inputs according to the inner index k in the ranges $k \in [1, \nu_p(x)]$, i.e., where $x/p^k \in \mathbb{Z}^+$, and then in the single index case where $k := \nu_p(x) + 1$. □

3. Applications to New Asymptotics for Power Scaled Partial Sums of the Generalized Sum-of-divisor Functions

We can use the new exact formula proved by Theorem 1.5 to asymptotically estimate partial sums, or *average orders*, of the respective arithmetic functions, of the next form for integers $x \geq 1$:

$$\Sigma^{(\alpha,\beta)}(x) := \sum_{n \leq x} \frac{\sigma_\alpha(n)}{n^\beta}, \alpha, \beta \geq 0. \tag{16}$$

We similarly define $\Sigma_\alpha(x) := \Sigma^{(\alpha,0)}(x)$ using the notation in Equation (16). In the special cases where $\alpha := 0, 1$, we restate a few more famous formulas providing classically well-known (and newer established) asymptotic bounds for sums of this form as follows where $\gamma \approx 0.577216$ is Euler's gamma constant, $d(n) \equiv \sigma_0(n)$ denotes the *divisor function*, and $\sigma(n) \equiv \sigma_1(n)$ is the (ordinary) *sum-of-divisors function* (see [9, 2, 8] and [13, cf. §27.11]):

$$\begin{aligned}
\Sigma_0(x) &:= \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O\left(x^{\frac{131}{416}}\right) \qquad (17)\\
\Sigma^{(0,1)}(x) &:= \sum_{n \leq x} \frac{d(n)}{n} = \frac{1}{2}(\log x)^2 + 2\gamma \log x + O\left(x^{-2/3}\right)\\
\Sigma_1(x) &:= \sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12}x^2 + O(x \log^{2/3} x).
\end{aligned}$$

We can extend the known classical result for the sums $\Sigma_\alpha(x)$ given by Equation (12) and for the special cases in Equation (17) to the cases of the modified summatory functions $\Sigma^{(\alpha,\beta)}(x)$ using the new formulas proved in Theorem 1.6. The next result provides the precise details of the limiting asymptotic relations we obtain for the sums $\Sigma^{(\alpha,\beta)}(x)$ over some restricted integer-order cases of $(\alpha, \beta) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ that arise in applications. The new results in Theorem 3.1 provide lower-order polynomial terms in the expansions of $\Sigma^{(\alpha,\beta)}(x)$ for integers $\alpha \geq 2$ and $2 \leq \beta \leq \alpha$.

Theorem 3.1 (Asymptotics for Summatory Functions)**.** *For integers $\alpha > 1$ and integer-valued $2 \leq \beta \leq \alpha$, we have that*

$$\begin{aligned}
\Sigma^{(\alpha,\beta)}(x) = {} & \frac{\zeta(\alpha+1)x^{\alpha+1-\beta}}{(\alpha+1-\beta)} \left(1 - C_1(\alpha) + C_{2,0}(\alpha) + C_3(\alpha) + C_6(\beta) + C_{7,0}(\alpha)\right)\\
& + \sum_{j=1}^{\alpha-\beta} \binom{\alpha+1-\beta}{j} \frac{B_j x^{\alpha+1-\beta-j}}{\alpha+1-\beta} \left(1 + C_{2,j}(\alpha) + C_{7,j}(\alpha)\right)\\
& + \sum_{j=0}^{\alpha-\beta} C_{4,j}(\alpha,\beta)\zeta(\alpha+1)x^j\\
& + \sum_{j=0}^{\alpha-\beta} \binom{\alpha-\beta}{j} \frac{C_5(\alpha,\beta)(-1)^{\alpha-\beta-j}E_j}{2^{2\alpha+2-\beta}} + O\left(\frac{x}{\log x}\right),
\end{aligned}$$

where B_n is a Bernoulli number and E_n denotes an Euler number. The parameterized absolute constants (depending only on α, β and m, respectively) in the last *expression are defined by the sums*

$$
\begin{aligned}
C_1(\alpha) &:= \sum_{p\geq 2} \frac{(p-2)}{p(p-1)(p^{\alpha+1}-1)}\left[\frac{(p-1)}{p(p^{\alpha}-1)} - \frac{1}{p^{\alpha}}\right] \\
C_{2,m}(\alpha) &:= \sum_{p\geq 2} \frac{(p-1)}{p^{\alpha+2-m}(p^{\alpha}-1)} \\
C_3(\alpha) &:= \sum_{p\geq 3} \frac{(p-2)}{2^{\alpha+1}p(p-1)(p^{\alpha+1}-1)}\left[\frac{(p-1)}{p(p^{\alpha}-1)} - \frac{1}{p^{\alpha}}\right] \\
C_{4,m}(\alpha,\beta) &:= \sum_{p\geq 2}\sum_{k=0}^{\alpha-\beta}\binom{\alpha-\beta}{k}\binom{\alpha-\beta-k}{m}\frac{(-1)^{k+m}E_k\cdot(p-1)}{2^{2\alpha+2-\beta-m}p^{\beta+1+m}(p^{\alpha}-1)} \\
C_5(\alpha,\beta) &:= \sum_{p\geq 2} \frac{(p-1)}{p^{\beta+1}(p^{\alpha}-1)} \\
C_6(\beta) &:= \sum_{p\geq 2} \frac{(p-2)}{p(p-1)(p^{\beta+1}-1)} \\
C_{7,m}(\alpha) &:= -\sum_{p\geq 2} \frac{1}{p^{\alpha+1-m}}.
\end{aligned}
$$

Lemma 3.2. *For any arithmetic functions $f, g, h : \mathbb{N} \to \mathbb{C}$, and natural numbers $x \geq 1$, we have the following pair of divisor sum summatory function identities:*

$$
\sum_{n=1}^{x} f(n)\sum_{d|n} g(d)h\left(\frac{n}{d}\right) = \sum_{d\leq x} g(d)\sum_{n=1}^{\left\lfloor \frac{x}{d}\right\rfloor} h(n)f(dn)
$$

$$
\sum_{d=1}^{x} f(d)\left(\sum_{r|(d,x)} g(r)h\left(\frac{d}{r}\right)\right) = \sum_{r|x} g(r)\left(\sum_{1\leq d\leq x/r} h(d)f(rd)\right).
$$

Since the proofs of these identities are not difficult, and are in fact fairly standard exercises, we will not prove the two formulas from Lemma 3.2 here.

Proof of Theorem 3.1. We decompose the proof into four separate tasks of estimating the component term sums from Theorem 1.6. We will evaluate the limiting asymptotics for each of the next summations:

$$\Sigma^{0}_{\alpha,\beta}(x) := \sum_{n\leq x} n^{\alpha-\beta} \times H_n^{(\alpha+1)} \tag{Step I}$$

$$\Sigma'_{\alpha,\beta}(x) := \sum_{n\leq x} n^{\alpha-\beta} \cdot S_1^{(-\alpha)}(n) \tag{Step II}$$

$$\Sigma''_{\alpha,\beta}(x) := \sum_{n\leq x} n^{\alpha-\beta} \cdot S_2^{(-\alpha)}(n) \tag{Step III}$$

$$\Sigma'''_{\alpha,\beta}(x) := \sum_{n\leq x} \tau_{-\alpha}(n) n^{\alpha-\beta}. \tag{Step IV}$$

The combined approximation we obtain is then given in terms of this notation by

$$\Sigma^{(\alpha,\beta)}(x) = \Sigma^{0}_{\alpha,\beta}(x) + \Sigma'_{\alpha,\beta}(x) + \Sigma''_{\alpha,\beta}(x) + \Sigma'''_{\alpha,\beta}(x).$$

Step (I). Leading Term Estimates: For $\alpha > 0$ and $x \geq 1$, we have that the $(\alpha+1)$-order harmonic numbers satisfy

$$H_x^{(\alpha+1)} = \zeta(\alpha+1) + O\left(x^{-(\alpha+1)}\right),$$

where $\zeta(s)$ is the *Riemann zeta function*. The leading terms in the formula for $\sigma_\alpha(n)$ from Theorem 1.6 then correspond to

$$\begin{aligned}
\sum_{n\leq x} n^{\alpha-\beta} \cdot H_n^{(\alpha+1)} &= \sum_{n\leq x} n^{\alpha-\beta}\left(\zeta(\alpha+1) + O\left(\frac{1}{n^{\alpha+1}}\right)\right) \\
&= \sum_{n\leq x}\left(\zeta(\alpha+1) n^{\alpha-\beta} + O\left(\frac{1}{n^{1+\beta}}\right)\right) \\
&= \zeta(\alpha+1)\left[\frac{x^{\alpha+1-\beta}}{\alpha+1-\beta} + \frac{x^{\alpha-\beta}}{2} + \sum_{j=2}^{\alpha-\beta}\binom{\alpha+1-\beta}{j} B_j \frac{x^{\alpha+1-\beta-j}}{\alpha+1-\beta}\right] \\
&\qquad + \begin{cases} O(\log x), & \text{if } \beta = 0; \\ O(1), & \text{if } \beta > 0, \end{cases}
\end{aligned} \tag{18}$$

by Faulhaber's formula stated in Equation (10) of the introduction, where the B_n denote the *Bernoulli numbers* for $n \geq 0$. This establishes the leading dominant term in the asymptotic expansion which confirms the classical result cited in Equation (12) above.

Step (II). Second Terms Estimate: We can expand the first component sum in Theorem 1.6 as

$$\begin{aligned}
S_1^{(-\alpha)}(n) &= \sum_{p\leq n}\left[\sum_{k=1}^{\nu_p(n)} \frac{(p-1)}{p}\left(\frac{\zeta(\alpha+1)}{p^{\alpha k}} + O\left(\frac{p^k}{n^{\alpha+1}}\right)\right) - \frac{\zeta(\alpha+1)}{p^{\alpha\nu_p(n)+\alpha+1}} + O\left(\frac{p^{\nu_p(n)}}{n^{\alpha+1}}\right)\right] \\
&= \sum_{p\leq n}\left[\frac{\zeta(\alpha+1)(p-1)}{p^{\alpha\nu_p(n)+1}(p^\alpha-1)} - \frac{\zeta(\alpha+1)(p-1)}{p^{\alpha+1}(p^\alpha-1)}\right][p|n]_\delta \\
&\qquad - \sum_{p\leq n} \frac{\zeta(\alpha+1)}{p^{\alpha\nu_p(n)+\alpha+1}} + O\left(\frac{1}{n^{\alpha-1}}\right).
\end{aligned}$$

The last error term results by observing that $\nu_p(n) \leq \log_p(n)$. Next, we use Abel summation together with a divergent asymptotic expansion for the exponential integral function to determine that for real $r > 0$, the next sums satisfy

$$\sum_{p\leq x} \frac{1}{p^{r+1}} = C_r + \frac{r+1}{r \cdot x^r \log x} + O\left(\frac{1}{x^r \log^2 x}\right)$$

$$\sum_{p\leq x}\frac{1}{p^{r+1}\log p} = D_r + \frac{(r+2)}{2x^r\log x} - \frac{(r+2)\log x+1}{2x^r\log^2 x} + O\left(\frac{1}{x^r\log^2 x}\right).$$

The terms C_r, D_r used to express the formulas in the previous equations are absolute constants depending only on r. Hence, we see that

$$\begin{aligned}\Sigma'_{\alpha,\beta}(x) &:= \sum_{n\leq x} n^{\alpha-\beta}\cdot S_1^{(-\alpha)}(n) \\ &= \sum_{p\leq x}\left[\sum_{n=p}^{x} n^{\alpha-\beta}\left(\frac{\zeta(\alpha+1)(p-1)}{p^{\alpha\nu_p(n)+1}(p^\alpha-1)} - \frac{\zeta(\alpha+1)(p-1)}{p^{\alpha+1}(p^\alpha-1)}\right)[p|n]_\delta\right] \qquad (19) \\ &\qquad - \sum_{p\leq x}\sum_{n=p}^{x}\frac{n^{\alpha-\beta}\zeta(\alpha+1)}{p^{\alpha\nu_p(n)+\alpha+1}} + O\left(\frac{1}{n^{\beta+1}}\right).\end{aligned}$$

For $\beta>1$, the error term in the last equation corresponds to

$$\sum_{p\leq x}\sum_{n=p}^{x} O\left(\frac{1}{n^{\beta+1}}\right) = O\left(\sum_{p\leq x}\left[\zeta(\beta)+\frac{1}{x^{\beta+1}}\right]\right) = O\left(\frac{x}{\log x}+\frac{1}{x^\beta\log x}\right) = O\left(\frac{x}{\log x}\right),$$

by applying the known asymptotic estimate for the *prime counting function* given by

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

The error terms estimated in the previous equation comprise the dominant error term in our complete asymptotic expression for $\Sigma_{\alpha,\beta}(x)$. As it turns out, this error term is fixed and independent in form for any of our restricted choices of the integer-valued parameters (α,β).

At this point we evaluate the main terms in Equation (19). We consider asymptotic approximations to sums of the next form at fixed primes p so we can use the resulting estimates with Abel summation.

$$T_{1,p}(x) := \sum_{n=p}^{x}\frac{n^{\alpha-\beta}[p|n]_\delta}{p^{\alpha\nu_p(n)}}$$

We can then form the approximations to the next summatory functions in the following way to approximate $T_{1,p}(x)$ for large x:

$$\begin{aligned}A_{1,p}(t) &= \sum_{i\leq t}\frac{[p|i]_\delta}{p^{\alpha\nu_p(i)}} = \sum_{i\leq t/p}\frac{1}{p^{\alpha\nu_p(p\cdot i)}} \\ &= \sum_{k=1}^{\log_p(t)}\frac{1}{p^{\alpha k}}\#\{i\leq t/p : \nu_p(i)=k\} \\ &= \sum_{k=0}^{\infty}\frac{1}{p^{\alpha(k+1)}}\left(\#\{i\leq t/p : p^{k+1}|i\} - \sum_{s=k+2}^{\infty}\#\{i\leq t/p : p^s|i\}\right)\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{1}{p^{\alpha(k+1)}} \left[\frac{t}{p^{k+2}} - \sum_{i=k+2}^{\infty} \frac{t}{p^{i+1}} \right] + O(1)$$
$$= \frac{(p-2)t}{p(p-1)\left(p^{\alpha+1}-1\right)} + O(1).$$

Then we have by Abel's summation formula that

$$T_{1,p}(x) = x^{\alpha-\beta} \cdot A_{1,p}(x) - \int_0^x A_{1,p}(t) D_t[t^{\alpha-\beta}] dt$$
$$\sim \frac{(p-2)x^{\alpha+1-\beta}}{(\alpha+1-\beta)p(p-1)\left(p^{\alpha+1}-1\right)}.$$

Similarly, we can estimate the asymptotic order of the sums

$$T_{2,p}(x) := \sum_{n=p}^{x} \frac{n^{\alpha-\beta}}{p^{\alpha\nu_p(n)+\alpha+1}}$$
$$\sim \frac{(p-2)x^{\alpha+1-\beta}}{(\alpha+1-\beta)(p-1)p^{\alpha+1}\left(p^{\alpha+1}-1\right)}.$$

Since $C_1(\alpha)$ is finite for any integers $\alpha \geq 2$, it follows that

$$\sum_{p \leq x} \left[\frac{\zeta(\alpha+1)(p-1)}{p(p^\alpha-1)} T_{1,p}(x) - \zeta(\alpha+1) T_{2,p}(x) \right] \sim \frac{C_1(\alpha)\zeta(\alpha+1)x^{\alpha+1-\beta}}{(\alpha+1-\beta)}.$$

We estimate the second (inner) sum forms in Equation (19) as follows:

$$\sum_{p \leq x} \sum_{n=p}^{x} n^{\alpha-\beta} \cdot \frac{\zeta(\alpha+1)(p-1)}{p^{\alpha+1}\left(p^\alpha-1\right)} [p|n]_\delta = \sum_{p \leq x} \sum_{n=1}^{x/p} (pn)^{\alpha-\beta} \cdot \frac{\zeta(\alpha+1)(p-1)}{p^{\alpha+1}\left(p^\alpha-1\right)}$$
$$= \sum_{j=0}^{\alpha-\beta} \binom{\alpha+1-\beta}{j} \frac{B_j x^{\alpha+1-\beta-j}}{(\alpha+1-\beta)} \cdot \frac{\zeta(\alpha+1)(p-1)}{p^{\alpha+1-j}\left(p^\alpha-1\right)}$$
$$\sim \sum_{j=0}^{\alpha-\beta} \binom{\alpha+1-\beta}{j} \frac{C_{2,j}(\alpha) B_j x^{\alpha+1-\beta-j}}{(\alpha+1-\beta)}.$$

In total, we obtain that

$$\Sigma'_{\alpha,\beta}(x) = \frac{C_1(\alpha)\zeta(\alpha+1)x^{\alpha+1-\beta}}{(\alpha+1-\beta)} - \sum_{j=0}^{\alpha-\beta} \binom{\alpha+1-\beta}{j} \frac{C_{2,j}(\alpha) B_j x^{\alpha+1-\beta-j}}{(\alpha+1-\beta)} + O\left(\frac{x}{\log x}\right). \tag{20}$$

Step (III). Third Component Terms Estimate: We have that

$$\Sigma''_{\alpha,\beta}(x) = \sum_{n \leq x} n^{\alpha-\beta} S_2^{(-\alpha)}(n)$$

$$= \sum_{p \leq x} \left[\sum_{n=p}^{x} \frac{(-1)^n \cdot n^{\alpha-\beta}}{2^{\alpha+1}} \left(\frac{\zeta(\alpha+1)(p-1)}{p^{\alpha\nu_p(n)+1}(p^\alpha - 1)} - \frac{\zeta(\alpha+1)(p-1)}{p^{\alpha+1}(p^\alpha - 1)} \right) [p|n]_\delta \right]$$
$$- \sum_{p \leq x} \sum_{n=p}^{x} \frac{(-1)^n \cdot n^{\alpha-\beta} \zeta(\alpha+1)}{2^{\alpha+1} p^{\alpha\nu_p(n)+\alpha+1}} + O\left(\frac{1}{n^\beta}\right).$$

Hence, our estimates in this case are based on evaulating the main components of the following formulas by Abel summation for large x:

$$U_{1,p}(x) := \sum_{n=p}^{x} \frac{(-1)^n \cdot n^{\alpha-\beta} [p|n]_\delta}{p^{\alpha\nu_p(n)}}.$$

To bound the main and error terms in this sum for large enough x, we proceed as before to form the summatory functions

$$\begin{aligned}
B_{1,2}(t) &= \sum_{i \leq t} \frac{(-1)^i \cdot [2|i]_\delta}{2^{\alpha\nu_2(i)}} = \sum_{1 \leq i \leq t/2} \frac{1}{2^{\alpha\nu_2(2i)}} \\
&= \sum_{k=1}^{\log_2(t)} \frac{1}{2^{\alpha k}} \#\{i \leq t/2 : \nu_2(i) = k\} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{\alpha(k+1)}} \left(\#\{i \leq t/2 : 2^{k+1}|i\} - \sum_{s=k+2}^{\infty} \#\{i \leq t/2 : 2^s|i\} \right) \\
&= \left. \frac{(p-2)t}{(p-1)(p^{\alpha+1}-1)} \right|_{p=2} \\
&= 0,
\end{aligned}$$

For $p \geq 3$ any odd prime, we expand analogously as

$$\begin{aligned}
B_{1,p}(t) &= \sum_{i \leq t} \frac{(-1)^i \cdot [p|i]_\delta}{p^{\alpha\nu_p(i)}} = -\sum_{i \leq t/p} \frac{1}{p^{\alpha\nu_p(p \cdot i)}} \\
&= -\sum_{k=1}^{\log_p(t)} \frac{1}{p^{\alpha k}} \#\{i \leq t/p : \nu_p(i) = k\} \\
&= -\sum_{k=0}^{\infty} \frac{1}{p^{\alpha(k+1)}} \left(\#\{i \leq t/p : p^{k+1}|i\} - \sum_{s=k+2}^{\infty} \#\{i \leq t/p : p^s|i\} \right) \\
&= -\sum_{k=0}^{\infty} \frac{1}{p^{\alpha(k+1)}} \left[\frac{t}{p^{k+2}} - \sum_{i=k+2}^{\infty} \frac{t}{p^{i+1}} \right] \\
&= -\frac{(p-2)t}{p(p-1)(p^{\alpha+1}-1)}.
\end{aligned}$$

It follows that

$$U_{1,p}(x) \sim -\frac{(p-2)x^{\alpha+1-\beta}}{(\alpha+1-\beta)p(p-1)\left(p^{\alpha+1}-1\right)} [p \geq 3]_{\delta}.$$

Then we have a resulting formula in the form of

$$\begin{aligned} U_{2,p}(x) &:= \sum_{n=p}^{x} \frac{(-1)^n \cdot n^{\alpha-\beta}}{2^{\alpha+1} p^{\alpha\nu_p(n)+\alpha+1}} \\ &\sim -\frac{(p-2)x^{\alpha+1-\beta}}{(\alpha+1-\beta)(p-1)p^{\alpha+1}\left(p^{\alpha+1}-1\right)} [p \geq 3]_{\delta}. \end{aligned}$$

When we sum the corresponding terms in these two auxiliary functions over all odd primes $p \leq x$, we obtain that

$$\sum_{3 \leq p \leq x} \left[\frac{\zeta(\alpha+1)(p-1)}{p(p^{\alpha}-1)} U_{1,p}(x) - \zeta(\alpha+1) U_{2,p}(x)\right] \sim -\frac{C_3(\alpha)\zeta(\alpha+1)x^{\alpha+1-\beta}}{(\alpha+1-\beta)}.$$

To complete the estimate for the full formula in this section (for Case III), it remains to estimate [13, cf. §24.4(iii)]

$$\begin{aligned} U_{3,p}(x) &:= \sum_{n=p}^{x} \frac{(-1)^n \cdot n^{\alpha-\beta}}{2^{\alpha+1}} \left(\frac{\zeta(\alpha+1)(p-1)}{p^{\alpha+1}\left(p^{\alpha}-1\right)}\right) [p|n]_{\delta} \\ &= \sum_{n=1}^{x/p} \frac{(-1)^n \cdot (pn)^{\alpha-\beta}}{2^{\alpha+1}} \left(\frac{\zeta(\alpha+1)(p-1)}{p^{\alpha+1}\left(p^{\alpha}-1\right)}\right) \\ &= \left[\sum_{k=0}^{\alpha-\beta} \binom{\alpha-\beta}{k} \frac{(-1)^{\alpha-\beta}E_k}{2^{k+\alpha+2}} \left(\frac{x}{p}-\frac{1}{2}\right)^{\alpha-\beta-k} + \frac{(-1)^{\alpha-\beta}}{2^{2\alpha+2-\beta}} \sum_{k=0}^{\alpha-\beta} \binom{\alpha-\beta}{k} (-1)^k E_k\right] \\ &\qquad \times \left(\frac{\zeta(\alpha+1)(p-1)}{p^{\beta+1}\left(p^{\alpha}-1\right)}\right) \end{aligned}$$

Thus in total, for Case (III) we have that

$$\begin{aligned} \Sigma''_{\alpha,\beta}(x) &\sim -\frac{C_3(\alpha)\zeta(\alpha+1)x^{\alpha+1-\beta}}{(\alpha+1-\beta)} + \sum_{r=0}^{\alpha-\beta} C_{4,r}(\alpha,\beta)x^r \\ &\quad + \frac{(-1)^{\alpha-\beta}}{2^{2\alpha+2-\beta}} \sum_{k=0}^{\alpha-\beta} \binom{\alpha-\beta}{k} (-1)^k E_k \times \sum_{p \geq 2} \frac{\zeta(\alpha+1)(p-1)}{p^{\beta+1}\left(p^{\alpha}-1\right)} + o(1), \text{ as } x \to \infty. \end{aligned} \tag{21}$$

Step (IV). Last Remaining Estimates: Finally, we have only one component in the sums from Theorem 1.6 left to bound. Namely, we must bound the sums

$$\Sigma'''_{\alpha,\beta}(x) := \sum_{n \leq x} \tau_{-\alpha}(n) n^{\alpha-\beta} = \sum_{n \leq x} \sum_{d=1}^{n} \frac{n^{\alpha-\beta}}{d^{\alpha+1}} H^{(\alpha+1)}_{\left\lfloor \frac{n}{d} \right\rfloor} c_d(n) \chi_{\mathrm{PP}}(d).$$

Now by expanding the previous sums according to the identities in Lemma 3.2, we obtain that

$$\begin{aligned}\Sigma'''_{\alpha,\beta}(x) &= \sum_{n\leq x}\sum_{d=1}^{n}\left(\sum_{r|(d,n)} r\mu(d/r)\right) H^{(\alpha+1)}_{\lfloor \frac{n}{d}\rfloor}\frac{\chi_{\mathrm{PP}}(d)}{d^{\alpha+1}} n^{\alpha-\beta} \\ &= \sum_{n\leq x}\sum_{d=1}^{n}\left(\sum_{r|(d,n)} r\mu(d/r)\right) H^{(\alpha+1)}_{\lfloor \frac{n}{d}\rfloor}\frac{n^{\alpha-\beta}}{d^{\alpha+1}} \\ &\quad - \sum_{n\leq x}\sum_{p\leq n}\sum_{k=1}^{\log_p(n)} C_{p^k}(n)\left(\zeta(\alpha+1)+O\left(\frac{p^{(\alpha+1)k}}{n^{\alpha+1}}\right)\right)\frac{n^{\alpha-\beta}}{p^{(\alpha+1)k}} \\ &= \sum_{n\leq x}\sum_{r|n}\sum_{1\leq d\leq r}\frac{r^{\alpha}}{n^{\beta}}\frac{\mu(d)}{d^{\alpha+1}} H^{(\alpha+1)}_{\lfloor \frac{r}{d}\rfloor} \\ &\quad - \sum_{p\leq x}\sum_{n=1}^{x/p}\sum_{k=1}^{\nu_p(n)-1} p^k \times \left(\zeta(\alpha+1)\frac{(p^k n)^{\alpha-\beta}}{p^{(\alpha+1)k}} + O\left(\frac{1}{n^{\beta+1}}\right)\right). \qquad (22)\end{aligned}$$

In the transition from the second to last to the previous equation, we have used a known fact about the *Ramanujan sums*, $c_{p^k}(n)$, at prime powers. In particular, $c_{p^k}(n)=0$ whenever $p^{k-1}\nmid n$, and where the function is given by $c_{p^k}(n)=-p^{k-1}$ if $p^{k-1}|n$, but $p^k\nmid n$ or $c_{p^k}(n)=\phi(p^k)=p^k-p^{k-1}$ if $p^k|n$. For the first sum terms in Equation (22), observe that a formula we obtain by applying ordinary summation by parts corresponds to the identity that

$$H^{(m)}_{\lfloor \frac{r+1}{d}\rfloor} - H^{(m)}_{\lfloor \frac{r}{d}\rfloor} = \left(\frac{r}{d}\right)^{-m}[d|r]_{\delta}\,.$$

Then we see that this first sum is bounded by

$$\begin{aligned}\sum_{n\leq x}\sum_{r|n}\sum_{1\leq d\leq r}\frac{r^{\alpha}}{n^{\beta}}\frac{\mu(d)}{d^{\alpha+1}} H^{(\alpha+1)}_{\lfloor \frac{r}{d}\rfloor} &= O\left(\sum_{n\leq x}\sum_{r=1}^{n}\left(r^{\alpha+1}+O(r^{\alpha})\right)\sum_{d|r}\frac{|\mu(d)|}{d^{\alpha+1}}\left(\frac{d}{r}\right)^{\alpha+1}\right) \\ &= O\left(\sum_{n\leq x}\frac{1}{n^{\beta}}\sum_{r=1}^{n}\sum_{d|r}|\mu(d)|\right) \\ &= O\left(\sum_{n\leq x}\frac{1}{n^{\beta}}\sum_{r=1}^{n}2^{\omega(r)}\right) \\ &= O\left(\sum_{n\leq x}\frac{1}{n^{\beta}}\sum_{k\geq 1}2^k\cdot\#\{1\leq r\leq n:\omega(r)=k\}\right).\end{aligned}$$

We can draw upon the result of Erdős in [5] to approximately sum (non-uniformly for $k>\log\log x$) the right-hand-side of the last equation as follows for any integers

$\beta > 1$:

$$O\left(\sum_{n\leq x} 2(1+o(1))n^{1-\beta}\log n\right) = \begin{cases} O\left(\log^2 x\right), & \text{if } \beta = 2; \\ O\left(\frac{2(\beta-2)\log x+1}{(\beta-2)^2 x^{\beta-2}}\right), & \text{if } \beta \geq 3. \end{cases}$$

We bound the order of the second sum in Equation (22) as

$$\begin{aligned} V_{2,p}(x) &:= \sum_{n=1}^{x/p} \sum_{k=1}^{\nu_p(n)-1} p^k \times \left(\zeta(\alpha+1)\frac{(p^k n)^{\alpha-\beta}\,[p|n]_\delta}{p^{(\alpha+1)k}} + O\left(\frac{1}{n^{\beta+1}}\right)\right) \\ &= \sum_{n=1}^{x/p} \frac{\zeta(\alpha+1)n^{\alpha-\beta}}{p^{\beta\nu_p(n)}} - \frac{1}{p^\beta}\sum_{j=0}^{\alpha-\beta}\binom{\alpha+1-\beta}{j}\frac{\zeta(\alpha+1)B_j}{(\alpha+1-\beta)}\left(\frac{x}{p}\right)^{\alpha+1-\beta-j} \\ &\quad + O\left(\frac{1}{(p-1)n^\beta} - \frac{p}{(p-1)n^{\beta+1}}\right), \end{aligned}$$

where we have again used the upper bound $\nu_p(n) \leq \log_p(n)$. To evaluate the first sum in the last equation, we can use Abel summation much like we have in the prior estimates of this proof. Namely, we form the summatory functions

$$\begin{aligned} A_{3,p}(t) &= \sum_{i\leq t} \frac{[p|i]_\delta}{p^{\beta\nu_p(i)}} = \sum_{i\leq t/p} \frac{1}{p^{\beta\nu_p(pi)}} \\ &= \sum_{k\geq 1} \frac{1}{p^{\beta k}} \#\{i \leq t/p : \nu_p(i) = k\} \\ &= \sum_{k\geq 1} \frac{1}{p^{\beta k}}\left[\frac{t}{p^{k+1}} - \sum_{i\geq k+1}\frac{t}{p^{i+1}}\right] \\ &= \frac{(p-2)t}{p(p-1)(p^{\beta+1}-1)}. \end{aligned}$$

The complete first sum above is given by

$$\sum_{p\leq x}\sum_{n=1}^{x/p} \frac{\zeta(\alpha+1)n^{\alpha-\beta}}{p^{\beta\nu_p(n)}} = \sum_{p\leq x} \frac{\zeta(\alpha+1)(p-2)x^{\alpha+1-\beta}}{(\alpha+1-\beta)p(p-1)(p^{\beta+1}-1)}.$$

We then obtain that

$$\begin{aligned} \sum_{p\leq x} V_{2,p}(x) &\sim \frac{C_6(\beta)\zeta(\alpha+1)x^{\alpha+1-\beta}}{(\alpha+1-\beta)} + \sum_{j=0}^{\alpha-\beta}\binom{\alpha+1-\beta}{j}\frac{C_{7,j}(\alpha)\zeta(\alpha+1)B_j x^{\alpha+1-\beta-j}}{(\alpha+1-\beta)} \\ &\quad + O\left(\frac{x}{\log x}\right). \end{aligned}$$

□

4. Conclusions

In this article, we started by considering the algebraic building blocks of the Lambert series generating functions for the sum-of-divisors functions in Equation (1). The new exact formulas for these special arithmetic functions are obtained in this case by our observations that reconcile the expansions of the series terms, $q^n \cdot (1-q^n)^{-1}$, with representations involving the cyclotomic polynomials and their logarithmic derivatives. We also employed the formulas for the logarithmic derivatives of the cyclotomic polynomials along with known formulas for reducing cyclotomic polynomials of the form $\Phi_{p^r m}(q)$ when $p \nmid m$ to establish the Lambert series term expansions cited in Equation (8). The expansions of our new exact formulas for the generalized sum-of-divisors functions are closely related to the distribution of the primes $p \leq x$ for any large $x \geq 2$.

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