SOME NESTED SUMS IN TERMS OF THE RIEMANN ZETA FUNCTION AND THE DIRICHLET BETA FUNCTION

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Abstract

We derive some formulas for the Riemann zeta value $\zeta(k)$ in terms of nested sums of depth k and depth $k-1$. Additionally, we derive formulas for the Dirichlet eta and beta values $\eta(k)$ and $\beta(k)$ in terms of depth k nested sums.

1. Main Results

Theorem 1. *Let $k \geq 2$ be an integer, and $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$. Then we have*

$$\zeta(k+1) = \frac{1}{k} \sum_{1 \leq i_1 \leq i_2 \cdots \leq i_k} \frac{1}{i_1 i_2 \cdots i_{k-1} \cdot i_k^2}, \tag{1}$$

$$\zeta(k) = \frac{1}{k-1} \sum_{1 \leq i_1 \leq i_2 \cdots \leq i_k} \frac{1}{i_1 i_2 \cdots i_k \cdot (i_k+1)}, \tag{2}$$

$$\zeta(k) = \frac{1}{2^k - 2} \sum_{1 \leq i_1 \leq i_2 \cdots \leq i_k} \frac{1}{i_1 i_2 \cdots i_k} \cdot \frac{\binom{2i_k}{i_k}}{2^{2i_k}}, \tag{3}$$

$$\eta(k) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^k} = 2^k \sum_{1 \leq i_1 \leq i_2 \cdots \leq i_k} \frac{1}{i_1 i_2 \cdots i_k} \cdot \frac{1}{2^{i_k/2}} \cos\left(\frac{\pi\, i_k}{4}\right), \tag{4}$$

$$\beta(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^k} = \sum_{1 \leq i_1 \leq i_2 \cdots \leq i_k} \frac{1}{i_1 i_2 \cdots i_k} \cdot \frac{1}{2^{i_k/2}} \sin\left(\frac{\pi\, i_k}{4}\right), \tag{5}$$

$$\beta(k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^k} = 1 - \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k} \frac{2^{i_k - 1}}{i_k \binom{2i_k}{i_k}} \cdot \frac{1}{(1+2i_1)(1+2i_2)\cdots(1+2i_k)}, \tag{6}$$

where the summation is over all k-tuples $(i_1, i_2, \cdots, i_k)$ of natural numbers satisfying $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k$ for fixed k, and $\eta(k)$ and $\beta(k)$ represent the Dirichlet eta and beta functions, respectively.

Remark 1. The equation (1) follows by the so-called *sum formula*, proven by Granville [1].

Equation (2) can then somehow be seen as a consequence of (1) by the stuffle regularization of multiple zeta values applied after a partial fractions decomposition. The formal cancellation of these divergences is permissible within the stuffle-regularization framework.

We give different proofs of (1) and (2). The equations (3), (4), (5), and (6) appear to be new.

Proof. The *polylogarithm function*, $\mathrm{Li}_k(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^k}$, has the following representation [2, 3]:

$$\mathrm{Li}_k(z) = -\sum_{1 \le i_1 \le i_2 \cdots \le i_k} \frac{1}{i_1 i_2 \cdots i_k} \left(\frac{-z}{1-z} \right)^{i_k}, \tag{7}$$

where $\Re(z) < 1/2$. The formula 3.2.1.6 in book [4] gives us the following:

$$\zeta(k+1) = \frac{-1}{k} \int_0^\infty \frac{\mathrm{Li}_k(-x)}{(x)(1+x)}\, dx, \tag{8}$$

$$\zeta(k) = -\frac{1}{k-1} \int_0^\infty \frac{\mathrm{Li}_k(-x)}{(1+x)^2}\, dx, \tag{9}$$

$$\zeta(k) = \frac{1}{\pi(2-2^k)} \int_0^\infty \frac{x^{-1/2}\,\mathrm{Li}_k(-x)}{1+x}\, dx. \tag{10}$$

The integral (8) gives us (1) after the fact that

$$\int_0^\infty \frac{x^{i_k-1}}{(1+x)^{i_k+1}}\, dx = \frac{1}{i_k}.$$

The integral (9) gives us (2) after the fact that

$$\int_0^\infty \frac{x^{i_k}}{(1+x)^{i_k+2}}\, dx = \frac{1}{i_k+1}.$$

The integral (10) gives us equation (3) after the following calculation:

$$\begin{aligned} \int_0^\infty \frac{x^{i_k-1/2}}{(1+x)^{i_k+1}}\, dx &= \frac{\sqrt{\pi}}{i_k!} \cdot \Gamma(i_k+1/2) \\ &= \frac{\pi}{i_k!} \left(\frac{1}{2}\right)^{(i_k)} \\ &= \frac{\pi}{i_k!} \frac{1 \cdot 3 \cdot 5 \cdots (2i_k-1)}{2^{i_k}} \\ &= \frac{\pi}{2^{2i_k}} \binom{2i_k}{i_k}. \end{aligned}$$

To prove (4) and (5) we recall that $\mathrm{Li}_k(-i) = -2^{-k}\eta(k) - i\beta(k)$ and then use series representation (7).

To prove (6) we use the following result from [3]:

$$\begin{aligned}\mathrm{Li}_s^{\alpha}(w) &= \sum_{n=1}^{\infty} \frac{w^n}{(\alpha+n)^k} \\ &= -\sum_{1\le i_1\le\cdots\le i_k} \frac{(i_k-1)!}{(\alpha+1)(\alpha+2)\cdots(\alpha+i_k)}\,\frac{1}{(\alpha+i_1)(\alpha+i_2)\cdots(\alpha+i_{k-1})}\cdots \\ &\quad \left(\frac{-w}{1-w}\right)^{i_k}.\end{aligned}$$

Letting $\alpha = 1/2$ and $w = -1$ in the above we get (6) at once. □

Remark 2. The integral relation

$$(m+n)\,\zeta(1+m+n) = \int_0^{\infty} \frac{\mathrm{Li}_m(-1/t)\ \mathrm{Li}_n(-t)}{t}\,dt$$

translates to the identity

$$(m+n)\,\zeta(1+m+n) = \sum_{\substack{1\le i_1\le\cdots\le i_n\\ 1\le j_1\le\cdots\le j_m}} \frac{1}{i_1\cdots i_n}\,\frac{\beta(i_n,j_m)}{j_1\cdots j_m}$$

in view of (7). Here $\beta(i,j)$ represents the Beta function.

Remark 3. Some relevant work highlighting the applications of the polylogarithm and related functions are [5, 6, 7, 8].

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