DIRICHLET SERIES WITH TRIGONOMETRIC COEFFICIENTS

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Abstract

In these short notes we give closed formulas for certain Dirichlet series at integer arguments with trigonometric coefficients. Our method uses a one-to-one correspondence between Dirichlet series with periodic coefficients and cotangent sums.

1. Introduction

In these notes we give short proofs for formulas like

$$\frac{\cot^2\left(\frac{\pi}{5}\right)}{1^2}+\frac{\cot^2\left(\frac{2\pi}{5}\right)}{2^2}+\frac{\cot^2\left(\frac{3\pi}{5}\right)}{3^2}+\frac{\cot^2\left(\frac{4\pi}{5}\right)}{4^2}+\frac{\cot^2\left(\frac{6\pi}{5}\right)}{6^2}+\cdots=\frac{28\pi^2}{125}. \qquad (1)$$

As we will see, identity (1) is the special case $N = 5$ of the general equation

$$\sum_{\substack{n>0\\ N\nmid n}} \frac{\cot^2\left(\frac{n\pi}{N}\right)}{n^2}=\frac{(N-1)(N-2)(N^2+3N+2)\pi^2}{90N^2}. \qquad (2)$$

The proof relies on some correspondence (see Theorem 1) between cotangent sums and a class of (generalized) L-series with periodic coefficients. Cotangent sums are given by finite sums involving trigonometric expressions. One classical example is the identity

$$\sum_{j=1}^{N-1}\cot^2\left(\frac{j\pi}{N}\right)=\frac{(N-1)(N-2)}{3}, \qquad (3)$$

valid for all integers $N > 0$. Berndt and Yeap [3] were able to extend (3) to the beautiful formula

$$\sum_{j=1}^{N-1}\cot^{2n}\left(\frac{\pi j}{N}\right)=(-1)^nN-(-1)^n2^{2n}\sum_{j_0=0}^{n}\left(\sum_{\substack{j_1,\dots,j_{2n}\geq 0\\ j_0+j_1+\cdots+j_{2n}=n}}\prod_{r=0}^{2n}\frac{B_{2j_r}}{(2j_r)!}\right)N^{2j_0} \qquad (4)$$

valid for all integers $n, N > 0$, where the Bernoulli numbers B_{2j} are defined in (5). Equation (4) follows from a reciprocity law for the general cotangent sum

$$s_m(h,k;a,b) := \sum_{0<r+b<k} \cot^m\left(\frac{r+b}{k}\pi\right)\cot\left(\left(\frac{h(r+b)}{k}-a\right)\pi\right)$$

which can be proved using contour integration. For example, the Dedekind sum is given by

$$s_1(h,k;0,0)+s_1(k,h;0,0)-\frac{1}{12hk}=\frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}-3\right).$$

Such reciprocity laws have also been studied in the context of moments of the Riemann zeta function by Bettin and Conrey [6] and were also extended by Bettin [4] to sums of the form

$$c_a(h,k) := k^a\sum_{r=1}^{k-1}\cot\left(\frac{hr\pi}{k}\right)\zeta\left(-a,\frac{r}{k}\right),$$

where $\zeta(s,x) := \sum_{n=1}^{\infty}(n+x)^{-s}$ denotes the Hurwitz zeta function. They have also proven to be of importance in the Nyman-Beurling criterion for the Riemann Hypothesis; see [6] and for more information on Nyman-Beurling, [1]. For example, if we define the so called Vasyunin sum by

$$V(h,k) := \sum_{r=1}^{k-1}\frac{r}{k}\cot\left(\frac{rh\pi}{k}\right),$$

we have the following interesting reciprocity formula:

$$\begin{aligned} V(h,k)+V(k,h) &= \frac{\log(2\pi)-\gamma}{\pi}(k+h)+\frac{k-h}{\pi}\log\left(\frac{h}{k}\right) \\ &\quad -\frac{\sqrt{hk}}{\pi^2}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left(\frac{h}{k}\right)^{it}\frac{\mathrm{d}t}{\frac{1}{4}+t^2}. \end{aligned}$$

The distribution behavior of the Vasyunin sums was investigated by Bettin [5]. In this context, we should also mention that cotangent sums are associated to the zeros of the Estermann zeta function $E\left(s,\frac{r}{b},\alpha\right)$, that can be defined via

$$E\left(s,\frac{r}{b},\alpha\right) := \sum_{n=1}^{\infty}\frac{d_\alpha(n)\exp(2\pi inr/b)}{n^s},$$

where $d_\alpha(n) := \sum_{d|n} d^\alpha$ denotes the generalized divisor sum. Cotangent sums associated to the Estermann zeta function were studied by Maier and Rassias [12]. Moreover, Balasubramanian, Conrey and Heath-Brown [2] used properties of the

Estermann zeta function to prove asymptotic formulas for mean-values of the product consisting of the Riemann zeta function and a Dirichlet polynomial $A(s)$

$$I(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|A\left(\frac{1}{2}+it\right)\right| \mathrm{d}t.$$

Asymptotics for functions of this form are useful for theorems providing lower bounds for the portion of zeros of the Riemann zeta function on the critical line; see for example [10]. Apart from that, a relation to quantum modular forms and the Generalized Riemann hypothesis has been shown by Lewis and Zagier [11].

In this paper we use cotangent sums to gather information about Dirichlet series with cotangent coefficients. Before we can state the main theorem, we need a bit of notation. We recall that the Bernoulli numbers B_n are a sequence of rational numbers defined by the generating series

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \qquad |z| < 2\pi. \tag{5}$$

Let χ be a (primitive) Dirichet character modulo N. We define the generalized Bernoulli number $B_{n,\chi}$ with respect to χ by

$$B_{n,\chi} := \sum_{j=1}^{N-1} \chi(j) \sum_{u=0}^{n} \binom{n}{u} B_u j^{n-u} N^{u-1}.$$

Of course, we can also define $B_{n,\chi}$ via the generating series

$$\sum_{a=1}^{m} \frac{\chi(a) z e^{az}}{e^{mz}-1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} z^n, \qquad |z| < 2\pi, \tag{6}$$

which is more useful in the context of Dirichlet L-functions, see also [13] on p. 94. As usual, we define $L(\chi; s)$ to be the Dirichlet L-function for the character χ

$$L(\chi; s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

In particular, when χ is the trivial character, we obtain the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

By elementary methods, using (5) and (6), one can derive closed formulas for $\zeta(j)$ and $L(\chi; j)$ if $j > 0$ and $j + \frac{1-\chi(-1)}{2} \equiv 0 \pmod 2$. In this case we obtain

$$\zeta(j) = (-1)^{\frac{j}{2}+1} \frac{(2\pi)^j B_j}{2j!} \tag{7}$$

and, more generally, for any Dirichlet character χ with conductor N:

$$L(\chi;j) = (-1)^{1+(j-(1-\chi(-1))/2)/2} \frac{\mathcal{G}(\chi)}{2i^{(1-\chi(-1))/2}} \frac{(2\pi)^j B_{j,\overline{\chi}}}{N^j j!}. \tag{8}$$

Here, $\mathcal{G}(\chi)$ is the so called Gauss sum of χ and given by

$$\mathcal{G}(\chi) := \sum_{j=1}^{N} \chi(j) e^{\frac{2\pi i j}{N}}.$$

Let $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$. We define the Stirling numbers of the second kind by

$$\left\{ {n \atop k} \right\} := \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n, \qquad 0 \le k \le n,$$

where $\left\{ {0 \atop 0} \right\} := 1$ and $\left\{ {n \atop k} \right\} := 0$ whenever $k > n$ or $k < 0$. Put

$$S^*(n,k) := k! \left\{ {n \atop k} \right\} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n, \qquad k \le n,$$

and

$$\Delta(\ell, u) := \binom{\ell}{u} - \binom{\ell}{u-1}.$$

A key role is played by two double sequences, which we will call $\delta_\nu(u)$ and $\delta_\nu^*(u)$. Both share the property of vanishing when $\nu + u$ is an odd number. Let $\delta_0(0) = \delta_1(0) = \delta_0(1) := 0$, and for integers $\nu, u \ge 0$ with $\nu + u \ge 2$:

$$\delta_\nu(u) := \frac{i^{\nu+u}}{(\nu-1)!} \sum_{\ell=u-1}^{\nu-1} (-1)^{\nu+\ell+u} 2^{\nu-1-\ell} S^*(\nu-1,\ell) \Delta(\ell,u). \tag{9}$$

Since we have $\Delta(\ell, u) = 0$ whenever $\ell < u - 1$, we may also use the more useful formula

$$\delta_\nu(u) = \frac{i^{\nu+u}}{(\nu-1)!} \sum_{\ell=-\infty}^{\nu-1} (-1)^{\nu+\ell+u} 2^{\nu-1-\ell} S^*(\nu-1,\ell) \Delta(\ell,u). \tag{10}$$

For non-negative integers ℓ and $k > 0$ put

$$\delta_{2k}^*(2\ell) := (-1)^{k+\ell} 2^{2k-2\ell} \sum_{\substack{j_1,\ldots,j_{2k} \ge 0 \\ \ell + j_1 + \cdots + j_{2k} = k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!} \tag{11}$$

$$\delta_{2k-1}^*(2\ell-1) := (-1)^{k+\ell} 2^{2k-2\ell} \sum_{\substack{j_1,\ldots,j_{2k-1} \ge 0 \\ 2\ell - 1 + 2j_1 + \cdots + 2j_{2k-1} = 2k-1}} \prod_{r=1}^{2k-1} \frac{B_{2j_r}}{(2j_r)!}. \tag{12}$$

The sequences δ and δ^* have some remarkable properties. Using cotangent sums, one can show that for all integers $m > 0$, the lower diagonal matrices $U := (\delta_\nu(u))_{1\leq\nu\leq m, 1\leq u\leq m}$ and $U^* := (\delta^*_\nu(u))_{1\leq\nu\leq m,1\leq u\leq m}$, satisfy $U^{-1} = U^*$. This is a consequence of the following theorem that expresses cotangent sums in terms of rational combinations of periodic L-type series, and vice versa.

Theorem 1. *Let $\beta : \mathbb{R} \to \mathbb{C}$ be a 1-periodic function with $\beta(0) = 0$ and finite support in $(0, 1)$. Then we have for all $k \geq 1$:*

$$\sum_{x\in\mathbb{R}^\times} \beta(x)x^{-k} = \pi^k \sum_{\ell=0}^{k} \delta_k(\ell) \sum_{x\in(0,1)} \beta(x)\cot^\ell(\pi x) \tag{13}$$

and conversely

$$\sum_{x\in(0,1)} \beta(x)\cot^k(\pi x) = \sum_{\ell=1}^{k} \delta^*_k(\ell)\left(\pi^{-\ell} \sum_{x\in\mathbb{R}^\times} \beta(x)x^{-\ell} - \delta_\ell(0) \sum_{x\in(0,1)} \beta(x)\right). \tag{14}$$

In the case $k = 1$ we put

$$\sum_{x\in\mathbb{R}^\times} \beta(x)x^{-1} := \lim_{N\to\infty} \sum_{0<|x|\leq N} \beta(x)x^{-1}.$$

A detailed proof of Theorem 1 proceeds by using contour integration, some special formulas for cotangent sums and by comparing coefficients between polynomials, see [8]. For the interested reader we mention that the framework behind Theorem 1 also arises in the context of a natural relation between rational functions and modular forms [7]. We also note that equation (13) reminds us of an identity proven by Ishibashi [9], where values $E\left(0, \frac{r}{b}, \alpha\right)$ of the Estermann zeta function are expressed as finite sums of cotangent functions.
Our first main theorem provides closed formulas for values of Dirichlet series with coefficients $\cot^m\left(\frac{n\pi}{N}\right)$.

Theorem 2. *Let $k > 0$ and $m \geq 0$ be integers, such that $k+m \equiv 0 \pmod 2$. Then we have the formula*

$$\sum_{\substack{n>0\\ N\nmid n}} \frac{\cot^m\left(\frac{n\pi}{N}\right)}{n^k} = \left(\frac{\pi}{N}\right)^k \sum_{j=1}^{\frac{m+k}{2}} a_{k,m}(2j)\zeta(2j)\pi^{-2j}\left(N^{2j} - 1\right), \tag{15}$$

where the rational numbers $a_{k,m}(j)$ are given by

$$a_{k,m}(j) = \sum_{\ell=0}^{k} \delta_k(\ell)\delta^*_{m+\ell}(j). \tag{16}$$

In particular, the values on the right-hand side of Theorem 2 all lie in $\mathbb{Q}\pi^k$. This is a special case of the following principle, that can be applied to series

$$\sum_{\substack{n=-\infty \\ N\nmid n}}^{\infty} \frac{f(n)}{n^k} \tag{17}$$

where $f(n)$ has period N (we formally assume $f(0) = 0$): If the limits of this series appear to be in $K\pi^k$ for $k > 0$, where K is some extension of $\mathbb{Q}$, we can conclude that the corresponding cotangent sums

$$\sum_{n=1}^{N-1} f(n) \cot^m\left(\frac{\pi n}{N}\right) \tag{18}$$

are in K for all $m \geq 0$, and vice versa. This can be verified applying Theorem 1 to the 1-periodic function $\beta(x) := f(Nx)$ (which shall be zero if $Nx \notin \mathbb{Z}$). Assume, that the cotangent sums (18) only have values in K. Then by (13) we conclude that (17) is an element of $K\pi^k$ for all values of k, since the coefficients $\delta_k(\ell)$ are all rational. The other direction is similar and uses (14).
For instance, the fact that the expression on the right side of (4) is a rational corresponds to the fact that $\zeta(2k) \in \mathbb{Q}\pi^{2k}$ for all integers $k > 0$, where $\zeta(s)$ denotes the Riemann zeta function. Note that in this context one uses the equation

$$\sum_{\substack{n>0 \\ N\nmid n}} \frac{1}{n^k} = \left(1 - \frac{1}{N^k}\right)\zeta(k).$$

Theorem 3. *Let $N > 1$, $k > 0$ and $m \geq 0$ be integers, and χ be a primitive Dirichlet character modulo N. Assume that $k + m + \frac{1-\chi(-1)}{2} \equiv 0 \pmod 2$. Then we have the formula*

$$\sum_{\substack{n>0 \\ N\nmid n}} \frac{\chi(n)\cot^m\left(\frac{n\pi}{N}\right)}{n^k} = \sum_{j=1}^{m+k} a_{k,m}(j) L(\chi; j)\pi^{k-j} N^{j-k} \tag{19}$$

where the rational numbers $a_{k,m}(j)$ are given in (16).

Remark 1. Note that only the terms in $L(\chi; j)$ with $j \equiv \frac{1-\chi(-1)}{2} \pmod 2$ appear on the right-hand side due to our condition $k + m + \frac{1-\chi(-1)}{2} \equiv 0 \pmod 2$. Indeed, let χ and m be even. If j is odd, it follows that $\delta_k(\ell)\delta^*_{m+\ell}(j) \neq 0$ implies that k must be odd too. But this contradicts $k+m+\frac{1-\chi(-1)}{2} \equiv 0 \pmod 2$. The argument is the same for all other combinations of χ and m.

We give a short proof of the main formula in Theorem 3 in the next section. The argument is somewhat remarkable in the sense, that it uses the formulas in

Theorem 1 *twice*: We begin with the well-known formulas for $L(\chi;k)$ in the case that $\chi(-1)(-1)^k = 1$, and applying Theorem 1 twice yields exact formulas

$$\left(\sum_{\substack{n=-\infty\\ N\nmid n}}^{\infty} \frac{\chi(n)}{n^k}\right)_{k\in\mathbb{N}} \rightsquigarrow \left(\sum_{n=1}^{N-1} \chi(n)\cot^m\left(\frac{n\pi}{N}\right)\right)_{m\in\mathbb{N}}$$
$$\rightsquigarrow \left(\sum_{\substack{n=-\infty\\ N\nmid n}}^{\infty} \frac{\chi(n)\cot^{m'}\left(\frac{\pi n}{N}\right)}{n^k}\right)_{\substack{k\in\mathbb{N}\\ m'\in\mathbb{N}}}.$$

In other words, the first application (which was already worked out in [8]) simply uses (14) and $\beta(n) = \chi(n)$ as periodic function, as well as the classical formulas for $L(\chi;k)$. The second application relies on decompositions $m = m' + m''$. It uses (13) and the resulting formula of the first application. After the proof, we will work out some corollaries and examples.

2. Proof of the Main Formulas

First, we prove formula (15) in Theorem 2. We obtain with Theorem 1

$$\begin{aligned}\sum_{\ell=1}^{N-1} \cot^{2n}\left(\frac{\pi\ell}{N}\right) &= \sum_{\ell=1}^{2n} \delta^*_{2n}(\ell)\left(\pi^{-\ell}\lim_{T\to\infty}\sum_{0<|x|\le T} x^{-\ell} - \delta_\ell(0)(N-1)\right)\\ &= 2\sum_{\ell=1}^{n} \delta^*_{2n}(2\ell)\zeta(2\ell)\pi^{-2\ell}\left(N^{2\ell}-1\right) - (N-1)\sum_{\ell=1}^{n}\delta^*_{2n}(2\ell)\delta_{2\ell}(0)\end{aligned}$$

and by the supplementary laws

$$\begin{aligned}&= 2\sum_{\ell=1}^{n} \delta^*_{2n}(2\ell)\zeta(2\ell)\pi^{-2\ell}\left(N^{2\ell}-1\right) - (N-1)(-1)^{n-1}\\ &= (N-1)(-1)^n + 2\sum_{\ell=1}^{n} \delta^*_{2n}(2\ell)\zeta(2\ell)\pi^{-2\ell}\left(N^{2\ell}-1\right).\end{aligned} \tag{20}$$

If $m + k \equiv 0 \pmod 2$, we obtain with Theorem 1

$$\begin{aligned}\sum_{\substack{n>0\\ N\nmid n}} \frac{\cot^m\left(\frac{n\pi}{N}\right)}{n^k} &= \frac{1}{2}N^{-k}\sum_{N\nmid n}\frac{\cot^m\left(\frac{n\pi}{N}\right)}{\left(\frac{n}{N}\right)^k}\\ &= \frac{1}{2}\left(\frac{\pi}{N}\right)^k\sum_{\ell=0}^{k}\delta_k(\ell)\sum_{n=1}^{N-1}\cot^{\ell+m}\left(\frac{\pi n}{N}\right).\end{aligned} \tag{21}$$

Now, substitute (20) into (21) and use $\delta^*_{m+\ell}(j) = 0$ if j is odd, to obtain the following term equal to the right-hand side:

$$\frac{1}{2}\left(\frac{\pi}{N}\right)^k \sum_{\substack{\ell=0\\ 2|(\ell+m)}}^{k} \delta_k(\ell)\left((-1)^{\frac{m+\ell}{2}}(N-1) + 2\sum_{j=1}^{\frac{m+\ell}{2}} \delta^*_{m+\ell}(2j)\frac{\zeta(2j)}{\pi^{2j}}\left(N^{2j}-1\right)\right). \quad (22)$$

We can evaluate the alternating sum $\sum_{\ell=0}^{k}(-1)^{\frac{m+\ell}{2}}\delta_k(\ell)$ and show that it always vanishes:

$$\begin{aligned}
\sum_{\ell=0}^{k}(-1)^{\frac{m+\ell}{2}}\delta_k(\ell) &= \sum_{\ell=0}^{k}(-1)^{\frac{m+\ell}{2}}\frac{i^{k+\ell}}{(k-1)!}\sum_{r=-\infty}^{k-1}(-1)^{k+r+\ell}2^{k-1-r}S^*(k-1,r)\Delta(r,\ell)\\
&= \frac{i^{m+k}}{(k-1)!}\sum_{r=-\infty}^{k-1}(-1)^{k+r}2^{k-1-r}S^*(k-1,r)\sum_{\ell=0}^{k}(-1)^{\ell}\Delta(r,\ell)\\
&= \frac{i^{m+k}}{(k-1)!}\sum_{r=-\infty}^{k-1}(-1)^{k+r}2^{k-1-r}S^*(k-1,r)\cdot 0\\
&= 0.
\end{aligned}$$

Hence

$$\sum_{\substack{n>0\\ N\nmid n}}\frac{\cot^m\left(\frac{n\pi}{N}\right)}{n^k} = \left(\frac{\pi}{N}\right)^k \sum_{\substack{\ell=0\\ 2|(\ell+m)}}^{k}\delta_k(\ell)\sum_{j=1}^{\frac{m+\ell}{2}}\delta^*_{m+\ell}(2j)\zeta(2j)\pi^{-2j}\left(N^{2j}-1\right). \quad (23)$$

Since $\delta^*_{m+\ell}(j) = 0$ if $j > m+\ell$, the inner sum on the right-hand side of (23) can be taken to $m+k$ instead of $m+\ell$. We obtain

$$\begin{aligned}
\sum_{\substack{n>0\\ N\nmid n}}\frac{\cot^m\left(\frac{n\pi}{N}\right)}{n^k} &= \left(\frac{\pi}{N}\right)^k \sum_{\substack{\ell=0\\ 2|(\ell+m)}}^{k}\delta_k(\ell)\sum_{j=1}^{\frac{m+k}{2}}\delta^*_{m+\ell}(2j)\zeta(2j)\pi^{-2j}\left(N^{2j}-1\right)\\
&= \left(\frac{\pi}{N}\right)^k \sum_{j=1}^{\frac{m+k}{2}}\sum_{\substack{\ell=0\\ 2|(\ell+m)}}^{k}\delta_k(\ell)\delta^*_{m+\ell}(2j)\zeta(2j)\pi^{-2j}\left(N^{2j}-1\right)\\
&= \left(\frac{\pi}{N}\right)^k \sum_{j=1}^{\frac{m+k}{2}} a_{k,m}(2j)\zeta(2j)\pi^{-2j}\left(N^{2j}-1\right)
\end{aligned}$$

This proves Theorem 2.

The proof of Theorem 3 works similarly. Let χ be a primitive character modulo $N > 1$. By (14) in Theorem 1 we obtain for $m + \frac{1-\chi(-1)}{2} \equiv 0 \pmod 2$:

$$\sum_{n=1}^{N-1} \chi(n) \cot^m \left(\frac{\pi n}{N}\right) = 2 \sum_{j=1}^{m} \delta_m^*(j) \left(\frac{N}{\pi}\right)^j L(\chi; j). \tag{24}$$

Thus, we obtain for integers $k > 0$ with $k + m + \frac{1-\chi(-1)}{2} \equiv 0 \pmod 2$:

$$\begin{aligned}
&\sum_{\substack{n>0 \\ N \nmid n}} \frac{\chi(n) \cot^m \left(\frac{\pi n}{N}\right)}{n^k} = \frac{1}{2} N^{-k} \sum_{N \nmid n} \frac{\chi(n) \cot^m \left(\frac{\pi n}{N}\right)}{\left(\frac{n}{N}\right)^k} \\
&= \frac{1}{2} \left(\frac{\pi}{N}\right)^k \sum_{\ell=0}^{k} \delta_k(\ell) \sum_{n=1}^{N-1} \chi(n) \cot^{\ell+m} \left(\frac{\pi n}{N}\right) \\
&= \left(\frac{\pi}{N}\right)^k \sum_{\ell=0}^{k} \delta_k(\ell) \sum_{j=1}^{m+\ell} \delta_{m+\ell}^*(j) \left(\frac{N}{\pi}\right)^j L(\chi; j).
\end{aligned}$$

Again, since $\delta_{m+\ell}^*(j) = 0$ if $j > m + \ell$, the inner sum can be taken to $m + k$ instead of $m + \ell$, hence this equals

$$\begin{aligned}
&\left(\frac{\pi}{N}\right)^k \sum_{\ell=0}^{k} \delta_k(\ell) \sum_{j=1}^{m+k} \delta_{m+\ell}^*(j) \left(\frac{N}{\pi}\right)^j L(\chi; j) \\
&= \sum_{j=1}^{m+k} \left(\sum_{\ell=0}^{k} \delta_k(\ell) \delta_{m+\ell}^*(j) \right) L(\chi; j) \pi^{k-j} N^{j-k}.
\end{aligned}$$

This proves Theorem 3.

3. Examples and Applications

First, we prove (2) as a corollary of the main formulas.

Corollary 1. *For integers $N > 0$ we have the identity*

$$\sum_{\substack{n>0 \\ N \nmid n}} \frac{\cot^2 \left(\frac{n\pi}{N}\right)}{n^2} = \frac{(N-1)(N-2)(N^2+3N+2)\pi^2}{90 N^2}.$$

Proof. We put $m = k = 2$ and use Theorem 2. Since we have $\delta_2(0) = \delta_2(2) = 1$,

$\delta_2^*(2) = \delta_4^*(4) = 1$ and $\delta_4^*(2) = -\frac{4}{3}$, we obtain $a_{2,2}(2) = -\frac{1}{3}$, $a_{2,2}(4) = 1$, and hence

$$\begin{aligned}\sum_{\substack{n>0\\N\nmid n}} \frac{\cot^2\left(\frac{n\pi}{N}\right)}{n^2} &= \left(\frac{\pi}{N}\right)^2 \left(\left(-\frac{1}{3}\right)\zeta(2)\pi^{-2}(N^2-1) + \zeta(4)\pi^{-4}(N^4-1)\right)\\ &= \frac{(N^4-5N^2+4)\pi^2}{90N^2},\end{aligned}$$

and the claim follows with $N^4 - 5N^2 + 4 = (N-1)(N-2)(N^2+3N+2)$. □

Note that (1) is immediate from Corollary 1.
Similarly, one obtains the identities

$$\begin{aligned}\sum_{\substack{n>0\\N\nmid n}} \frac{\cot\left(\frac{n\pi}{N}\right)}{n^3} &= \frac{(N^4-5N^2+4)\pi^3}{90N^3},\\ \sum_{\substack{n>0\\N\nmid n}} \frac{\cot^2\left(\frac{n\pi}{N}\right)}{n^4} &= \frac{(N^6-7N^4+14N^2-8)\pi^4}{945N^4},\end{aligned}$$

and

$$\sum_{\substack{n>0\\N\nmid n}} \frac{\cot^2\left(\frac{n\pi}{N}\right)}{n^6} = \frac{(3N^8-20N^6+21N^4+60N^2-64)\pi^6}{28350N^6}.$$

From this we can also extract information of the asymptotic growth of these series as N becomes large. We present one detailed example.

Corollary 2. *For even integers $k \geq 4$ we have*

$$\sum_{\substack{n>0\\N\nmid n}} \frac{\cot^2\left(\frac{n\pi}{N}\right)}{n^k} = \frac{\zeta(k+2)}{\pi^2}N^2 - \frac{2}{3}\zeta(k) + O\left(\frac{1}{N^2}\right).$$

Proof. With the formulas

$$\begin{aligned}S^*(n,n) &= n!,\\ S^*(n,n-1) &= \frac{1}{2}(n-1)!n(n-1),\\ S^*(n,n-2) &= \frac{1}{24}(n-2)!n(n-1)(n-2)(3n-5),\end{aligned}$$

valid for all $n > 1$, we obtain for even $k \geq 4$ with (9)

$$\begin{aligned}\delta_k(k-2) = \frac{(-1)^{k-1}}{(k-1)!}&(-4S^*(k-1,k-3)\Delta(k-3,k-2)\\ &+ 2S^*(k-1,k-2)\Delta(k-2,k-2) - S^*(k-1,k-1)\Delta(k-1,k-2)) = \frac{k}{3}.\end{aligned}$$

On the other hand, we find

$$\delta^*_{k+2}(k) = (-1)^{\frac{2k+2}{2}} 2^{k+2-k} \sum_{\substack{j_1,j_2,\ldots,j_{k+2}\geq 0 \\ \frac{k}{2}+j_1+\cdots+j_{k+2}=\frac{k}{2}+1}} \frac{B_2}{2} = -\frac{k+2}{3}.$$

We put all of this into the formula (16) to obtain

$$a_{k,2}(k) = \delta_{k-2}(k)\delta^*_k(k) + \delta_k(k)\delta^*_{k+2}(k) = \frac{k}{3} - \frac{k+2}{3} = -\frac{2}{3},$$

where we have used $\delta^*_k(k) = \delta_k(k) = 1$. We clearly have

$$a_{k,2}(k+2) = \delta_k(k)\delta^*_{k+2}(k+2) = 1,$$

and hence we follow by Theorem 15

$$\begin{aligned}\sum_{\substack{n>0 \\ N\nmid n}} \frac{\cot^2\left(\frac{n\pi}{N}\right)}{n^k} &= \left(\frac{\pi}{N}\right)^k \sum_{j=1}^{\frac{k+2}{2}} a_{k,2}(2j)\zeta(2j)\pi^{-2j}\left(N^{2j}-1\right) \\ &= \frac{\zeta(k+2)}{\pi^2}N^2 - \frac{2}{3}\zeta(k) + O(N^{-k}) \\ &+ \left(\frac{\pi}{N}\right)^k \sum_{j=1}^{\frac{k+2}{2}-2} a_{k,2}(2j)\zeta(2j)\pi^{-2j}\left(N^{2j}-1\right)\end{aligned}$$

which easily proves the claim. □

Finally, we present an example for a twisted series with a non-trivial character.

Example 3.1. *For $k = 3$, $m = 2$ and the Legendre symbol modulo 7 we obtain*

$$\frac{\cot^2\left(\frac{\pi}{7}\right)}{1^3} + \frac{\cot^2\left(\frac{2\pi}{7}\right)}{2^3} - \frac{\cot^2\left(\frac{3\pi}{7}\right)}{3^3} + \frac{\cot^2\left(\frac{4\pi}{7}\right)}{4^3} - \frac{\cot^2\left(\frac{5\pi}{7}\right)}{5^3} - \cdots = \frac{2^7\pi^3}{7^{\frac{7}{2}}}. \quad (25)$$

For the proof of Example 3.1, note that $\delta_3(1) = \delta_3(3) = \delta^*_3(3) = \delta^*_5(1) = \delta^*_5(5) = 1$ and $\delta^*_5(3) = -\frac{5}{3}$. Hence $a_{3,2}(1) = 0$, $a_{3,2}(3) = -\frac{1}{3}$ and $a_{3,2}(5) = \frac{1}{2}$, respectively, for the values in (16). The formula (25) now follows with $L(\chi;3) = \frac{2^5\pi^3}{7^{\frac{5}{2}}}$, $L(\chi;5) = \frac{2^6\pi^5}{3\cdot 7^{\frac{9}{2}}}$ and (19).

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