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ISOMORPHISM CLASSES OF DISTANCE GRAPHS IN $\mathbb{Q}^3$

Matt Noble
Department of Mathematics and Statistics, Middle Georgia State University, Macon, Georgia
matthew.noble@mga.edu

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Abstract

For $d > 0$, designate by $G(\mathbb{Q}^3, d)$ the graph whose set of vertices is the rational space $\mathbb{Q}^3$, with any two vertices being adjacent if and only if they are Euclidean distance d apart. Deem such a graph to be "non-trivial" if d is actually realized as a distance between points of $\mathbb{Q}^3$. In this work, we prove that non-trivial graphs $G(\mathbb{Q}^3, d_1)$ and $G(\mathbb{Q}^3, d_2)$ are isomorphic if and only if d_1, d_2 are rational multiples of each other. This determination of the isomorphism classes of graphs with vertex set $\mathbb{Q}^3$ answers a question posed by Johnson.

1. Introduction

This work will be an extension of [11], and as such, features a mixture of graph theory, elementary number theory, and geometry. For the most part, we will adhere to the standard graph theory terminology and notation, and for reference, one may consult [3]. Also present in our discussion will be a number of concepts from classical number theory – quadratic residues, reciprocity laws, the Chinese remainder theorem, Dirichlet's theorem on primes in arithmetic progressions. We expect readers to be familiar with such topics, but for a refresher, see [8] or virtually any introductory text. As for the needed geometric background, in all honesty, one can probably get by with just their wits and what is learned in high school.

For graphs G_1 and G_2, define G_1 to be *isomorphic* to G_2, and write $G_1 \simeq G_2$, if and only if there exists a bijective function φ from $V(G_1)$ to $V(G_2)$ such that for any $a, b \in V(G_1)$, $ab \in E(G_1)$ if and only if $\varphi(a)\varphi(b) \in E(G_2)$. A *graph invariant* is any property of a graph G that is guaranteed to be preserved under isomorphism. Common invariants that we will touch upon are the chromatic number $\chi(G)$ and the clique number $\omega(G)$. As well, we will employ one bit of non-standard notation. For graph G, $\ell \in \mathbb{Z}^+$, and $a, b \in V(G)$, define a function $f_\ell(a, b)$ that counts the number of distinct paths of length ℓ beginning at a and terminating at b. For graphs G_1, G_2 with isomorphism $\varphi : G_1 \to G_2$, it is easily seen that $f_\ell(a, b) = f_\ell(\varphi(a), \varphi(b))$ for each ℓ and all $a, b \in V(G_1)$.

Of central importance in our work is the notion of the *Euclidean distance graph* (see [13] for a detailed history, along with [2] and [12] for some recent developments). Let $S \subset \mathbb{R}^n$ and $d > 0$. Define $G(S,d)$ to be the graph whose vertices are the points of S, with any two vertices being adjacent if and only if they are a Euclidean distance d apart. Such a graph is deemed *non-trivial* if d is actually realized as a distance between points of S, as otherwise, $G(S,d)$ has an empty edge set, and is not of interest. Throughout, for any points $u, v \in S$, we will notate by $|u - v|$ the Euclidean distance between u, v. Note that the graphs $G(S, d_1)$ and $G(S, d_2)$ are isomorphic if and only if there exists an automorphism φ of S such that for any $u, v \in S$, $|u - v| = d_1$ if and only if $|\varphi(u) - \varphi(v)| = d_2$. Given some S, define an *isomorphism class* as a maximal set $P \subset (0, \infty)$ with the property that $G(S, d_1) \simeq G(S, d_2)$ for any $d_1, d_2 \in P$. Now, in the case of $S = \mathbb{R}^n$, it is trivial to see that the only isomorphism class is the interval $(0, \infty)$ itself, as for any $d_1, d_2 > 0$, the graphs $G(\mathbb{R}^n, d_1)$ and $G(\mathbb{R}^n, d_2)$ are isomorphic by an obvious scaling argument. When S is taken to be some proper subset of $\mathbb{R}^n$, like say, the rational space $\mathbb{Q}^n$, there may be significantly more to the story. However, in the case of d_1, d_2 being rational multiples of each other, such a scaling argument does apply, and we have $G(\mathbb{Q}^n, d_1) \simeq G(\mathbb{Q}^n, d_2)$. This is, of course, a straightforward observation, and for quick reference, we state it as Lemma 1 below. Note also that this implies each non-trivial isomorphism class of $\mathbb{Q}^n$ contains a distance d of the form $d = \sqrt{r}$ where r is some square-free positive integer.

Lemma 1. *Let $d > 0$ and $q \in \mathbb{Q}^+$. For any $n \in \mathbb{Z}^+$, $G(\mathbb{Q}^n, d) \simeq G(\mathbb{Q}^n, qd)$.*

In a compendium work [7] concerning Euclidean distance graphs with vertex set $\mathbb{Q}^n$, Johnson asks for a characterization of all isomorphism classes of such graphs. A somewhat successful attack on this problem was made in [1], in which the following result appeared.

Theorem 1. *Let $n \in \mathbb{Z}^+$ and $d_1, d_2 > 0$ both realized as distances between points of $\mathbb{Q}^n$. Then $G(\mathbb{Q}^n, d_1) \simeq G(\mathbb{Q}^n, d_2)$ under either of the following conditions.*

(*i*) *n is equal to 1, 2, or a multiple of 4.*

(*ii*) *n is even and $d_1 = \sqrt{r_1}, d_2 = \sqrt{r_2}$ where r_1, r_2 are both representable as a sum of two rational squares.*

It was also proven in [1] that for $n \geq 3$ with $n \not\equiv 0 \pmod 4$, there exist specific selections of d_1, d_2 which result in the non-trivial graphs $G(\mathbb{Q}^n, d_1), G(\mathbb{Q}^n, d_2)$ being non-isomorphic. This was shown by finding d_1, d_2 such that the clique number $\omega(\mathbb{Q}^n, d_1) \neq \omega(\mathbb{Q}^n, d_2)$, hence $G(\mathbb{Q}^n, d_1) \not\simeq G(\mathbb{Q}^n, d_2)$. Unfortunately, barring some number of unforeseen innovations, this method of proof cannot be used to answer Johnson's question in general.

In our current work, we will consider a different graph invariant – the function $f_\ell(u, v)$ defined above – and use it, along with a geometric construction, to fully resolve Johnson's question in the case of $n = 3$. Our line of proof follows that presented in [11], where it is shown that for distinct d_1, d_2, the non-trivial graphs $G(\mathbb{Z}^2, d_1), G(\mathbb{Z}^2, d_2)$ are not isomorphic. It should be said, however, that considerably more nuance and finesse is required in the arguments when making the jump from the integer lattice $\mathbb{Z}^2$ to the rational space $\mathbb{Q}^3$. Our main result is that for the case of $n = 3$, Lemma 1 is not only sufficient for $G(\mathbb{Q}^n, d_1) \simeq G(\mathbb{Q}^n, d_2)$, it is necessary as well. In other words, for all pairs of distinct square-free positive integers r_1, r_2 such that $\sqrt{r_1}, \sqrt{r_2}$ are each realized as distances between points of $\mathbb{Q}^3$, $G(\mathbb{Q}^3, \sqrt{r_1}) \not\simeq G(\mathbb{Q}^3, \sqrt{r_2})$.

2. Preliminaries

In this section, we briefly outline the method used in [11] to prove Theorem 2 below. Afterward, we will describe what modifications need to be done to translate this line of proof to the setting of $\mathbb{Q}^3$, and then develop a few lemmas to fit the task. To simplify the discussion, throughout this section and the next, it will be assumed, for any distance graph $G(S, d)$, that d actually is a distance realized between points of S.

Theorem 2. *Let $d_1, d_2 > 0$ with $d_1 \neq d_2$ and denote $G_1 = G(\mathbb{Z}^2, d_1)$ and $G_2 = G(\mathbb{Z}^2, d_2)$. Then $G_1 \not\simeq G_2$.*

The proof given in [11] of Theorem 2 consisted of the steps enumerated below.

1) First, pare down the set of distances we need to consider. It turns out that to show $G_1 \not\simeq G_2$, it suffices to only consider $d_1 = \sqrt{r_1}$, $d_2 = \sqrt{r_2}$ where $r_1, r_2 \in \mathbb{Z}^+$ have prime factorizations consisting solely of factors congruent to 1 modulo 4.

2) Write $d_1 = \sqrt{r_1}$, $d_2 = \sqrt{r_2}$ and assume to the contrary the existence of an isomorphism $\varphi : G_1 \to G_2$. Without loss of generality, assume φ fixes the origin.

3) Show that for any points $p, q \in \mathbb{Z}^2$, each at distance $\sqrt{r_1}$ from the origin, not only are $\varphi(p), \varphi(q)$ at distance $\sqrt{r_2}$ from the origin, but moreover, $\varphi(ip) = i\varphi(p)$ and $\varphi(iq) = i\varphi(q)$ for all $i \in \mathbb{Z}^+$.

4) Prove that for some pair of vectors $p, q \in \mathbb{Z}^2$, each of length $\sqrt{r_1}$, the angle θ between p, q is not realized between any pair of $\mathbb{Z}^2$ vectors, each having length $\sqrt{r_2}$.

5) Observe that for vectors p, q in the previous step, for arbitrarily large integers m, n, the absolute value of the difference between the quantities $\frac{|mp-nq|}{\sqrt{r_1}}$ and $\frac{|\varphi(mp)-\varphi(nq)|}{\sqrt{r_2}}$ is made arbitrarily large as well.

6) Prove that for some selection of integers m, n, there exists $\ell \in \mathbb{Z}^+$ such that exactly one of $f_\ell(mp, nq)$, $f_\ell(\varphi(mp), \varphi(nq))$ is non-zero. As f_ℓ is invariant, conclude that $G_1 \not\cong G_2$.

A similar structure can be followed to attain the main result of this paper. Indeed, in the rest of this section, we will systematically describe what needs to be done to execute these steps concerning graphs with vertex set $\mathbb{Q}^3$. To begin, note that Step 1) has already been done, as in the previous section it was observed that we need only consider graphs of the form $G(\mathbb{Q}^3, \sqrt{r})$ where r is a square-free positive integer.

Regarding Step 2), designate $G_1 = G(\mathbb{Q}^3, \sqrt{r_1})$ and $G_2 = G(\mathbb{Q}^3, \sqrt{r_2})$ where r_1, r_2 are square-free positive integers. As we desire each of G_1, G_2 to be non-trivial, by a classical result of Gauss concerning integers representable as a sum of three squares, we have neither of r_1, r_2 congruent to 7 modulo 8. Assume $\varphi : G_1 \to G_2$ is an isomorphism, and suppose φ maps the origin to itself. Letting $i \in \{1, 2\}$, from the results of Johnson [6] and Chow [5], respectively, we have that $\chi(G_i) = 2$ if r_i is odd, and $\chi(G_i) \geq 3$ if r_i is even. We may therefore assume as well that $r_1 \equiv r_2 \pmod{2}$.

We now establish in Lemma 3 a claim regarding the isomorphism φ defined above that is similar to that of Step 3). Its proof will employ Lemma 2, which will also be utilized in Section 4.

Lemma 2. *Let C be a circle that is centered at a point of $\mathbb{Q}^3$, contains at least one point $p \in \mathbb{Q}^3$, and lies in a plane having normal vector $n \in \mathbb{Q}^3$. Then points of $\mathbb{Q}^3$ are dense on C.*

Proof. Assume C is centered at the origin, and also that C has radius $\sqrt{r}$ for some $r \in \mathbb{Q}^+$. Denote $p = (x_0, y_0, z_0)$, and let $n = \langle n_1, n_2, n_3 \rangle$. The plane containing C is then given by the equation $n_1x + n_2y + n_3z = 0$, and thus any point (x, y, z) on C satisfies Equation 1 below:

$$x^2 + y^2 + \left(\frac{n_1x + n_2y}{n_3}\right)^2 - r = 0. \tag{1}$$

Let function $f(x, y)$ be the left-hand side of Equation 1, and set $f(x, y) = 0$. Consider a line ℓ given by $y = t(x - x_0) + y_0$ where $t \in \mathbb{Q}$. By selecting any t such that ℓ is not tangent to the graph of $f(x, y) = 0$, we have that ℓ intersects its graph in two points, one of which being (x_0, y_0). By substituting $y = t(x - x_0) + y_0$ into Equation 1, we obtain a quadratic equation (of variable x) with rational coefficients, and if such an equation has one rational solution (here, x_0), its other solution must be rational as well. Therefore, as t ranges over $\mathbb{Q}$, we have that the points of $\mathbb{Q}^3$ are dense on C. □

Lemma 3. *Let $p \in \mathbb{Q}^3$ be at distance $\sqrt{r_1}$ from the origin. Then for each $i \in \mathbb{Z}^+$, $\varphi(ip) = i\varphi(p)$.*

Proof. Let $u = (0, 0, 0)$. For any $v \in V(G_1)$ and $n \in \mathbb{Z}^+$, $f_n(u, v) = 1$ if and only if v lies on a sphere of radius $n\sqrt{r_1}$ and centered at u. The "if" part of this statement is obvious, and to see why the "only if" direction holds, first note that if $|u - v| > n\sqrt{r_1}$, we have $f_n(u, v) = 0$. If $|u - v| < n\sqrt{r_1}$, and there exist points $u = p_0, p_1, \ldots, p_n = v$ constituting the vertices of a path of length n in G_1, then there exists some $j \in \{0, \ldots, n - 2\}$ with $|p_j - p_{j+2}| < 2\sqrt{r_1}$. The set of all points simultaneously at distance $\sqrt{r_1}$ from p_j and p_{j+2} is a circle C that fits the hypotheses of Lemma 2. There exist infinitely many points of $\mathbb{Q}^3$ on C, and we can choose one of them, call it q where $q \neq p_{j+1}$, such that $p_0, \ldots, p_j, q, p_{j+2}, \ldots, p_n$ form a path of length n in G_1, showing that in this case, $f_n(u, v) > 1$.

We now proceed by way of induction on i. For the base step, $i = 1$ gives $\varphi(p) = \varphi(p)$. Suppose that the statement holds for all $i \leq k - 1$. We have $|\varphi(kp) - \varphi((k - 1)p)| = \sqrt{r_2}$, or in other words, $\varphi(kp)$ lies on a sphere C_1 centered at $\varphi((k-1)p)$ and having radius $\sqrt{r_2}$. Since $f_k(u, \varphi(kp)) = f_k(u, kp) = 1$, $\varphi(kp)$ must also lie on a sphere C_2 of radius $k\sqrt{r_2}$ centered at the origin. The spheres C_1, C_2 intersect at exactly the point $k\varphi(p)$. This completes the induction step and with it, the proof of the lemma. □

The task stipulated by Step 4) was straightforward to achieve for $\mathbb{Z}^2$. In fact, its proof, given in [11], took up only a few lines. However, proving a similar claim for the rational space Q^3 appears to be a much more involved process, and its proof will take up the majority of the next section of this paper. For now, we will just remark that proving there exist vectors $p, q \in \mathbb{Q}^3$, each of length $\sqrt{r_1}$, such that the angle θ realized between p, q is not realized between any pair of $\mathbb{Q}^3$ vectors of length $\sqrt{r_2}$, is equivalent to showing the existence of a real number c such that the isosceles triangle with side lengths $\sqrt{r_1}, \sqrt{r_1}, c\sqrt{r_1}$ can be drawn with its vertices being points of $\mathbb{Q}^3$, but the isosceles triangle with side lengths $\sqrt{r_2}, \sqrt{r_2}, c\sqrt{r_2}$ cannot.

Step 5) is easy to intuit. Suppose we are given triangles T_1 and T_2 where each has a side of length a and a side of length b, and the respective angles between those sides are θ_1, θ_2 with $\theta_1 < \theta_2$. One can visualize (or prove, using say, the law of cosines) that if c_1 is the third side of T_1 and c_2 is the third side of T_2, it must be the case that $c_1 < c_2$. See Figure 1 below. Additionally, if θ_1, θ_2 are fixed, and a, b grow arbitrarily large, we have $c_2 - c_1$ growing arbitrarily large as well.

Regarding Step 6), we will need to establish Lemma 4 giving an upper bound on the minimum length of a path between two vertices of a graph $G(\mathbb{Q}^3, \sqrt{r})$, supposing

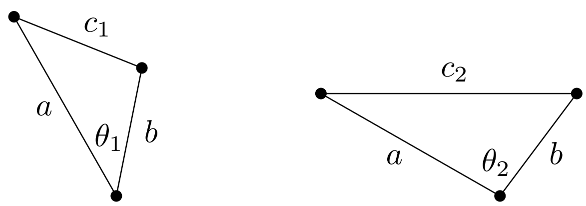

Figure 1

that they are in the same component. The proof of Lemma 4 will involve Theorem 3, which is an extension of a result of Chilakamarri [4], and which may by itself be of some use to those interested in this subject matter.

Theorem 3. *Let r be a square-free positive integer, and let $p \in \mathbb{Q}^3$ where $p = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$ for $a, b, c, d \in \mathbb{Z}$ with $\gcd(a, b, c, d) = 1$. In the graph $G(\mathbb{Q}^3, \sqrt{r})$, p is in the same component as the origin if and only if d is odd and one of the following hold:*

(*i*) $r \equiv 1 \pmod 4$,

(*ii*) $r \equiv 2 \pmod 4$ *and* $a + b + c \equiv 0 \pmod 2$,

(*iii*) $r \equiv 3 \pmod 4$ *and a, b, c are all even or all odd.*

Proof. First, we show that d must be odd. Consider a vector $v \in \mathbb{Q}^3$ of length $\sqrt{r}$. Write $v = (\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$ where $\gcd(x, y, z, w) = 1$ and note that $x^2 + y^2 + z^2 = w^2 r$. Having w even yields $x^2+y^2+z^2 \equiv 0 \pmod 4$, and since the only quadratic residues of 4 are 0 and 1, we have the contradiction of x, y, z even as well. No finite collection of vectors, each of length $\sqrt{r}$ and expressed in the same form as v (with w odd), can possibly sum to p with d being even. For the rest of the proof we therefore may assume d odd.

We now show that the conditions given in Cases (*ii*) and (*iii*) are necessary. Let $r \equiv 2 \pmod 4$ and $v_1, v_2 \in \mathbb{Q}^3$ with $|v_1| = |v_2| = \sqrt{r}$. For $i \in \{1, 2\}$, write $v_i = (\frac{x_i}{w_i}, \frac{y_i}{w_i}, \frac{z_i}{w_i})$ where $x_i, y_i, z_i, w_i \in \mathbb{Z}$ and $\gcd(x_i, y_i, z_i, w_i) = 1$. We have $x_i^2+y_i^2+z_i^2 = w_i^2 r \equiv 2 \pmod 4$ which means exactly two of x_i, y_i, z_i are odd. This gives us $x_i+y_i+z_i \equiv 0 \pmod 2$. Note also that $v_1+v_2 = (\frac{x_1w_2+x_2w_1}{w_1w_2}, \frac{y_1w_2+y_2w_1}{w_1w_2}, \frac{z_1w_2+z_2w_1}{w_1w_2})$ and $x_1w_2+x_2w_1+y_1w_2+y_2w_1+z_1w_2+z_2w_1 = w_1(x_2+y_2+z_2)+w_2(x_1+y_1+z_1) \equiv 0 \pmod 2$. It follows that for p to be in the same component as the origin, we must have $a + b + c \equiv 0 \pmod 2$.

Now suppose $r \equiv 3 \pmod 4$, and let $v = (\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$ where $|v| = \sqrt{r}$, $x, y, z, w \in \mathbb{Z}$, and $\gcd(x, y, z, w) = 1$. We have $x^2 + y^2 + z^2 = w^2 r \equiv 3 \pmod 4$ which implies each of x, y, z are odd. Consider vectors $v_1 = (\frac{x_1}{w_1}, \frac{y_1}{w_1}, \frac{z_1}{w_1}), v_2 = (\frac{x_2}{w_2}, \frac{y_2}{w_2}, \frac{z_2}{w_2})$ where x_1, y_1, z_1 are each even or each odd, and x_2, y_2, z_2 are each even or each odd. It is easily seen that the vector $v_1 + v_2 = (\frac{x_1w_2+x_2w_1}{w_1w_2}, \frac{y_1w_2+y_2w_1}{w_1w_2}, \frac{z_1w_2+z_2w_1}{w_1w_2})$ also has its numerators all even or all odd. It follows that for p to be in the same component as the origin, it must be that a, b, c are all even or all odd.

To see that these conditions are sufficient, first observe that a path in $G(\mathbb{Q}^3, \sqrt{r})$ beginning at $(0, 0, 0)$ and terminating at p can be shown to exist by displaying the existence of a path in the graph $G(\mathbb{Z}^3, \sqrt{rd^2})$ that begins at $(0, 0, 0)$ and terminates at $v = (a, b, c)$. By a classical result of Legendre, there exist non-negative $x, y, z \in \mathbb{Z}$ such that $x^2+y^2+z^2 = rd^2$ and $\gcd(x, y, z) = 1$. Designate $w = \langle x, y, z\rangle$, and let Φ_w be the set of all vectors that can be formed by permuting or negating component entries of w. Let $W \subseteq \mathbb{Z}^3$ be the group of vectors generated by Φ_w under the usual vector addition. Note that for any $\alpha \in W$, any vector formed by permuting or negating component entries of α is also an element of W. Also observe that $\langle x, y, z\rangle+\langle x, -y, -z\rangle = \langle 2x, 0, 0\rangle \in W$, and similarly $\langle 2y, 0, 0\rangle, \langle 2z, 0, 0\rangle \in W$ as well. Since $\gcd(x, y, z) = 1$, there exist $n_1, n_2, n_3 \in \mathbb{Z}^+$ such that $n_1x + n_2y - n_3z = 1$. We then have $n_1\langle 2x, 0, 0\rangle + n_2\langle 2y, 0, 0\rangle + n_3\langle -2z, 0, 0\rangle = \langle 2, 0, 0\rangle \in W$.

Now consider cases. If $r \equiv 1 \pmod 4$, then exactly one of x, y, z (say, x) is odd. We may then start with vector $\langle x, y, z\rangle$, and, as many times as necessary, subtract vectors $\langle 2, 0, 0\rangle$, $\langle 0, 2, 0\rangle$, and $\langle 0, 0, 2\rangle$ to construct the vector $\langle 1, 0, 0\rangle$. This means that in this case, we have W equal to $\mathbb{Z}^3$, so certainly $\langle a, b, c\rangle \in W$.

If $r \equiv 2 \pmod 4$, then exactly two of x, y, z (say, x and y) are odd. Just as in the previous case, we may start with vector $\langle x, y, z\rangle$, and, as many times as necessary, subtract vectors $\langle 2, 0, 0\rangle$, $\langle 0, 2, 0\rangle$, and $\langle 0, 0, 2\rangle$ to construct the vector $\langle 1, 1, 0\rangle$. As each of $\langle \pm1, \pm1, 0\rangle$, $\langle \pm1, 0, \pm1\rangle$, $\langle 0, \pm1, \pm1\rangle$ are elements of W, an appropriate sum using those vectors can be used to construct the desired $\langle a, b, c\rangle$.

Finally, if $r \equiv 3 \pmod 4$, then all of x, y, z are odd. We may then begin with either $\langle 0, 0, 0\rangle$ or x, y, z (depending on whether a, b, c are all even or all odd), and iteratively add or subtract $\langle 2, 0, 0\rangle$, $\langle 0, 2, 0\rangle$, and $\langle 0, 0, 2\rangle$ to create the vector $\langle a, b, c\rangle$. □

Lemma 4. *Let r be a square-free positive integer, and let $p = (\frac{a}{d}, \frac{b}{d}, \frac{c}{d})$ for $a, b, c, d \in \mathbb{Z}$ with $\gcd(a, b, c, d) = 1$. In the graph $G(\mathbb{Q}^3, \sqrt{r})$, suppose that p and the origin lie in the same component. Then there exists a path in $G(\mathbb{Q}^3, \sqrt{r})$, beginning at the origin and terminating at p, and whose length is bounded from above by a function $h(r, d)$.*

Proof. We will prove the claim for the case $r \equiv 1 \pmod 4$ and remark that the cases of $r \equiv 2, 3 \pmod 4$ can be handled in a similar fashion. The upper bound we will display for the length of the desired path is $h(r, d) = r^3d^6$. As in the proof of Theorem 3, consider a path in the graph $G(\mathbb{Z}^3, \sqrt{rd^2})$ that begins at the origin and terminates at (a, b, c). Let x, y, z be non-negative integers where $x^2 + y^2 + z^2 = rd^2$ and $\gcd(x, y, z) = 1$, and assume x odd, with y, z both even. Let $n_1, n_2, n_3 \in \mathbb{Z}$ such that $n_1x+n_2y+n_3z = 1$. We have $|n_1|+|n_2|+|n_3| < xy+xz+yz \leq x^2+y^2+z^2 = rd^2$.

Note also in the proof of Theorem 3 that the vector $\langle 2,0,0\rangle$ is formed as a sum of $2(|n_1|+|n_2|+|n_3|)$ vectors of length $\sqrt{rd^2}$.

Now consider the vector sum in Equation 2 below:

$$\langle x,y,z\rangle - \left(\frac{x-1}{2}\right)\langle 2,0,0\rangle - \left(\frac{y}{2}\right)\langle 0,2,0\rangle - \left(\frac{z}{2}\right)\langle 0,0,2\rangle = \langle 1,0,0\rangle. \tag{2}$$

In the above equation, $\langle 1,0,0\rangle$ is created by summing $1+(\frac{x-1}{2}+\frac{y}{2}+\frac{z}{2})[2(|n_1|+|n_2|+|n_3|)]$ vectors of length $\sqrt{rd^2}$, and since $\langle a,b,c\rangle$ can be formed by summing $|a|+|b|+|c|$ copies of $\langle \pm 1,0,0\rangle$, $\langle 0,\pm 1,0\rangle$, $\langle 0,0,\pm 1\rangle$, we have the total number of length $\sqrt{rd^2}$ vectors used to form $\langle a,b,c\rangle$ given by the quantity $[1+(x+y+z-1)(|n_1|+|n_2|+|n_3|)](|a|+|b|+|c|)$. By hypothesis, $a^2+b^2+c^2<rd^2$, so we are left with $[1+(x+y+z-1)(|n_1|+|n_2|+|n_3|)](|a|+|b|+|c|)<r^3d^6$. □

The rationale behind Lemma 4 (and its intended use later in the paper) is as follows. Let $p_1,p_2\in\mathbb{Q}^3$ where $|p_1-p_2|<\sqrt{r}$. Suppose that in the graph $G(\mathbb{Q}^3,\sqrt{r})$, for some positive integer x_1, there exists a path having x_1 edges that begins at the origin and terminates at p_1, and similarly, there exists a path having x_2 edges that begins at the origin and terminates at p_2. Well, to travel in $G(\mathbb{Q}^3,\sqrt{r})$ from p_1 to p_2, one could simply begin at p_1, backtrack to the origin along the first path described above, and then traverse the edges of the second path to arrive at p_2. However, if x_1,x_2 are large, this may be terribly inefficient. Instead, Lemma 4 indicates that we may take a "more direct" route from p_1 to p_2, with the length of this path depending on r and the vector $\overrightarrow{p_1p_2}$, and not depending on x_1,x_2 themselves.

We close in this section by remarking that the statement of Lemma 4 is acceptable for the role it will play in the proof of our main result in Section 4. That said, the function $h(r,d)=r^3d^6$ is almost certainly a drastic overshot of what a tight upper bound should be. In fact, we will offer this as a question for future investigation.

Question 1. *Let r be a square-free positive integer and $p\in\mathbb{Q}^3$ where $|p|<\sqrt{r}$. In the graph $G(\mathbb{Q}^3,\sqrt{r})$, suppose that p lies in the same component as the origin. Does there exist an integer n such that there is guaranteed to be a path of length less than or equal to n that begins at the origin and terminates at p? If so, is $n=3$ sufficient?*

The stipulated n in the above question is universal in the sense that it does not depend on p. Also, it is easily seen that $n=2$ will not work by applying Theorem 4 in the next section.

3. Isosceles Triangles in $\mathbb{Q}^3$

In this section we prove that for distinct positive, square-free integers r_1,r_2, it is possible to choose $s,t\in\{1,2\}$ such that there exist vectors $p,q\in\mathbb{Q}^3$, each of length

$\sqrt{r_s}$, such that the angle θ realized between p, q is not realized between any pair of $\mathbb{Q}^3$ vectors of length $\sqrt{r_t}$. Our main tool is a characterization of isosceles triangles which can be drawn with their vertices being points of $\mathbb{Q}^3$, originally given by the author in [10]. It is Theorem 4. Incidentally, in its original appearance in [10], the variables r and d were given different assignments, however we present Theorem 4 as it is given below to synch up with the notation used in [11] and in the previous section. Hopefully no confusion results.

Theorem 4. *Let $r, d \in \mathbb{Z}^+$ where $\sqrt{r}$ and $\sqrt{d}$ are both realized as distances in $\mathbb{Q}^3$. Let $d = a^2 + b^2 + c^2$ for $a, b, c \in \mathbb{Q}$ with a, b not both equal to zero. Then the isosceles triangle with side lengths $\sqrt{d}, \sqrt{r}, \sqrt{r}$ is embeddable in $\mathbb{Q}^3$ if and only if the Diophantine equation*

$$x^2 + dy^2 - (4r - d)(a^2 + b^2)z^2 = 0 \tag{3}$$

has a non-trivial integer solution.

The solvability (or insolvability) of Equation 3 can be shown by applying a classical result of Legendre, which we give as Theorem 5. For a proof, see page 219 of [9].

Theorem 5. *Let a, b, c be non-zero integers, not each positive or each negative, and suppose that abc is square-free. Then the Diophantine equation*

$$ax^2 + by^2 + cz^2 = 0$$

has a non-trivial integer solution (x, y, z) if and only if each of the following are satisfied:

(i) $-ab$ is a quadratic residue of c,

(ii) $-ac$ is a quadratic residue of b,

(iii) $-bc$ is a quadratic residue of a.

Before moving on, it will be useful to also summarize the commentary given in [10] regarding Equation 3. For example, when writing $d = a^2 + b^2 + c^2$, it is not transparent that the solvability of Equation 3 is independent of the selection of a, b. We address those concerns with the following observations. When considering a homogeneous quadratic Diophantine equation, if any coefficient is not square-free, we may freely "absorb" the repeated prime factor into that coefficient's corresponding variable. For example, an equation of the form $\alpha x^2 + \beta y^2 + \gamma z^2 = 0$ where $\alpha = \alpha_0 s^2$, is solvable if and only if $\alpha_0 x^2 + \beta y^2 + \gamma z^2 = 0$ is solvable. In our specific case, since $d - (a^2 + b^2) = c^2$, $-(a^2 + b^2)$ is always a residue of d and d is always a residue of $(a^2 + b^2)$. Furthermore, by a well-known characterization of integers that are representable as a sum of two squares (see [8] for elaboration), we have that the square-free part of $(a^2 + b^2)$, call it α, can contain no prime factor congruent to 3 (mod 4). This means that -1 is a residue of α, and in turn, that $-d$ is a residue of α as well.

Theorem 6. *Let r_1, r_2 be distinct square-free positive integers where $r_1 \equiv r_2 \pmod 2$. Suppose that one of the following conditions holds:*

(i) r_1, r_2 are both even,

(ii) r_1, r_2 are both odd, and at least one has a prime factorization of the form $p_1 \cdots p_\alpha q_1 \ldots q_\beta$ where each $p_i \equiv 1 \pmod 4$, each $q_j \equiv 3 \pmod 4$, and β is even.

Then for some selection of $s, t \in \{1, 2\}$, there exist $m, n \in \mathbb{Z}$ such that the isosceles triangle with side lengths $n\sqrt{r_s}$, $n\sqrt{r_s}$, $m\sqrt{r_s}$ is embeddable in $\mathbb{Q}^3$, but the isosceles triangle with side lengths $n\sqrt{r_t}$, $n\sqrt{r_t}$, $m\sqrt{r_t}$ is not.

Proof. First, suppose r_1, r_2 both even. Without loss of generality, assume there exists a prime δ such that δ divides r_2, but δ does not divide r_1. Let $-\delta_0$ be a quadratic non-residue of δ. Write $r_1 = 2p_1 \cdots p_\alpha q_1 \ldots q_\beta$ where each prime $p_i \equiv 1 \pmod 4$ and each prime $q_j \equiv 3 \pmod 4$. Let T_1 be the isosceles triangle with side lengths $n\sqrt{r_1}$, $n\sqrt{r_1}$, $m\sqrt{r_1}$ and T_2 that with side lengths $n\sqrt{r_2}$, $n\sqrt{r_2}$, $m\sqrt{r_2}$. In light of Theorem 4, we have T_1 embeddable in $\mathbb{Q}^3$ if and only if Equation 4 is solvable, where $a_1^2 + b_1^2 + c_1^2 = m^2 r_1$ with $a_1, b_1, c_1 \in \mathbb{Z}$:

$$x^2 + m^2 r_1 y^2 - (4n^2 r_1 - m^2 r_1)(a_1^2 + b_1^2)z^2 = 0. \tag{4}$$

Note that no requirements are given on the parity of m, so we may write $m = 2m_0$, and then substitute to obtain Equation 5:

$$x^2 + 4m_0^2 r_1 y^2 - (4n^2 r_1 - 4m_0^2 r_1)(a_1^2 + b_1^2)z^2 = 0. \tag{5}$$

It is easily seen that Equation 5 has a non-trivial solution if and only if Equation 6 has a non-trivial solution:

$$r_1 x^2 + y^2 - (n^2 - m_0^2)(a_1^2 + b_1^2)z^2 = 0. \tag{6}$$

Similarly, triangle T_2 is embeddable in $\mathbb{Q}^3$ if and only if Equation 7 is solvable, where $a_2^2 + b_2^2 + c_2^2 = m^2 r_2$ with $a_1, b_1, c_1 \in \mathbb{Z}$:

$$r_2 x^2 + y^2 - (n^2 - m_0^2)(a_2^2 + b_2^2)z^2 = 0. \tag{7}$$

The Chinese remainder theorem guarantees the existence of a positive integer k_0 satisfying the following system of linear congruences:

(*i*) $k_0 \equiv 1 \pmod 8$ if β is even or $k_0 \equiv 5 \pmod 8$ if β is odd,

(*ii*) $k_0 \equiv \delta_0 \pmod \delta$,

(*iii*) for each p_i, $k_0 \equiv 1 \pmod{p_i}$,

(*iv*) for each q_j, $k_0 \equiv -1 \pmod{q_j}$.

Consider the sequence $k_0 + 8\delta p_1 \cdots p_\alpha q_1 \cdots q_\beta \ell$ for $\ell = 1, 2, 3, \ldots$. By Dirichlet's theorem, this sequence contains a prime, which we will designate k. Note also that k can be assumed to be larger than both r_1, r_2. Any odd integer can be expressed as a difference of two squares, so let $n^2 - m_0^2 = k$. As $-k$ is not a residue of r_2, Equation 7 is not solvable.

Now consider Equation 6. For solvability, it is sufficient to have $-k$ being a residue of r_1, and $-r_1$ being a residue of k. For each p_i, the Legendre symbol $\left(\frac{-k}{p_i}\right) = \left(\frac{-1}{p_i}\right) = 1$. For each q_j, we have $\left(\frac{-k}{q_j}\right) = \left(\frac{-1}{q_j}\right)\left(\frac{-1}{q_j}\right) = (-1)(-1) = 1$. This shows that $-k$ is indeed a residue of r_1.

We analyze the Legendre symbol $\left(\frac{-r_1}{k}\right)$ in two separate cases. If β is even, we apply the reciprocity laws and have $\left(\frac{-r_1}{k}\right) = \left(\frac{-1}{k}\right)\left(\frac{2}{k}\right)\left(\frac{p_1}{k}\right)\cdots\left(\frac{p_\alpha}{k}\right)\left(\frac{q_1}{k}\right)\cdots\left(\frac{q_\beta}{k}\right) = (1)(1)\left(\frac{k}{p_1}\right)\cdots\left(\frac{k}{p_\alpha}\right)\left(\frac{k}{q_1}\right)\cdots\left(\frac{k}{q_\beta}\right) = (-1)^\beta = 1$. If β is odd, since $k \equiv 5 \pmod 8$, we instead have $\left(\frac{2}{k}\right) = -1$. After a similar application of quadratic reciprocity, we are left with $\left(\frac{-r_1}{k}\right) = (-1)(-1)^\beta = 1$. We therefore conclude that Equation 6 is solvable.

Now assume condition (*ii*) holds. Here, there are two possibilities to consider. For the first case, let r_1 be of the required form – that is, divisible by an even total number of primes congruent to 3 modulo 4 – and suppose that δ is a prime that divides r_2 but not r_1. A resolution of this case will proceed similarly to the above of when condition (*i*) held, so assume $-\delta_0$ to be a non-residue of δ. Express r_1 in the form $r_1 = p_1 \cdots p_\alpha q_1 \ldots q_\beta$ where each $p_i \equiv 1 \pmod 4$, each $q_j \equiv 3 \pmod 4$. Again, we have T_1, T_2 embeddable in $\mathbb{Q}^3$ if and only if Equations 6 and 7, respectively, are solvable. Use Dirichlet's theorem in conjunction with the Chinese remainder theorem to find a prime k satisfying the following system of linear congruences:

(*i*) $k \equiv 1 \pmod 4$,

(*ii*) $k \equiv \delta_0 \pmod \delta$,

(*iii*) for each p_i, $k \equiv 1 \pmod{p_i}$,

(*iv*) for each q_j, $k \equiv -1 \pmod{q_j}$.

With this selection of $k = n^2 - m_0^2$, Equation 7 is not solvable as $-k$ is not a residue of r_2. Thus T_2 is not embeddable in $\mathbb{Q}^3$. Now considering Equation 6, we have $-k$ being a residue of r_1 for the same reason outlined in previous arguments. As well, $\left(\frac{-r_1}{k}\right) = \left(\frac{-1}{k}\right)\left(\frac{p_1}{k}\right)\cdots\left(\frac{p_\alpha}{k}\right)\left(\frac{q_1}{k}\right)\cdots\left(\frac{q_\beta}{k}\right) = (1)\left(\frac{k}{p_1}\right)\cdots\left(\frac{k}{p_\alpha}\right)\left(\frac{k}{q_1}\right)\cdots\left(\frac{k}{q_\beta}\right) = (-1)^\beta = 1$. This shows that $-r_1$ is a residue of k, hence Equation 6 is solvable.

The other possibility we need to analyze is that of r_1 being in the stipulated form of condition (i), r_2 not being expressible in this form, but r_1 being a multiple of r_2. In other words, there does not exist a prime corresponding to δ above. Fortunately, this is only a minor inconvenience. Express r_1 as before, and let k be a prime satisfying the following system of congruences:

(i) $k \equiv 1 \pmod 4$,

(ii) for each p_i, $k \equiv 1 \pmod{p_i}$,

(iii) for each q_j, $k \equiv -1 \pmod{q_j}$.

We may exactly mimic the previous arguments to see that with $k = n^2 - m_0^2$, T_1 is embeddable in $\mathbb{Q}^3$. Now write $r_2 = p_{s_1}\cdots p_{s_\gamma} q_{t_1}\cdots q_{t_\eta}$ where $\gamma \le \alpha$, $\eta < \beta$, and $\{p_{s_1},\ldots,p_{s_\gamma}\}$, $\{q_{t_1},\ldots,q_{t_\eta}\}$ are subsets of the respective $\{p_1,\ldots,p_\alpha\}$, $\{q_1,\ldots,q_\beta\}$. For T_2 to be embeddable in $\mathbb{Q}^3$, we must have $-r_2$ being a residue of k. However, here, t_η is odd, and we have $\left(\frac{-r_2}{k}\right) = \left(\frac{-1}{k}\right)\left(\frac{p_{s_1}}{k}\right)\cdots\left(\frac{p_{s_\gamma}}{k}\right)\left(\frac{q_{t_1}}{k}\right)\cdots\left(\frac{q_{t_\eta}}{k}\right) = (-1)^{t_\eta} = -1$. This concludes the proof of the theorem. □

In Section 2, it was noted that showing the existence of an angle θ realized between two vectors $p, q \in \mathbb{Q}^3$, each of length $\sqrt{r_1}$, but not realized between any pair of $\mathbb{Q}^3$ vectors of length $\sqrt{r_2}$ is equivalent to showing the existence of $c \in \mathbb{R}^+$ such that the triangle with side lengths $\sqrt{r_1}$, $\sqrt{r_1}$, $c\sqrt{r_1}$ is embeddable in $\mathbb{Q}^3$, but the triangle with side lengths $\sqrt{r_2}$, $\sqrt{r_2}$, $c\sqrt{r_2}$ is not. Theorem 6 shows that when r_1, r_2 fit the given hypotheses, $c = \frac{m}{n}$ will do the job. Having c rational is quite convenient for the proof of our main result, as it essentially lets us take a shortcut to the end of the proof without applying all of the lemmas and other machinery put into place in the previous section. Unfortunately, when r_1, r_2 do not fit either of the two given conditions in the statement of Theorem 6, it turns out that for all $c \in \mathbb{Q}^+$, neither of the isosceles triangles with respective side lengths $\sqrt{r_1}$, $\sqrt{r_1}$, $c\sqrt{r_1}$ or $\sqrt{r_2}$, $\sqrt{r_2}$, $c\sqrt{r_2}$ can be embedded in $\mathbb{Q}^3$. We omit proof of this fact as it will not help any in the proof of our main result. Rather, we are just taking this time to alert the reader as to why the following Theorem 7 is needed.

Theorem 7. *Let distinct $r_1, r_2 \in \mathbb{Z}^+$ be odd and square-free, and suppose that each of $\sqrt{r_1}, \sqrt{r_2}$ is realized as a distance between points of $\mathbb{Q}^3$. Suppose that in each of the respective prime factorizations of r_1, r_2, there is an odd total number of prime factors congruent to 3 modulo 4. Then there exists $n \in \mathbb{Z}^+$ such that for some selection of $s, t \in \{1, 2\}$, the isosceles triangle with side lengths $\sqrt{r_s}, \sqrt{r_s}, \sqrt{\frac{8r_s}{n}}$ is embeddable in $\mathbb{Q}^3$, but the triangle with side lengths $\sqrt{r_t}, \sqrt{r_t}, \sqrt{\frac{8r_t}{n}}$ is not.*

Proof. Without loss of generality, assume there exists a prime δ such that δ divides r_2 but does not divide r_1. This assumption means that r_1 will take the place of the r_s described in the statement of the theorem. Let $-\delta_0$ be a quadratic non-residue of δ. Let T_1 be the isosceles triangle with side lengths $\sqrt{r_1}, \sqrt{r_1}, \sqrt{\frac{8r_1}{n}}$, and let T_2 be that with side lengths $\sqrt{r_2}, \sqrt{r_2}, \sqrt{\frac{8r_2}{n}}$. By Theorem 4, T_1 is embeddable in $\mathbb{Q}^3$ if and only if Equation 8 has a non-trivial integer solution. Here, $\frac{8r_1}{n} = a_1^2 + b_1^2 + c_1^2$ for some selection of $a_1, b_1, c_1 \in \mathbb{Q}$:

$$x^2 + \left(\frac{8r_1}{n}\right) y^2 - \left(4r_1 - \frac{8r_1}{n}\right)\left(a_1^2 + b_1^2\right) z^2 = 0. \tag{8}$$

Write $k = n - 2$, and after a small amount of manipulation, one can see that Equation 8 is solvable if and only if Equation 9 is solvable:

$$r_1 n x^2 + 2y^2 - k(a_1^2 + b_1^2) z^2 = 0. \tag{9}$$

Similarly, triangle T_2 is embeddable in $\mathbb{Q}^3$ if and only if Equation 10 is solvable, where $\frac{8r_2}{n} = a_2^2 + b_2^2 + c_2^2$ for some selection of $a_2, b_2, c_2 \in \mathbb{Q}$:

$$r_2 n x^2 + 2y^2 - k(a_1^2 + b_1^2) z^2 = 0. \tag{10}$$

Of course, our goal is to select n so that Equation 9 is solvable, but Equation 10 is not. We will employ a plan similar to that in the proof of Theorem 6, and use the Chinese remainder theorem along with Dirichlet's theorem to find a prime k satisfying a system of linear congruences. Write $r_1 = p_1 \cdots p_\alpha q_1 \dots q_\beta$ where each prime $p_i \equiv 1 \pmod 4$ and each prime $q_j \equiv 3 \pmod 4$, and then select k so that each of the following is true:

(*i*) $k \equiv 3 \pmod 4$,

(*ii*) $k \equiv 1 \pmod p$ for each $p \in \{p_1, \dots, p_\alpha\}$ with $p \equiv 1 \pmod 8$,

(*iii*) $k \equiv 2 \pmod p$ for each $p \in \{p_1, \dots, p_\alpha\}$ with $p \equiv 5 \pmod 8$,

(*iv*) $k \equiv 1 \pmod q$ for each $q \in \{q_1, \dots, q_\beta\}$ with $q \equiv 3 \pmod 8$,

(*v*) $k \equiv -1 \pmod q$ for each $q \in \{q_1, \dots, q_\beta\}$ with $q \equiv 7 \pmod 8$,

(*vi*) $k \equiv \delta_0 \pmod \delta$.

By the same reasoning used in the proof of Theorem 6, condition (*vi*) guarantees that Equation 10 is not solvable, and thus T_2 is not embeddable in $\mathbb{Q}^3$. To show that Equation 9 is solvable, Theorem 5, along with the discussion in the paragraph following the statement of Theorem 5, indicates that we need to verify three things:

that $-2k$ is a residue of n, that $-2k$ is a residue of r_1, and that $-2r_1n$ is a residue of k.

The first is immediate as $k \equiv -2 \pmod{n}$, and thus $-2k \equiv 4 \pmod{n}$ implies that $-2k$ is a residue of n. To see that the second holds, we compute Legendre symbols. Note that if $p|r_1$ with $p \equiv 1 \pmod 8$, then $\left(\frac{-2k}{p}\right) = \left(\frac{-2}{p}\right) = 1$, and if $p|r_1$ with $p \equiv 5 \pmod 8$, then $\left(\frac{-2k}{p}\right) = \left(\frac{-4}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{4}{p}\right) = 1$. If $q|r_1$ with $q \equiv 3 \pmod 8$, then $\left(\frac{-2k}{q}\right) = \left(\frac{-2}{q}\right) = \left(\frac{-1}{q}\right)\left(\frac{2}{q}\right) = (-1)(-1) = 1$, and if $q|r_1$ with $q \equiv 3 \pmod 8$, then $\left(\frac{-2k}{q}\right) = \left(\frac{2}{q}\right) = 1$. These observations show that $-2k$ is a residue of r_1.

To see that the third holds, we start by noting that since $n \equiv 2 \pmod{k}$, we have $\left(\frac{-2r_1n}{k}\right) = \left(\frac{-1}{k}\right)\left(\frac{r_1}{k}\right) = \left(\frac{-1}{k}\right)\left(\frac{p_1}{k}\right)\cdots\left(\frac{p_\alpha}{k}\right)\left(\frac{q_1}{k}\right)\cdots\left(\frac{q_\beta}{k}\right)$. Partition $\{p_1, \ldots, p_\alpha\} = P_1 \cup P_5$ where for $i \in \{1, 5\}$, $p \in P_i$ implies $p \equiv i \pmod 8$. Similarly, partition $\{q_1, \ldots, q_\beta\} = Q_3 \cup Q_7$ where for $j \in \{1, 5\}$, $q \in Q_j$ implies $q \equiv j \pmod 8$. We again compute Legendre symbols, noting that $p \in P_1$ implies $\left(\frac{p}{k}\right) = \left(\frac{k}{p}\right) = \left(\frac{1}{p}\right) = 1$ and $p \in P_5$ implies $\left(\frac{p}{k}\right) = \left(\frac{k}{p}\right) = \left(\frac{2}{p}\right) = -1$. We also have $q \in Q_3$ implies $\left(\frac{q}{k}\right) = (-1)\left(\frac{k}{q}\right) = (-1)\left(\frac{1}{q}\right) = -1$, and $q \in Q_7$ implies $\left(\frac{q}{k}\right) = (-1)\left(\frac{k}{q}\right) = (-1)\left(\frac{-1}{q}\right) = (-1)(-1) = 1$. Since $k \equiv 3 \pmod 4$, we are left with $\left(\frac{-1}{k}\right)\left(\frac{r_1}{k}\right) = (-1)(-1)^{|P_5|}(-1)^{|Q_3|}$.

We claim that $|P_5| + |Q_3|$ must be odd. To see this, suppose to the contrary that it is even, or in other words, that $|P_5| \equiv |Q_3| \pmod 2$. If both $|P_5|, |Q_3|$ are even, since β is odd, we have as well that $|Q_7|$ is odd. The square of any odd integer is congruent to 1 modulo 8, so this leaves us with $r_1 \equiv 7 \pmod 8$ and contradicts the fact that $\sqrt{r_1}$ is a distance realized between points of $\mathbb{Q}^3$. If instead both $|P_5|, |Q_3|$ are odd, we have that $|Q_7|$ is even. Again, this gives the same contradiction of $r_1 \equiv 7 \pmod 8$.

After establishing that $|P_5| + |Q_3|$ is odd, we are left with $(-1)(-1)^{|P_5|}(-1)^{|Q_3|} = 1$, which shows that $-2r_1n$ is a residue of k. We conclude that Equation 9 is solvable, and that T_1 is embeddable in $\mathbb{Q}^3$. □

4. Main Result

We now have all the tools in place to establish our main result, thus giving a full description of the isomorphism classes of Euclidean distance graphs with vertex set $\mathbb{Q}^3$.

Theorem 8. *Let distinct $r_1, r_2 \in \mathbb{Z}^+$, each square-free, such that $\sqrt{r_1}, \sqrt{r_2}$ are both realized as distances between points of $\mathbb{Q}^3$. Then $G(\mathbb{Q}^3, \sqrt{r_1}) \not\cong G(\mathbb{Q}^3, \sqrt{r_2})$.*

Proof. Let $u = (0, 0, 0)$. Denote $G_1 = G(\mathbb{Q}^3, \sqrt{r_1})$ and $G_2 = G(\mathbb{Q}^3, \sqrt{r_2})$. As mentioned in Section 2, we may assume $r_1 \equiv r_2 \pmod 2$. Assume the existence of an isomorphism $\varphi : G_1 \to G_2$, and suppose that φ maps u to itself. We will consider two cases, and in both, apply the lemmas and observations given in the previous two sections.

For the first case, assume that r_1, r_2 satisfy the hypotheses of Theorem 6. Without loss of generality, assume there exist positive integers m, n such that the isosceles triangle T_1 with side lengths $n\sqrt{r_1}$, $n\sqrt{r_1}$, $m\sqrt{r_1}$ is embeddable in $\mathbb{Q}^3$, but triangle T_2 with side lengths $n\sqrt{r_2}$, $n\sqrt{r_2}$, $m\sqrt{r_2}$ is not. Orient T_1 as in Figure 2, where $p, q \in \mathbb{Q}^3$ with $|p| = |q| = \sqrt{r_1}$.

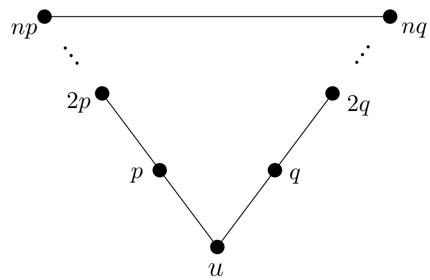

Figure 2

By Lemma 3, $\varphi(np) = n\varphi(p)$ and $\varphi(nq) = n\varphi(q)$. This means that the triangle with vertices $u, \varphi(np), \varphi(nq)$ has side lengths $n\sqrt{r_2}, n\sqrt{r_2}, s$ with $s \neq m\sqrt{r_2}$. As $|np - nq| = m\sqrt{r_1}$, in the graph G_1, we have $f_m(np, nq) = 1$. However, $|\varphi(np) - \varphi(nq)| \neq m\sqrt{r_2}$ implies that in G_2, $f_m(\varphi(np), \varphi(nq)) \neq 1$. Since f_m is guaranteed to be preserved by φ, this is a contradiction.

For the second case, assume that r_1, r_2 satisfy neither condition (i) nor condition (ii) in the statement of Theorem 6. We instead apply Theorem 7, and without loss of generality, assume the existence of vectors $p, q \in \mathbb{Q}^3$, each of length $\sqrt{r_1}$, such that the angle θ_1 realized between p and q is not realized between any pair of $\mathbb{Q}^3$ vectors of length $\sqrt{r_2}$. Let θ_2 be the angle realized between $\varphi(p)$ and $\varphi(q)$. We will also assume that $\theta_1 < \theta_2$, however, a virtually identical argument would suffice in the case of $\theta_2 < \theta_1$.

Let $\mathcal{P}$ be the plane containing p, q, and u. Let ℓ be the ray containing points $u, q, 2q, 3q, \ldots$, and form support lines ℓ_1, ℓ_2 by translating ℓ by the respective vectors $t, -t$, each perpendicular to ℓ and lying in plane $\mathcal{P}$ and having length $\frac{\sqrt{3r_1}}{2}$. For each $i \in \mathbb{Z}^+$, let C_i denote the circle of radius $\sqrt{r_1}$, centered at point iq, and lying in $\mathcal{P}$. For a visual depiction, see Figure 3.

Write $p = \langle \frac{a_p}{d_p}, \frac{b_p}{d_p}, \frac{c_p}{d_p} \rangle$ and $q = \langle \frac{a_q}{d_q}, \frac{b_q}{d_q}, \frac{c_q}{d_q} \rangle$. Now consider a circle C centered at p, having radius $\sqrt{r_1}$, and lying in $\mathcal{P}$. This circle fits the hypotheses of Lemma 2, and we may select a rational point $s \in C$ such that the line containing points p

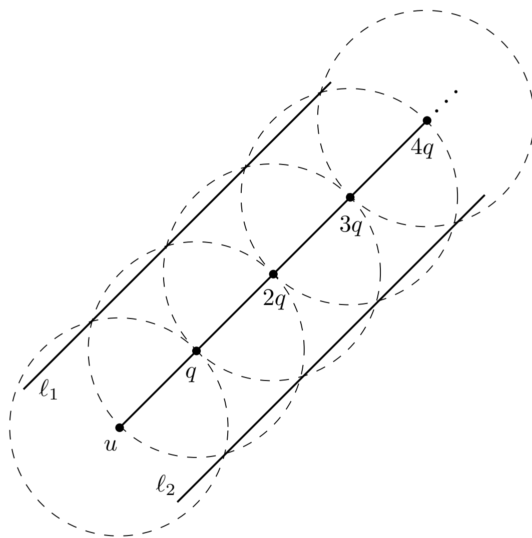

Figure 3

and s intersects ℓ at some point other than the origin u. Let v be the vector with initial point p and terminal point s, and write $v = \langle \frac{a_v}{d_v}, \frac{b_v}{d_v}, \frac{c_v}{d_v} \rangle$. Let $n \in \mathbb{Z}^+$ and form a sequence of points $np, np+v, np+2v, np+3v, \ldots$. There exists a positive integer j (corresponding to our selection of n) such that the point $np+jv$ lies inside a circle C_i, for some $i \in \mathbb{Z}^+$. Denote by α the vector with initial point $np+jv$ and terminal point iq, and note that we can write $\alpha = \langle \frac{a_\alpha}{d_\alpha}, \frac{b_\alpha}{d_\alpha}, \frac{c_\alpha}{d_\alpha} \rangle$ where $d_\alpha = d_p d_q d_v$. By Lemma 4, there exists a path in G_1 that begins at $np+jv$, terminates at iq, and has length bounded from above by a function $h(r, d_\alpha)$.

Now, designate $k = j + h(r, d_\alpha)$. This means that in the graph G_1, we have $f_k(np, iq) > 0$. However, for n taken sufficiently large, the points $\varphi(np), \varphi(iq)$ are such that $|\varphi(np) - \varphi(iq)| > k\sqrt{r_2}$. In the graph G_2, we then have that $f_k(\varphi(np), \varphi(iq)) = 0$. This contradiction completes the proof, ultimately showing that $G_1 \not\simeq G_2$. □

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