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THE BI-PERIODIC FIBONACCI-HORADAM MATRIX

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Abstract

The Fibonacci sequence has been generalized in many ways. Some generalizations are done by altering the recursive equation while others are obtained by changing its initial values. In this work, we look at particular generalizations of this sequence by incorporating a periodic parameter in the first term and a constant coefficient in the second term of the recursive formula. We study two sequences given by $G_{n+1} = p_nG_n + qG_{n-1}$ and $H_{n+1} = p_{n+1}H_n + qH_{n-1}$, where $p_n = a$ when n is odd, and $p_n = b$ when n is even, with initial values $G_0 = 0$, $G_1 = 1$, $H_0 = 2$, $H_1 = a$. Here, a, b, and q are nonzero real numbers. We call these sequences the bi-periodic Fibonacci-Horadam sequence and the bi-periodic Lucas-Horadam sequence, respectively. The initial values correspond to those of the bi-periodic Fibonacci sequence and bi-periodic Lucas sequence, described in the works of Edson et al. (2009) and Bilgici (2014). In contrast to the use of generating functions, elementary matrix analysis will be used to formulate identities of the said new recursions. The bi-periodic Fibonacci-Horadam matrix G is constructed which is then used in the derivations of the different properties of the bi-periodic Horadam sequences including summation identities.

1. Introduction

The famous Fibonacci sequence has been observed and studied for a very long time. From a simple rabbit problem came the birth of the well-known recursion given by

$$f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2$$

with initial terms $f_0 = 0$ and $f_1 = 1$. This sequence struck the interest of many mathematicians which made way to the discovery and/or construction of other sequences with similar structures. Below are three of the most famous variations of this sequence:

$$\begin{aligned} \text{Lucas sequence:} \quad & l_n = l_{n-1} + l_{n-2} && \text{with } \ l_0 = 2,\ l_1 = 1; \\ \text{Pell sequence:} \quad & p_n = 2p_{n-1} + p_{n-2} && \text{with } \ p_0 = 0,\ p_1 = 1; \\ \text{Jacobsthal sequence:} \quad & j_n = j_{n-1} + 2j_{n-2} && \text{with } \ j_0 = 0,\ j_1 = 1. \end{aligned}$$

The Lucas sequence has the same recursion as the Fibonacci sequence but with different initial conditions, while the last two sequences share the same initial conditions as the Fibonacci sequence but has a factor of 2 in the first or second term. Other combination sequences can be formulated such as the Pell-Lucas sequence, whose recursion is that of the Pell sequence, and the Jacobsthal-Lucas sequence, whose recursion is that of the Jacobsthal sequence, both with the initial condition similar to that of the Lucas sequence. For a more in-depth discussion on the history of the Fibonacci sequence and its properties, one may consult [7] and the references therein.

In 1965, A.F. Horadam [5] gave a generalization of these recurrences . He presented the formula

$$h_n = ph_{n-1} + qh_{n-2} \quad \text{for } n \geq 2$$

with arbitrary initial conditions $h_0 = r$, $h_1 = s$, where p, q, r, s are constants. This sequence is now called the Horadam sequence. Surveys and updates on Horadam sequences are well discussed in [9] and [8]. These articles include the early works of Horadam along with summaries of the various methods, techniques, and approaches used for the analysis of different recursions, gathered from a vast list of references for Horadam sequences and other similar or related topics.

There have been other modifications or generalizations of the Fibonacci sequence, one of which is the conditional sequence introduced by Edson et al. [4] in 2009. Similar to the Pell sequence, they inserted a factor in the first term of the recursive formula of the classical Fibonacci sequence. They incorporated a 2-periodic parameter that changes value depending on the parity of the sequence term. Given nonzero constants a and b, the n^{th} term of the bi-periodic Fibonacci sequence is given by

$$F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)} & \text{when } n \text{ is even} \\ \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)} & \text{when } n \text{ is odd} \end{cases} \tag{1}$$

for $n \geq 2$ with $F_0^{(a,b)} = 0$, and $F_1^{(a,b)} = 1$. Similarly, a Lucas counterpart of this sequence was presented by Bilgici [1] in 2014. He yielded the bi-periodic Lucas sequence modeled by the recursion

$$L_n^{(a,b)} = \begin{cases} bL_{n-1}^{(a,b)} + L_{n-2}^{(a,b)} & \text{when } n \text{ is even} \\ \\ aL_{n-1}^{(a,b)} + L_{n-2}^{(a,b)} & \text{when } n \text{ is odd} \end{cases} \tag{2}$$

with $L_0^{(a,b)} = 2$, and $L_1^{(a,b)} = a$. For both bi-periodic sequences, the authors utilized the idea of generating functions to formulate identities involving these two sequences and/or the relationships between them. Recent studies involving the bi-periodic Fibonacci sequence and its analogue, where combinatorial approaches are used, are exhibited in some other papers such as of Tan [11] and Ramírez [10].

The main purpose of this research is to introduce particular generalizations of these sequences namely, the bi-periodic Fibonacci sequence and bi-periodic Lucas sequence, and to use a different method in formulating identities involving these new sequences. In particular, basic matrix properties or simple matrix analysis will be applied to derive analogues of well-known identities and establish other formulas. Researches involving this kind of method were done in a number of literatures relating to Fibonacci sequences. The first one would be the work of C. King [6] in 1960 where he constructed the matrix

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

coined as a $Q-$matrix, to produce properties of Fibonacci numbers. Another example would be that of Cerda [2] in 2012, where he presented the matrix [1]

$$\begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$$

to derive identities involving two types of Horadam sequences - that whose initial values are the same with the Fibonacci sequence ($h_0 = 0$, $h_1 = 1$), and that whose initial values are similar with the Lucas sequence ($h_0 = 2, h_0 = p$). Some similar works in matrices relating to Fibonacci numbers and the like can be seen in the work of the authors in [3] and [12], among others.

In this paper, a combination of the bi-periodic sequences and Horadam sequences will be introduced. We define the bi-periodic Horadam sequences to be $G_n^{(a,b,q)}$ and $H_n^{(a,b,q)}$ corresponding to the initial values of the bi-periodic Fibonacci and bi-periodic Lucas sequences, respectively. More precisely, for nonzero constants a, b, and q, the n^{th} bi-periodic Fibonacci-Horadam number is defined as

$$G_n^{(a,b,q)} = \begin{cases} aG_{n-1}^{(a,b,q)} + qG_{n-2}^{(a,b,q)} & \text{when } n \text{ is even} \\ \\ bG_{n-1}^{(a,b,q)} + qG_{n-2}^{(a,b,q)} & \text{when } n \text{ is odd} \end{cases} \tag{3}$$

[1] The author actually used $-q$ instead of q, maybe for computational purposes.

for $n \geq 2$ with $G_0^{(a,b,q)} = 0$, and $G_1^{(a,b,q)} = 1$, while the n^{th} bi-periodic Lucas-Horadam number is given by

$$H_n^{(a,b,q)} = \begin{cases} bH_{n-1}^{(a,b,q)} + qH_{n-2}^{(a,b,q)} & \text{when } n \text{ is even} \\ \\ aH_{n-1}^{(a,b,q)} + qH_{n-2}^{(a,b,q)} & \text{when } n \text{ is odd} \end{cases} \tag{4}$$

with $H_0^{(a,b,q)} = 2$, and $H_1^{(a,b,q)} = a$. Notice that when $q = 1$, the bi-periodic sequences studied by Edson et al. and Bilgici are obtained, while when $a = b$, the two Horadam sequences studied by Cerda are produced. In fact, a more general case of this sequence has been observed in [12] but has shown results only for particular parity cases and has considered identities in the positive indices. Here, all cases are accounted for.

Moreover, one may recall that an equation related to the Fibonacci sequence is the quadratic equation

$$x^2 - x - 1 = 0$$

whose roots are given by $\phi = \frac{1+\sqrt{5}}{2}$ and $1 - \phi = \frac{1-\sqrt{5}}{2}$. The number ϕ is called the golden ratio which is famous for its ubiquity. As observed, the ratio of consecutive Fibonacci numbers approaches this number. Generalizations of this equation together with its roots are presented in the literatures mentioned above. For this study, an equivalent quadratic equation is given by

$$x^2 - abx - abq = 0$$

whose roots are

$$\alpha = \frac{ab + \sqrt{ab(ab+4q)}}{2} \qquad \text{and} \qquad \beta = \frac{ab - \sqrt{ab(ab+4q)}}{2}.$$

Indeed, $a = b = 1 = q$ gives the classical cases.

Lastly, definitions of these sequences can be extended to negative indices by defining

$$G_{-n}^{(a,b,q)} := q^{-n}(-1)^{n+1}G_n^{(a,b,q)} \qquad \text{and} \qquad H_{-n}^{(a,b,q)} := q^{-n}(-1)^n H_n^{(a,b,q)}.$$

One may verify that these extensions still abide by their corresponding recursions. With these formulations, the first few terms of each sequence are given in the Appendix.

2. Bi-Periodic Fibonacci-Horadam Matrix

To avoid cumbersome notations, fix the constants a, b, and q, and simply use G_n and H_n to indicate the bi-periodic Horadam sequences. Also, the authors in

[4] introduced the parity function $\xi(k) = k - 2\lfloor k/2 \rfloor$ which shows up in most of the properties they formulated. Properties such as $\xi(k) = \xi(k+2j)$, $\xi(-k) = \xi(k)$, $\xi(k+j) = \xi(k-j)$, and $\xi(k) + \xi(k+1) = 1$ for integers k and j, were presented.

For the rest of this paper, the use of terms with ξ or greatest integer function is avoided. Instead, the following definitions will be used. First, define the sequence of alternating a and b as

$$p_n := a^{\xi(n)} b^{\xi(n+1)}.$$

Notice that with this notation, one can write Equation (3) and (4) as

$$G_n = p_{n-1}G_{n-1} + qG_{n-2} \qquad \text{and} \tag{5}$$

$$H_n = p_n H_{n-1} + qH_{n-2}, \tag{6}$$

respectively. Clearly, $p_n p_{n+1} = ab$ for any integer n.

Other sequences involving a, b, and the $\xi-$function that will be used throughout this paper are shown in the following: for any integer n, define $\{r_n\}$ to be the sequence with alternatingly increasing power of b and a given by

$$r_n := a^{\frac{n-\xi(n)}{2}} b^{\frac{n+\xi(n)}{2}}.$$

The first few terms are shown below:

$$\ldots,\ r_{-2} = a^{-1}b^{-1},\ r_{-1} = a^{-1},\ r_0 = 1,\ r_1 = b,\ r_2 = ab,\ r_3 = ab^2,\ r_4 = a^2b^2,\ \ldots.$$

Lastly, for any integer m and n, set the following:

$$s_{m,n} := a^{\frac{\xi(m+n)-\xi(m)-\xi(n)}{2}} b^{\frac{-\xi(m+n)+\xi(m)+\xi(n)}{2}} \qquad \text{and} \qquad s_n := s_{n,n} = a^{-\xi(n)} b^{\xi(n)}.$$

One might notice that the value of $s_{m,n}$ alternates only between $a^{-1}b$ and 1. In particular, $s_{m,n} = a^{-1}b$ whenever m and n are both odd and $s_{m,n} = 1$ otherwise. Any other properties of these sequences p_n, r_n, and $s_{m,n}$ are inherited from the properties of the ξ-function. These are listed in the following proposition whose proof is omitted.

Proposition 1. For integers k, j, n, and m,

1. $p_k r_k = r_{k+1}$
2. $(ab)^k r_{n-k} = r_{n+k}$
3. $(ab)^k s_k = r_k^2$
4. $s_{j,k} = s_{jk} = s_{k,j}$
5. $s_{-j,k} = s_{j,k} = s_{j,-k}$
6. $s_{j+2m,k} = s_{j,k} = s_{j,k+2n}$
7. $s_{j+1,k} = s_{j-1,k}$
8. $p_k = as_{k+1}$
9. $r_{-k}r_{k+1} = b$ and
10. $r_{-k}r_{k-1} = a^{-1}$.

Now, the first two results which are the most fundamental component of this research are ready to be presented. The first one is a generalization of the work of Demirtürk [3], while the second one is an analogue of the result of Cerda [2] for Horadam sequences. The former is stated as follows.

Lemma 1. Let a, b, q be nonzero constants. If X is an invertible matrix such that

$$X^2 - abX - abqI = 0,$$

then for any $n \in \mathbb{Z}$,

$$X^n = r_{n-1}G_nX + r_nqG_{n-1}I.$$

Proof. Let X be an invertible matrix with

$$X^2 - abX - abqI = 0. \tag{7}$$

Using the properties in Proposition 1, mainly items 1 and 2, one can show that Lemma 1 is true for nonnegative integers by induction.

Now, from the property of the matrix, $X(X - abI) = abqI$ which implies that $X - abI = abqX^{-1}$. Hence, the matrix $Y := abI - X = -abqX^{-1}$ is invertible. Squaring both sides of the first equation gives us

$$Y^2 = (ab)^2I - 2abX + X^2.$$

Using (7), one can check that $Y^2 = abY + abqI$, or equivalently, matrix Y satisfies the conditions of the lemma. Hence by the previous result,

$$Y^n = r_{n-1}G_nY + r_nqG_{n-1}I$$

for any nonnegative integer n. Using $Y = -abqX^{-1} = abI - X$ and the properties in Proposition 1, the following equation holds.

$$\begin{aligned}(-abq)^nX^{-n} &= r_{n-1}G_n(abI - X) + r_nqG_{n-1}I\\ &= -r_{n-1}G_nX + r_n(p_nG_n + qG_{n-1})I\\ &= -r_{n-1}G_nX + r_nG_{n+1}I.\end{aligned}$$

Finally, because $G_{-m} = q^{-m}(-1)^{m+1}G_m$ and $(ab)^kr_{j-k} = r_{j+k}$ for integers m, k, and j, dividing both sides of the above equation by $(-abq)^n$ results to

$$X^{-n} = r_{-n-1}G_{-n}X + qr_{-n}G_{-n-1}I,$$

which completes the proof. □

Define the *bi-periodic Fibonacci-Horadam (FH) matrix* by

$$G := \begin{bmatrix} ab & ab \\ q & 0 \end{bmatrix} \tag{8}$$

where a, b, q are nonzero real numbers. The characteristic equation of this matrix is given by $\lambda^2 - ab\lambda - abq = 0$. Notice that when $a = b = 1 = q$, $G = Q$, where Q is the classical Fibonacci $Q-$matrix defined by King. Using the Cayley-Hamilton Theorem for matrices, the n^{th} power of G can be computed to show the bi-periodic Horadam numbers within its components. This is stated in the following theorem.

Theorem 1. Let G be the bi-periodic Fibonacci-Horadam matrix as in (8). Then for any integer n,

$$G^n = r_n \begin{bmatrix} G_{n+1} & p_n G_n \\ p_{n+1}^{-1} q G_n & qG_{n-1} \end{bmatrix}.$$

Proof. First note that $\det(G) = -abq \neq 0$ which implies that G is invertible. In addition, G has the characteristic equation $\lambda^2 - ab\lambda - abq = 0$ which, by the Cayley-Hamilton Theorem, implies that $G^2 - abG - abqI = 0$. Thus, Lemma 1 ensures

$$G^n = r_{n-1}G_nG + qr_nG_{n-1}I = r_{n-1}\begin{bmatrix} abG_n & abG_n \\ qG_n & 0 \end{bmatrix} + qr_n \begin{bmatrix} G_{n-1} & 0 \\ 0 & G_{n-1} \end{bmatrix}.$$

Since $abr_{n-1} = r_{n+1} = p_n r_n$, the matrix sum above can be computed as

$$G^n = r_n \begin{bmatrix} p_nG_n + qG_{n-1} & p_nG_n \\ (ab)^{-1}p_nqG_n & qG_{n-1} \end{bmatrix}.$$

With $G_{n+1} = p_nG_n + qG_{n-1}$ and $(ab)^{-1}p_n = p_{n+1}^{-1}$, the result follows. $\square$

Now, this result will be used to derive generalizations of some well-known identities involving bi-periodic Horadam numbers, starting with the Binet formula (explicit formula) of the bi-periodic Fibonacci-Horadam numbers. The case when $q = 1$ was already presented by Edson et al. in [4]. Using generating functions, the authors derived the *Binet Formula for bi-periodic Fibonacci number* to be

$$q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor n/2 \rfloor}}\right)\frac{\alpha_0^n - \beta_0^n}{\alpha_0 - \beta_0}, \tag{9}$$

where q_n is the n^{th} bi-periodic Fibonacci number and α_0, β_0 are the roots of the equation $x^2 - abx - ab = 0$. In our case, the $r-$notation will be used in the formula which will be proven using matrices, analogous to the work of Cerda in [2] for Horadam sequences. The generalized result is stated as follows.

Theorem 2. For any $n \in \mathbb{Z}$ and $a, b, q \in \mathbb{R} \setminus \{0\}$ with $ab + 4q \neq 0$, the Binet formula for the bi-periodic Fibonacci number is given by

$$G_n = r_{n-1}^{-1} \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \dfrac{ab + \sqrt{a^2b^2 + 4abq}}{2}$ and $\beta = \dfrac{ab - \sqrt{a^2b^2 + 4abq}}{2}$.

Proof. Let G be the bi-periodic FH matrix as in (8). Note that the characteristic equation of G is $\lambda^2 - ab\lambda - abq = 0$, which gives the eigenvalues

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4abq}}{2} \quad \text{and} \quad \beta = \frac{ab - \sqrt{a^2b^2 + 4abq}}{2}.$$

Since a, b, q, and $ab + 4q$ are all nonzero, it must be that α and β are nonzero, distinct, and whose corresponding eigenvectors are $\vec{v_\alpha} = (\alpha, q)^T$ and $\vec{v_\beta} = (\beta, q)^T$, respectively. Thus, matrix G can be diagonalized as $D = P^{-1}GP$, where D is the diagonal matrix equal to $\operatorname{diag}(\alpha, \beta)$ and $P = \begin{bmatrix} \vec{v_\alpha} & \vec{v_\beta} \end{bmatrix}$. Easily, one can compute the inverse of P to be

$$P^{-1} = \begin{bmatrix} \frac{1}{\alpha-\beta} & \frac{1}{q} \cdot \frac{-\beta}{\alpha-\beta} \\ \frac{-1}{\alpha-\beta} & \frac{1}{q} \cdot \frac{\alpha}{\alpha-\beta} \end{bmatrix}$$

and that $D^n = P^{-1}G^nP$, for any nonnegative integer n. Thus, we can write

$$G^n = PD^nP^{-1} = \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha-\beta} & \frac{1}{q} \cdot \frac{-\beta}{\alpha-\beta} \\ \frac{-1}{\alpha-\beta} & \frac{1}{q} \cdot \frac{\alpha}{\alpha-\beta} \end{bmatrix},$$

which can be simplified as

$$G^n = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha^{n+1} - \beta^{n+1} & -\frac{1}{q} \cdot \alpha\beta(\alpha^n - \beta^n) \\ q(\alpha^n - \beta^n) & -\alpha\beta(\alpha^{n-1} - \beta^{n-1}) \end{bmatrix}.$$

Looking at the $(2,1)-$entry of this matrix and comparing that of G^n as in Theorem 1, one can check that

$$r_n p_{n+1}^{-1} q G_n = q \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Because $p_{n+1}^{-1} = (ab)^{-1} p_n$, $r_{n+1} = p_n r_n$, and $(ab)^{-1} r_{n+1} = r_{n-1}$, it follows that

$$G_n = r_{n-1}^{-1} \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for nonnegative n. Now, the inverses of G and D can be solved respectively as

$$G^{-1} = \begin{bmatrix} 0 & q^{-1} \\ (ab)^{-1} & -q^{-1} \end{bmatrix} \quad \text{and} \quad D^{-1} = \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{bmatrix}.$$

Also, $D^{-1} = P^{-1}G^{-1}P$ and because $(A^{-1})^n = A^{(-n)}$, it must be that $G^{-n} = PD^{-n}P^{-1}$. So the result for negative indices can be similarly shown. $\square$

The next corollary is a consequence of Theorem 1. For its proof, the following remark is observed.

Remark 1. For any integer n,

$$G\begin{bmatrix} p_{n+1}^{-1}G_n \\ (ab)^{-1}qG_{n-1} \end{bmatrix} = \begin{bmatrix} G_{n+1} \\ p_{n+1}^{-1}qG_n \end{bmatrix} \text{ and } G\begin{bmatrix} p_n^{-1}H_n \\ (ab)^{-1}qH_{n-1} \end{bmatrix} = \begin{bmatrix} H_{n+1} \\ p_n^{-1}qH_n \end{bmatrix}.$$

These are evident because of the recursion of the bi-periodic Fibonacci- and Lucas-Horadam sequences. The preceding remark gives the following.

Corollary 1. For any integers m, n, and k we have

(i) $G^m\begin{bmatrix} p_n^{-1}H_n \\ (ab)^{-1}qH_{n-1} \end{bmatrix} = G^{m-(2k-1)}(ab)^{k-1}\begin{bmatrix} H_{n+2k-1} \\ p_n^{-1}qH_{n+2k-2} \end{bmatrix}$,

(ii) $G^m\begin{bmatrix} p_n^{-1}H_n \\ (ab)^{-1}qH_{n-1} \end{bmatrix} = G^{m-(2k)}(ab)^{k-1}\begin{bmatrix} p_{n+1}H_{n+2k} \\ qH_{n+2k-1} \end{bmatrix}$,

(iii) $G^m\begin{bmatrix} p_{n+1}^{-1}G_n \\ (ab)^{-1}qG_{n-1} \end{bmatrix} = G^{m-(2k-1)}(ab)^{k-1}\begin{bmatrix} G_{n+2k-1} \\ p_{n+1}^{-1}qG_{n+2k-2} \end{bmatrix}$, and

(iv) $G^m\begin{bmatrix} p_{n+1}^{-1}G_n \\ (ab)^{-1}qG_{n-1} \end{bmatrix} = G^{m-(2k)}(ab)^{k-1}\begin{bmatrix} p_nG_{n+2k} \\ qG_{n+2k-1} \end{bmatrix}$.

Proof. Provided only are the proofs of items (i) and (ii). The proofs of (iii) and (iv) are found in a similar fashion. The proof of the first part will be done in two steps. First, fix m and n and use induction on k. By Remark 1,

$$G^m\begin{bmatrix} p_n^{-1}H_n \\ (ab)^{-1}qH_{n-1} \end{bmatrix} = G^{m-1}(ab)^{1-1}\begin{bmatrix} H_{n+1} \\ p_n^{-1}qH_n \end{bmatrix}.$$

Thus, the equation holds for $k = 1$. Now, suppose that the equality also holds for some $k \geq 1$, that is

$$LHS := G^m\begin{bmatrix} p_n^{-1}H_n \\ (ab)^{-1}qH_{n-1} \end{bmatrix} = G^{m-(2k-1)}(ab)^{k-1}\begin{bmatrix} H_{n+2k-1} \\ p_n^{-1}qH_{n+2k-2} \end{bmatrix}.$$

Given that $G^{m-(2k-1)} = G^{m-(2k+1)}G^2$, by simply computing for G^2, one can verify that

$$LHS = G^{m-(2k+1)}(ab)^k \begin{bmatrix} ab+q & ab \\ q & q \end{bmatrix} \begin{bmatrix} H_{n+2k-1} \\ p_n^{-1}qH_{n+2k-2} \end{bmatrix}.$$

Upon multiplying and using $ab = p_n p_{n+1}$, one arrives at the following computations:

$$LHS = G^{m-(2k+1)}(ab)^k \begin{bmatrix} qH_{n+2k-1} + p_{n+1}(p_n H_{n+2k-1} + qH_{n+2k-2}) \\ p_n^{-1}q(p_n H_{n+2k-1} + qH_{n+2k-2}) \end{bmatrix}.$$

Finally, using $p_{z+2k} = p_z$ given integer z, the above equation can now be written as

$$LHS = G^{m-(2k+1)}(ab)^k \begin{bmatrix} H_{n+2k+1} \\ p_n^{-1}qH_{n+2k} \end{bmatrix}.$$

Hence, (i) is true for positive k. It can be similarly shown to be true for nonpositive integer k. Part (ii) follows directly from part (i), since $G^{m-(2k-1)} = G^{m-2k}G$ and

$$G \begin{bmatrix} H_{n+2k-1} \\ p_n^{-1}qH_{n+2k-2} \end{bmatrix} = \begin{bmatrix} abH_{n+2k-1} + p_{n+1}qH_{n+2k-2} \\ qH_{n+2k-1} \end{bmatrix} = \begin{bmatrix} p_{n+1}H_{n+2k} \\ qH_{n+2k-1} \end{bmatrix}.$$

□

Theorem 3. For any integers m and n,

(i) $H_{n+m} = s_{(m+1)(n+1)}G_m H_{n+1} + s_{mn}qG_{m-1}H_n$ and

(ii) $G_{n+m} = s_{(m+1)n}G_m G_{n+1} + s_{m(n+1)}qG_{m-1}G_n$.

Proof. If m is odd, then by Corollary 1(i) with $m = 2k-1$,

$$G^m \begin{bmatrix} p_n^{-1}H_n \\ (ab)^{-1}qH_{n-1} \end{bmatrix} = G^{m-m}(ab)^{[(m+1)/2]-1} \begin{bmatrix} H_{n+m} \\ p_n^{-1}qH_{n+m-1} \end{bmatrix}$$
$$= (ab)^{(m-1)/2} \begin{bmatrix} H_{n+m} \\ p_n^{-1}qH_{n+m-1} \end{bmatrix}. \tag{10}$$

On the other hand, the second part of the remark assures that

$$G^m \begin{bmatrix} p_n^{-1}H_n \\ (ab)^{-1}qH_{n-1} \end{bmatrix} = G^{m-1} \begin{bmatrix} H_{n+1} \\ p_n^{-1}qH_n \end{bmatrix}$$

which, by Theorem 1, equates to

$$G^m \begin{bmatrix} p_n^{-1}H_n \\ (ab)^{-1}qH_{n-1} \end{bmatrix} = (ab)^{(m-1)/2} \begin{bmatrix} G_m & bG_{m-1} \\ (1/a)qG_{m-1} & qG_{m-2} \end{bmatrix} \begin{bmatrix} H_{n+1} \\ p_n^{-1}qH_n \end{bmatrix}$$
$$= (ab)^{(m-1)/2} \begin{bmatrix} G_mH_{n+1} + a^{-\xi(n)}b^{1-\xi(n+1)}qG_{m-1}H_n \\ q[(1/a)G_{m-1}H_{n+1} + p_n^{-1}qG_{m-2}H_n] \end{bmatrix}. \tag{11}$$

Equating the $(1,1)-$entries of the matrices in Equations (10) and (11), one can easily verify the result to be

$$H_{n+m} = G_mH_{n+1} + s_nqG_{m-1}H_n$$

for odd m. If m is even, i.e., $m = 2k$, then with Corollary 1(ii),

$$LHS := G^m \begin{bmatrix} p_n^{-1}H_n \\ (ab)^{-1}qH_{n-1} \end{bmatrix} = G^{m-m}(ab)^{(m/2)-1} \begin{bmatrix} p_{n+1}H_{n+m} \\ qH_{n+m-1} \end{bmatrix}$$
$$= (ab)^{(m-2)/2} \begin{bmatrix} p_{n+1}H_{n+m} \\ qH_{n+m-1} \end{bmatrix}. \tag{12}$$

Similarly, Remark 1 and Theorem 1 guarantee that

$$LHS = a^{(m-2)/2}b^{m/2} \begin{bmatrix} G_m & aG_{m-1} \\ (1/b)qG_{m-1} & qG_{m-2} \end{bmatrix} \begin{bmatrix} H_{n+1} \\ p_n^{-1}qH_n \end{bmatrix}$$
$$= a^{(m-2)/2}b^{m/2} \begin{bmatrix} G_mH_{n+1} + a^{\xi(n+1)}b^{-\xi(n+1)}qG_{m-1}H_n \\ q[(1/b)G_{m-1}H_{n+1} + a^{-\xi(n)}b^{-\xi(n+1)}qG_{m-2}H_n] \end{bmatrix}. \tag{13}$$

By equating the $(1,1)-$entries of Equations (12) and (13), one can arrive at

$$H_{n+m} = a^{-\xi(n+1)}b^{1-\xi(n)}(G_mH_{n+1} + a^{\xi(n+1)}b^{-\xi(n+1)}qG_{m-1}H_n).$$

Equivalently,

$$H_{n+m} = s_{n+1}G_mH_{n+1} + qG_{m-1}H_n$$

for even m. For such cases, since $s_{2jk} = 1$ and $s_{(2j+1)k} = s_k$, then

$$H_{n+m} = s_{(m+1)(n+1)}G_mH_{n+1} + s_{mn}qG_{m-1}H_n.$$

The proof of (ii) is similar by applying parts (iii) and (iv) of Corollary 1. $\square$

Corollary 2. For any integer m,

(i) $H_m = G_{m+1} + qG_{m-1}$,

(ii) $H_{2m} = s_{m+1}G_m H_{m+1} + s_m qG_{m-1}H_m$, and

(iii) $H_{2m-1} = G_m H_m + qG_{m-1}H_{m-1}$.

Proof. With $n = 0$, use part (i) of Theorem 3 and obtain

$$H_m = s_{m+1}G_m H_1 + s_0 qG_{m-1}H_0.$$

Since $H_1 = a$, $H_0 = 2$, and $s_0 = 1$, we have $H_m = as_{m+1,m+1}G_m + 2qG_{m-1}$. Note that $as_{m+1,m+1} = p_m s_{m,m+1}$, but $s_{m,m+1} = 1$, thus, $H_m = p_m G_m + 2qG_{m-1}$. Regrouping terms and using the recursion of bi-periodic Fibonacci-Horadam sequence will arrive at the desired result.

The second and third equalities can be directly obtained by replacing $n = m$ and $n = m - 1$ in part (i) of Theorem 3, respectively. □

Notice that part (i) of Corollary 2 relates any bi-periodic Lucas-Horadam numbers to the neighbors of the bi-periodic Fibonacci-Horadam number corresponding to it. This property can be utilized to derive the closed form of the bi-periodic Lucas-Horadam number. Here, the proof for the Binet formula of the recursion will be provided using the matrix method and will generalize the one introduced by Bilgici in [1], given by

$$l_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor (n+1)/2 \rfloor}}(\alpha_0^n + \beta_0^n) \tag{14}$$

where l_n is the n^{th} bi-periodic Lucas number and α_0, β_0 are still the roots of the quadratic equation $x^2 - abx - ab = 0$. Similar to [4], Bilgici uses generating functions to derive the above formula. Here, the generalized formula of (14) will be written in $r-$notation and the proof will be an analogue of the work of the author in [2].

Theorem 4. Let $n \in \mathbb{Z}$ and $a, b, q \in \mathbb{R} \setminus \{0\}$ with $ab + 4q \neq 0$. The Binet formula for the bi-periodic Lucas-Horadam number is given by

$$H_n = r_n^{-1}(\alpha^n + \beta^n),$$

where $\alpha = \dfrac{ab + \sqrt{a^2b^2 + 4abq}}{2}$ and $\beta = \dfrac{ab - \sqrt{a^2b^2 + 4abq}}{2}$.

Proof. From elementary matrix analysis and Theorem 1, the characteristic equation of G^n can be realized from the equation

$$\det(\lambda I - G^n) = \lambda^2 - \text{tr}(G^n)\lambda + \det(G^n) = \lambda^2 - r_n(G_{n+1} + qG_{n-1})\lambda + \det(G^n). \tag{15}$$

The characteristic equation is obtained by equating (15) to 0. Note that by Corollary 2, $G_{n+1} + qG_{n-1} = H_n$. Also, recall that $\det(G) = -abq$, which simply implies that $\det(G^n) = (\det G)^n = (-1)^n(abq)^n$. Hence the characteristic equation of G^n is

$$\lambda^2 - r_nH_n\lambda + (-1)^n(abq)^n = 0.$$

So the generalized characteristic values must be $\lambda_{1,2} = \frac{1}{2}(r_nH_n \pm \sqrt{d})$, where $d = r_n^2H_n^2 - 4(-1)^n(abq)^n$. Consequently, since these λ_i's must be the n^{th} power of the eigenvalues of G, then

$$\alpha^n = \frac{r_nH_n + \sqrt{d}}{2} \quad \text{and} \quad \beta^n = \frac{r_nH_n - \sqrt{d}}{2}.$$

Hence, $\alpha^n + \beta^n = r_nH_n$, or equivalently, $H_n = r_n^{-1}(\alpha^n + \beta^n)$. □

The next step is to formulate more identities that generalize well-known ones. By maximizing Theorem 1, analogous equations for the bi-periodic Horadam sequences can be further derived. Starting with the generalized Cassini's identity, the following theorem is stated with $s-$notations.

Theorem 5. (Cassini's Identity) For every integer n, we have

$$s_nG_{n+1}G_{n-1} - s_{n+1}G_n^2 = (-1)^nq^{n-1}.$$

Proof. Note that $\det(G^n) = (-1)^n(abq)^n$ because $\det(G) = -abq$. Also, the determinant of the matrix G^n as in Theorem 1 is given by

$$\det(G^n) = r_n^2q\left(G_{n+1}G_{n-1} - p_{n+1}^{-1}p_nG_n^2\right).$$

Finally, because $r_n^2 = (ab)^ns_n$ and $p_ns_n = p_{n+1}s_{n+1}$ and using the equation above, the result follows. □

When $a = b = 1 = q$, the formula in the previous theorem reduces back to the formula for classical Fibonacci numbers given by Cassini in around 1680:

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n.$$

For the previous results, only item (i) of Theorem 3 is being used. Here is where part (ii) will be utilized. The next results will be the generalizations of the so-called convolution property and d' Ocagne's identity. Bi-periodic generalizations were already presented in [4]. Bi-periodic Horadam analogues are stated here with $s-$notation as a consequence of Theorem 3 item (ii).

Corollary 3. For any integers m, n, and k,

(i) (Convolution Property)

$$G_k = s_{k,m+1} G_{k-m+1} G_m + s_{k,m} q G_{k-m} G_{m-1} \quad \text{and}$$

(ii) (d' Ocagne's Identity)

$$(-1)^{n+1} q^m G_{n-m} = s_{n,m+1} G_{n+1} G_m - s_{m,n+1} G_n G_{m+1}.$$

Proof. The convolution property follows directly from Theorem 3 by replacing m by $k-n$. Using item 4 of Proposition 1 we have $s_{k-n,n+1} = s_{[k(n+1)-n(n+1)]} = s_{k,n+1}$ and $s_{k-n+1,n} = s_{[kn-(n-1)n]} = s_{k,n}$, since $j(j+1)$ is always even for any integer j. Similarly, the analogue to d' Ocagne's identity can be obtained by replacing n by $-n$ of Theorem 3 with $G_{-n} = q^{-n}(-1)^{n+1} G_n$, $G_{-n-1} = q^{-n-1}(-1)^n G_{n+1}$, and applying the properties of the $s-$notation. $\square$

Also, some cases are worth noting, such as when specific values of the index m in Theorem 3 are evaluated. The next corollary deals with these cases, where odd and even indices are obtained.

Corollary 4. For every integer n, we have

(i) $G_{2n+1} = s_n G_{n+1}^2 + s_{n+1} q G_n^2$,

(ii) $G_{2n} = G_n H_n$, and

(iii) $p_n G_{2n} = G_{n+1}^2 - q^2 G_{n-1}^2$.

Proof. For part (i) and (ii), use Theorem 3 (i) with $m = n+1$ and $m = n$, to respectively get

$$G_{2n+1} = s_{(n+2)n} G_{n+1}^2 + s_{(n+1)^2} q G_n^2 = s_n G_{n+1}^2 + s_{n+1} q G_n^2$$

and

$$\begin{aligned} G_{2n} &= s_{(n+1)n} G_n G_{n+1} + s_{n(n+1)} q G_{n-1} G_n \\ &= G_n (G_{n+1} + q G_{n-1}) \\ &= G_n H_n. \end{aligned} \tag{16}$$

The preceding equations are consequences of $s_{k(k+2)} = s_{k^2} = s_k$, $s_{k(k+1)} = 1$ and Corollary 2. Part (iii) follows from Equation (16) since $G_{n+1} = p_n G_n + q G_{n-1}$, i.e.,

$$p_n G_n = G_{n+1} - q G_{n-1}.$$

$\square$

Note that when $a = b = 1$, parts (i) and (iii) become the identities for classical Fibonacci numbers obtained by Lucas in 1876, that is,

$$f_{2m+1} = f_{m+1}^2 + f_m^2 \quad \text{and} \quad f_{2m} = f_{m+1}^2 - f_{m-1}^2.$$

The first equation by Lucas states that any Fibonacci number with an odd index can be written as a sum of squares of consecutive Fibonacci numbers. It is also true that five times any Fibonacci number with an odd index can be written as a sum of squares of consecutive Lucas numbers, i.e., $5f_{2n+1} = l_{n+1}^2 + l_n^2$. The next corollary is a generalization of this result.

Corollary 5. For any integer n,

$$(ab + 4q)G_{2n+1} = s_{n+1}H_{n+1}^2 + s_n q H_n^2.$$

Proof. Recall from Corollary 2 that $H_{n+1} = G_{n+2} + qG_n$ and $H_n = G_{n+1} + qG_{n-1}$. The definition of the bi-periodic Fibonacci-Horadam sequence implies that

$$H_{n+1} = p_{n+1}G_{n+1} + 2qG_n \quad \text{and} \quad H_n = 2G_{n+1} - p_n G_n.$$

Squaring both sides of both equations gives us

$$H_{n+1}^2 = p_{n+1}^2 G_{n+1}^2 + 4p_{n+1}qG_{n+1}G_n + 4q^2G_n^2 \text{ and } H_n^2 = 4G_{n+1}^2 - 4p_nG_{n+1}G_n + p_n^2G_n^2.$$

From these, it can be shown that

$$p_nH_{n+1}^2 + p_{n+1}qH_n^2 = (ab + 4q)a[a^{-\xi(n)}b^{\xi(n)}G_{n+1}^2 + a^{-\xi(n+1)}b^{\xi(n+1)}qG_n^2].$$

By definition, $s_k = a^{-\xi(k)}b^{\xi(k)}$. So using Corollary 4(i), the above equation can be simplified to

$$p_nH_{n+1}^2 + p_{n+1}qH_n^2 = a(ab + 4q)G_{2n+1}.$$

Finally, using the property that $a^{-1}p_k = s_{k+1}$, the corollary follows. □

3. Sum of Fibonacci-Horadam Numbers

In this section, identities involving sums of terms of the bi-periodic Horadam sequences are determined. In particular, sums with coefficients in $r-$notation and in combinatorial form are explored. First, recall that $r_0 = 1$, $r_1 = b, r_2 = ab$, and in general

$$r_n = a^{\frac{n-\xi(n)}{2}} b^{\frac{n+\xi(n)}{2}}.$$

Here, the results are obtained by using the bi-periodic Fibonacci-Horadam matrix G and factorization of matrices. The following theorem generalizes the classical identity:

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1.$$

Theorem 6. For any positive integer n,

$$q\sum_{k=0}^{n} r_k^{-1}G_k = r_n^{-1}G_{n+2} - a.$$

Proof. Let A be an invertible matrix. Then

$$(A - rI)(A^n + rA^{n-1} + \cdots + r^{n-2}A^2 + r^{n-1}A + r^nI) = A^{n+1} - r^{n+1}I, \tag{17}$$

where r is scalar and I is the identity matrix. Recall that G satisfies its own characteristic equation, i.e., $G^2 - abG - abqI = 0$. From this, one can infer that $G^{-1} = (1/abq)G - (1/q)I$. Hence, by setting $A := (1/abq)G$ and $r := 1/q$ and multiplying both sides of Equation (17) by G, one obtains the matrix equation

$$\sum_{k=0}^{n} \frac{1}{q^{n-k}}\frac{1}{(abq)^k}G^k = \frac{1}{(abq)^{n+1}}G^{n+2} - \frac{1}{q^{n+1}}G.$$

Simplifying terms and equating the $(1,2)-$ entry of both sides using Theorem 1 gives us

$$\sum_{k=0}^{n} \frac{1}{(ab)^k}\left(r_kp_kG_k\right) = \frac{1}{q}\left[\frac{1}{(ab)^{n+1}}\left(r_{n+2}p_{n+2}G_{n+2}\right) - ab\right].$$

Now, $p_jr_j = r_{j+1}$ and $(ab)^{-k}r_{j+k} = r_{j-k}$ for integers j and k, thus

$$\sum_{k=0}^{n} r_{1-k}G_k = \frac{1}{q}[r_{1-n}G_{n+2} - ab].$$

Also, by item (9) of Proposition 1, $b^{-1}r_{1-k} = r_k^{-1}$. Hence, dividing both sides of the above equation by b proves our claim. $\square$

The next step is to utilize the application of the binomial theorem for matrices. For square matrices that commute,

$$(A + B)^n = \sum_{k=0}^{n}\binom{n}{k}A^{n-k}B^k$$

for any positive integer n. Particularly, if A is invertible, r is constant, and $B = rA^{-1}$, then

$$(A + rA^{-1})^n = \sum_{k=0}^{n}\binom{n}{k}r^kA^{n-2k}.$$

Now, also recall that for any integer k,

$$G_{-2k} = -q^{-2k}G_{2k}, \quad G_{-(2k+1)} = q^{-(2k+1)}G_{2k+1}, \quad \text{and}$$
$$H_{-2k} = q^{-2k}H_{2k}, \quad H_{-(2k+1)} = -q^{-(2k+1)}H_{2k+1}.$$

Matrix G is invertible provided that a, b, and q are all nonzero and by Theorem 1,

$$G^{-1} = \begin{bmatrix} 0 & q^{-1} \\ (ab)^{-1} & -q^{-1} \end{bmatrix}.$$

Using these, define the following:

$$X_G := G + abqG^{-1} = \begin{bmatrix} ab & 2ab \\ 2q & -ab \end{bmatrix} \qquad \text{and} \tag{18}$$

$$Y_G := G - abqG^{-1} = abI. \tag{19}$$

One may observe that taking a power of these matrices takes a rather interesting form. This is stated in the following remark.

Remark 2. Let X_G and Y_G be the matrices defined as in (18) and (19), respectively. Then for $n \in \mathbb{N}$, we have the following:

(i) $X_G^{2n} = [ab(ab + 4q)]^n I$ (ii) $Y_G^n = (ab)^n I$.

Here, an even power of X_G is taken to obtain a multiple of the identity matrix I. Obviously, $X_G^{2n+1} = X_G^{2n} X_G = [ab(ab + 4q)]^n X_G$. These matrices will be used to formulate summations with combinatorial coefficients. The first theorem is given as follows.

Remark 3. For positive integer n,

(i) $\displaystyle\sum_{k=1}^{n} \binom{2n+1}{k} q^k G_{2(n-k)+1} = (ab + 4q)^n - G_{2n+1}$

(ii) $\displaystyle\sum_{k=1}^{n} \frac{2n - 2k + 2}{2n - k + 2} \binom{2n+1}{k} q^k G_{2(n-k)+2} = a(ab + 4q)^n - G_{2n+2}.$

Proof. By Lemma 2, $X_G^{2n} = [ab(ab + 4q)]^n I$ and $X_G^{2n+1} = [ab(ab + 4q)]^n X_G$. Now by binomial expansion for matrices,

$$X_G^m = (G + abqG^{-1})^m = \sum_{k=0}^{m} \binom{m}{k} (abq)^k G^{m-2k} \tag{20}$$

for a positive integer m. Replacing m with $2n$, Equation (20) becomes

$$\sum_{k=0}^{2n} \binom{2n}{k} (abq)^k G^{2n-2k} = [ab(ab + 4q)]^n I.$$

In view of Theorem 1, the $(1,1)-$entry of the previous matrix equation is

$$\sum_{k=0}^{2n}\binom{2n}{k}(abq)^k r_{2n-2k}G_{2n-2k+1}=[ab(ab+4q)]^n.$$

Note that $(ab)^k r_{2n-2k}=r_{2n}=(ab)^n$. Moreover, when $k=0$, $\binom{2n}{k}=1$. Hence,

$$\sum_{k=1}^{2n}\binom{2n}{k}q^k G_{2n-2k+1}=(ab+4q)^n-G_{2n+1}. \tag{21}$$

Notice that the terms in the summation will have positive indices from $k=1$ to $k=n$; otherwise, the terms will have negative indices from $k=n+1$ to $k=2n$. Furthermore, for an integer i with $1\le i\le n$, it is true that $n+1\le 2n-i+1\le 2n$ and $2n-2(2n-i+1)+1=-(2n-2i+1)$. Also, since $G_{-(2m+1)}=q^{-(2m+1)}G_{2m+1}$ for any integer m, it follows that $G_{2n-2(2n-i+1)+1}=q^{-(2n-2i+1)}G_{2n-2i+1}$. Hence, the left side of Equation (21) can be written as

$$\begin{aligned}\sum_{k=1}^{2n}\binom{2n}{k}q^k G_{2n-2k+1}&=\sum_{k=1}^{n}\left[\binom{2n}{k}q^k G_{2n-2k+1}+\binom{2n}{N}q^N G_{2n-2(N)+1}\right]\\&=\sum_{k=1}^{n}\left[\binom{2n}{k}+\binom{2n}{N}\right]q^k G_{2n-2k+1},\end{aligned}$$

where $N=2n-k+1$. Using the combinatorial properties, $\binom{2n}{2n-k+1}=\binom{2n}{k-1}$ and $\binom{2n}{k}+\binom{2n}{k-1}=\binom{2n+1}{k}$, it follows that

$$\sum_{k=1}^{2n}\binom{2n}{k}q^k G_{2n-2k+1}=\sum_{k=1}^{n}\binom{2n+1}{k}q^k G_{2n-2k+1}. \tag{22}$$

Comparing the results of Equation (21) and (22), part (i) holds true. The proof of part (ii) is similar. Use the fact that whenever $1\le i\le n$, then $n+2\le 2n-i+2\le 2n+1$. Also, $2n-2(2n-i+2)+2=-(2n-2i+2)$, $G_{2n-2(2n-i+2)+2}=-q^{2n-i+2}G_{2n-2i+2}$, and when $k=n+1$, $G_{2n-2k+2}=0$. Moreover,

$$\binom{2n+1}{k}-\binom{2n+1}{2n-k+2}=\frac{2n-2k+2}{2n-k+2}\binom{2n+1}{k}.$$

Using these, the second result follows. $\square$

Lastly, summation with alternating terms will be discussed here. For this, the binomial expansion

$$(A-rA^{-1})^n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k r^k A^{n-2k}$$

is applied with the $Y-$matrix as in (19).

Theorem 7. For any positive integer n,

$$\sum_{k=0}^{n} \binom{n}{k} (-q)^k G_{n-2k+1} = (ab)^n r_n^{-1}.$$

Proof. Remark 2 states that $Y_G^n = (G - abqG^{-1})^n = (ab)^n I$. Now by the expansion,

$$(G - abqG^{-1})^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (abq)^k G^{n-2k}.$$

As a result of Theorem 1, we have

$$\sum_{k=0}^{n} \binom{n}{k} (-q)^k (ab)^k r_{n-2k} G_{n-2k+1} = (ab)^n.$$

But $(ab)^k r_{n-2k} = r_n$ and so the theorem follows. □

4. Conclusion

In summary, a generalized Q-matrix is formulated hence producing new and generalized identities involving the bi-periodic Horadam sequences with any integer indices. This is done with the aid of the defined sequences $\{p_n\}, \{r_n\}$, and $\{s_{m,n}\}$, together with elementary matrix properties such as the Cayley-Hamilton Theorem. The binomial theorem is also utilized to formulate summation identities with terms in the bi-periodic Horadam sequences.

For future work, one may consider making the coefficients of the second term in the sequences as periodic parameters as well. That is, consider the sequences

$$G_n = p_{n-1} G_{n-1} + q_{n-1} G_{n-2} \quad \text{and} \quad H_n = p_n H_{n-1} + q_n H_{n-2}.$$

Such generalizations are already mentioned in [1] and [4] in positive indices. One can extend the Fibonacci-Horadam matrix and explore matrix properties to find identities of such sequences that extends to any integer indices.

On the other hand, matrix G is called the Fibonacci-Horadam matrix since any power contains the Fibonacci-Horadam terms as entries, thereby producing identities mostly of Fibonacci-Horadam sequences. The next task is to formulate a Lucas-Horadam matrix that can generate properties that focuses solely on Lucas-Horadam terms, or to produce even more identities relating the Lucas- and Fibonacci-Horadam sequences, such as in the work of Cerda [2].

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Appendix

n	G_n
-5	$(a^2b^2+3abq+q^2)q^{-5}$
-4	$(-a^2b-2aq)q^{-4}$
-3	$(ab+q)q^{-3}$
-2	$-aq^{-2}$
-1	q^{-1}
0	0
1	1
2	a
3	$ab+q$
4	a^2b+2aq
5	$a^2b^2+3abq+q^2$
6	$a^3b^2+4a^2bq+3aq^2$
7	$a^3b^3+5a^2b^2q+6abq^2+q^3$
8	$a^4b^3+6a^3b^2q+10a^2bq^2+4aq^3$
9	$a^4b^4+7a^3b^3q+15a^2b^2q^2+10abq^3+q^4$
10	$a^5b^4+8a^4b^3q+21a^3b^2q^2+20a^2bq^3+5aq^4$

Table 1: Bi-periodic Fibonacci-Horadam numbers for $n=-5:10$

n	H_n
-5	$(-a^3b^2-5a^2bq-5aq^2)q^{-5}$
-4	$(a^2b^2+4abq+2q^2)q^{-4}$
-3	$(-a^2b-3aq)q^{-3}$
-2	$(ab+2q)q^{-2}$
-1	$-aq^{-1}$
0	2
1	a
2	$ab+2q$
3	a^2b+3aq
4	$a^2b^2+4abq+2q^2$
5	$a^3b^2+5a^2bq+5aq^2$
6	$a^3b^3+6a^2b^2q+9abq^2+2q^3$
7	$a^4b^3+7a^3b^2q+14a^2bq^2+7aq^3$
8	$a^4b^4+8a^3b^3q+20a^2b^2q^2+16abq^3+2q^4$
9	$a^5b^4+9a^4b^3q+27a^3b^2q^2+30a^2bq^3+9aq^4$
10	$a^5b^5+10a^4b^4q+35a^3b^3q^2+50a^2b^2q^3+25abq^4+2q^5$

Table 2: Bi-periodic Lucas-Horadam numbers for $n=-5:10$