SHIFTED CONVOLUTION SUMS OF ARITHMETIC FUNCTIONS OF TWO VARIABLES AND RAMANUJAN EXPANSIONS

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Abstract

Let $f, g : \mathbb{N}^2 \to \mathbb{C}$ be two arithmetic functions of two variables. We study the shifted convolution sum defined by $\sum_{n_1 \leqq N_1,\ n_2 \leqq N_2} f(n_1, n_2) g(n_1 + h_1, n_2 + h_2)$, where h_1 and h_2 are nonnegative integers. We use the method of Ramanujan expansions used by Gadiyar, Murty, and Padma, who treated the case of arithmetic functions of one variable. We extend their results to the case of arithmetic functions of two variables.

1. Introduction

For an arithmetic function $f : \mathbb{N} \to \mathbb{C}$, the mean value $M(f)$ is defined to be the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n \leqq N} f(n),$$

if this limit exists. There are many results concerning $M(f)$. For example, from the well-known formula $\sum_{n \leqq N} \varphi(n) = (3/\pi^2)N^2 + O(N \log N)$ where φ is Euler's totient function, we have by partial summation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n \leqq N} \frac{\varphi(n)}{n} = \frac{6}{\pi^2}.$$

Similarly, from the well-known formula $\sum_{n \leqq N} \sigma(n) = (\pi^2/12)N^2 + O(N \log N)$ where $\sigma(n) = \sum_{d|n} d$, we have by partial summation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n \leqq N} \frac{\sigma(n)}{n} = \frac{\pi^2}{6}.$$

On the other hand, Gadiyar, Murty, and Padma [2] studied the shifted convolution sum defined by

$$\sum_{n \leqq N} f(n)g(n+h),$$

where f and g are two arithmetic functions, and h is a nonnegative integer. They obtained

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leqq N} \frac{\varphi(n)}{n} \frac{\varphi(n+h)}{n+h} = \prod_{p\in\mathcal{P}} (1 - \frac{2}{p^2}) \prod_{p|h} \frac{p^3-2p+1}{p(p^2-2)}, \tag{1}$$

and, if $\alpha, \beta > 1/2$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leqq N} \frac{\sigma_\alpha(n)}{n^\alpha} \frac{\sigma_\beta(n+h)}{(n+h)^\beta} = \frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)} \sigma_{-\alpha-\beta-1}(h), \tag{2}$$

where $\mathcal{P}$ is the set of prime numbers, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, and $\zeta(s)$ is the Riemann zeta function.

In this paper, we study the shifted convolution sum of arithmetic functions of two variables defined by

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}} f(n_1,n_2)g(n_1+h_1,n_2+h_2),$$

where f, g are two arithmetic functions of two variables, and h_1, h_2 are fixed nonnegative integers. We extend the method of Ramanujan expansions used by Gadiyar, Murty, and Padma [2] in the case of arithmetic functions of one variable, to the case of arithmetic functions of two variables. We give some examples including extensions of (1) and (2).

2. Ramanujan Sums and the Method of Gadiyar, Murty and Padma

Let $c_r(n)$ be the Ramanujan sums defined in [4] by

$$c_r(n) = \sum_{\substack{k=1\\ \gcd(k,r)=1}}^{q} \exp(\frac{2\pi i k n}{r}),$$

where $\gcd(k,r)$ is the greatest common divisor of k and r. Let $f : \mathbb{N} \mapsto \mathbb{C}$ be an arithmetic function. Ramanujan [4] investigated its Ramanujan-Fourier series which is an infinite series of the form

$$f(n) = \sum_{r=1}^{\infty} \widehat{f}(r) c_r(n), \tag{3}$$

where $\widehat{f}(r)$ are called the Ramanujan-Fourier coefficients of f, and he obtained, for example, the following results.

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1)\sum_{r=1}^{\infty}\frac{c_r(n)}{r^{s+1}}, \tag{4}$$

$$\frac{\varphi_s(n)}{n^s} = \frac{1}{\zeta(s+1)}\sum_{r=1}^{\infty}\frac{\mu(r)}{\varphi_{s+1}(r)}c_r(n), \tag{5}$$

where $\varphi_s(n) = n^s\prod_{p|n}(1-1/p^s)$ and μ is the Möbius function.

Let $\tau(n) = \sum_{d|n} 1$. In their paper, Gadiyar, Murty, and Padma [2, Theorem 5] obtained the following theorem.

Theorem (Gadiyar, Murty, and Padma)**.** *Suppose that f and g are two arithmetic functions with absolutely convergent Ramanujan-Fourier series:*

$$f(n) = \sum_{r=1}^{\infty}\widehat{f}(r)c_r(n), \quad g(n) = \sum_{s=1}^{\infty}\widehat{g}(s)c_s(n),$$

respectively. If

$$\sum_{r,s=1}^{\infty}|\widehat{f}(r)\widehat{g}(s)|\sqrt{rs}\tau(r)\tau(s) < \infty,$$

then, as N tends to infinity,

$$\sum_{n\leqq N} f(n)g(n+h) \sim N\sum_{r=1}^{\infty}\widehat{f}(r)\widehat{g}(r)c_r(h).$$

As a corollary of this theorem, Gadiyar, Murty, and Padma [2] obtained (1) and (2). We will extend their results to the case of arithmetic functions of two variables and obtain some results later.

3. Lemmas

In this section we prepare some lemmas. Gadiyar, Murty, and Padma [2, Lemma 2] proved

$$\sum_{n\leqq N} c_r(n)c_s(n+h) = \delta_{r,s}Nc_r(h) + O(rs\log(rs)),$$

where $h \in \mathbb{Z}$ and $\delta_{r,s} = 1$ or 0 according to whether $r = s$ or not. The following lemma is a straightforward extension of this formula.

Lemma 1. *If $h_1, h_2 \in \mathbb{Z}$, then*

$$\sum_{\substack{n_1 \leqq N_1 \\ n_2 \leqq N_2}} c_{r_1}(n_1)c_{r_2}(n_2)c_{s_1}(n_1+h_1)c_{s_2}(n_2+h_2)$$
$$= (\delta_{r_1,s_1}N_1c_{r_1}(h_1) + O(r_1s_1\log(r_1s_1)))(\delta_{r_2,s_2}N_2c_{r_2}(h_2) + O(r_2s_2\log(r_2s_2))).$$

Gadiyar, Murty, and Padma [2, Lemma 3] proved

$$|\sum_{n \leqq N} c_r(n)c_s(n+h)| \leqq \sqrt{N(N+|h|)rs}\,\tau(r)\tau(s),$$

where $h \in \mathbb{Z}$. The following lemma is a straightforward extension of this inequality.

Lemma 2. *If $h_1, h_2 \in \mathbb{Z}$, then*

$$|\sum_{\substack{n_1 \leqq N_1 \\ n_2 \leqq N_2}} c_{r_1}(n_1)c_{r_2}(n_2)c_{s_1}(n_1+h_1)c_{s_2}(n_2+h_2)|$$
$$\leqq \sqrt{N_1(N_1+|h_1|)N_2(N_2+|h_2|)r_1r_2s_1s_2}\,\tau(r_1)\tau(r_2)\tau(s_1)\tau(s_2).$$

We also need the following two lemmas.

Lemma 3. *If $\alpha \geqq 0$ and $s - 2\alpha > 1$, then*

$$\sum_{r_1,r_2=1}^{\infty} \frac{(r_1r_2)^{\alpha}}{(\mathrm{lcm}(r_1,r_2))^s} < \infty,$$

where $\mathrm{lcm}(r_1, r_2)$ is the least common multiple of r_1 and r_2.

Proof. Setting $d = \gcd(r_1, r_2)$, $r_1 = dr_1'$, and $r_2 = dr_2'$, we have

$$\sum_{r_1,r_2=1}^{\infty} \frac{(r_1r_2)^{\alpha}}{(\mathrm{lcm}(r_1,r_2))^s} = \sum_{d=1}^{\infty} \sum_{\substack{r_1',r_2'=1 \\ \gcd(r_1',r_2')=1}}^{\infty} \frac{(r_1'r_2'd^2)^{\alpha}}{(r_1'r_2'd)^s}$$
$$= \sum_{d=1}^{\infty} \frac{1}{d^{s-2\alpha}} \sum_{\substack{r_1',r_2'=1 \\ \gcd(r_1',r_2')=1}}^{\infty} \frac{1}{(r_1')^{s-\alpha}(r_2')^{s-\alpha}}$$
$$\leqq \sum_{d=1}^{\infty} \frac{1}{d^{s-2\alpha}} \sum_{r_1'=1}^{\infty} \frac{1}{(r_1')^{s-\alpha}} \sum_{r_2'=1}^{\infty} \frac{1}{(r_2')^{s-\alpha}} < \infty.$$

□

Lemma 4. *If $s > 1$, then for any $n_1, n_2 \in \mathbb{N}$*

$$\sum_{r_1,r_2=1}^{\infty} \frac{1}{(\mathrm{lcm}(r_1,r_2))^s} |c_{r_1}(n_1)c_{r_2}(n_2)| < \infty.$$

Proof. From the well known formula $c_r(n) = \sum_{d|\gcd(r,n)} d\mu(r/d)$ we have

$$|c_r(n)| = |\sum_{d|\gcd(r,n)} d\mu(r/d)| \leqq \sum_{d|n} d = \sigma_1(n).$$

Therefore we obtain by Lemma 3

$$\sum_{r_1,r_2=1}^{\infty} \frac{1}{(\mathrm{lcm}(r_1,r_2))^s} |c_{r_1}(n_1)c_{r_2}(n_2)| \leqq \sum_{r_1,r_2=1}^{\infty} \frac{1}{(\mathrm{lcm}(r_1,r_2))^s} \sigma_1(n_1)\sigma_1(n_2)$$
$$= \sigma_1(n_1)\sigma_1(n_2) \sum_{r_1,r_2=1}^{\infty} \frac{1}{(\mathrm{lcm}(r_1,r_2))^s} < \infty.$$

□

4. Theorem

In this section, we prove the following theorem which is an extension of Gadiyar-Murty-Padma's theorem introduced in Section 2 to the case of two variables.

Theorem 1. *Let f, g be arithmetic functions of two variables. Suppose that*

$$f(n_1,n_2) = \sum_{r_1,r_2=1}^{\infty} \widehat{f}(r_1,r_2)c_{r_1}(n_1)c_{r_2}(n_2),$$
$$g(n_1,n_2) = \sum_{s_1,s_2=1}^{\infty} \widehat{g}(s_1,s_2)c_{s_1}(n_1)c_{s_2}(n_2),$$

are absolutely convergent and

$$\sum_{r_1,r_2,s_1,s_2=1}^{\infty} |\widehat{f}(r_1,r_2)\widehat{g}(s_1,s_2)| \sqrt{r_1 s_1 r_2 s_2}\, \tau(r_1)\tau(r_2)\tau(s_1)\tau(s_2) < \infty.$$

Then we have for nonnegative integers h_1, h_2

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\substack{n_1 \leqq N_1 \\ n_2 \leqq N_2}} f(n_1,n_2)g(n_1+h_1,n_2+h_2)$$
$$= \sum_{r_1,r_2=1}^{\infty} \widehat{f}(r_1,r_2)\widehat{g}(r_1,r_2)c_{r_1}(h_1)c_{r_2}(h_2).$$

Proof. The proof proceeds along the same lines as the proof of Gadiyar-Murty-Padma's theorem. We first see that

$$\sum_{\substack{n_1 \leqq N_1 \\ n_2 \leqq N_2}} f(n_1, n_2) g(n_1 + h_1, n_2 + h_2)$$
$$= \sum_{r_1, r_2, s_1, s_2 = 1}^{\infty} \widehat{f}(r_1, r_2) \widehat{g}(s_1, s_2) \sum_{\substack{n_1 \leqq N_1 \\ n_2 \leqq N_2}} c_{r_1}(n_1) c_{r_2}(n_2) c_{s_1}(n_1 + h_1) c_{s_2}(n_2 + h_2).$$

We split the outer sum over r_1, r_2, s_1, s_2 into four parts

$$\sum_{\substack{r_1 s_1 \leqq U_1 \\ r_2 s_2 \leqq U_2}}, \quad \sum_{\substack{r_1 s_1 \leqq U_1 \\ r_2 s_2 > U_2}}, \quad \sum_{\substack{r_1 s_1 > U_1 \\ r_2 s_2 \leqq U_2}}, \quad \sum_{\substack{r_1 s_1 > U_1 \\ r_2 s_2 > U_2}}, \tag{6}$$

where $U_1 = \sqrt{N_1}$ and $U_2 = \sqrt{N_2}$. As for the first part, we have by Lemma 1

$$\sum_{\substack{r_1 s_1 \leqq U_1 \\ r_2 s_2 \leqq U_2}} \widehat{f}(r_1, r_2) \widehat{g}(s_1, s_2) \sum_{\substack{n_1 \leqq N_1 \\ n_2 \leqq N_2}} c_{r_1}(n_1) c_{r_2}(n_2) c_{s_1}(n_1 + h_1) c_{s_2}(n_2 + h_2)$$
$$= \sum_{\substack{r_1 s_1 \leqq U_1 \\ r_2 s_2 \leqq U_2}} \widehat{f}(r_1, r_2) \widehat{g}(s_1, s_2) (\delta_{r_1, s_1} N_1 c_{r_1}(h_1) + O(r_1 s_1 \log(r_1 s_1)))$$
$$\times (\delta_{r_2, s_2} N_2 c_{r_2}(h_2) + O(r_2 s_2 \log(r_2 s_2)))$$
$$= \sum_{\substack{r_1 s_1 \leqq U_1 \\ r_2 s_2 \leqq U_2}} \widehat{f}(r_1, r_2) \widehat{g}(s_1, s_2) \delta_{r_1, s_1} \delta_{r_2, s_2} N_1 N_2 c_{r_1}(h_1) c_{r_2}(h_2)$$
$$+ \sum_{\substack{r_1 s_1 \leqq U_1 \\ r_2 s_2 \leqq U_2}} \widehat{f}(r_1, r_2) \widehat{g}(s_1, s_2) O(r_1 s_1 \log(r_1 s_1)) \delta_{r_2, s_2} N_2 c_{r_2}(h_2)$$
$$+ \sum_{\substack{r_1 s_1 \leqq U_1 \\ r_2 s_2 \leqq U_2}} \widehat{f}(r_1, r_2) \widehat{g}(s_1, s_2) \delta_{r_1, s_1} N_1 c_{r_1}(h_1) O(r_2 s_2 \log(r_2 s_2))$$
$$+ \sum_{\substack{r_1 s_1 \leqq U_1 \\ r_2 s_2 \leqq U_2}} \widehat{f}(r_1, r_2) \widehat{g}(s_1, s_2) O(r_1 s_1 r_2 s_2 \log(r_1 s_1) \log(r_2 s_2))$$
$$=: I_1 + I_2 + I_3 + I_4.$$

As for I_1 we have

$$I_1 = \sum_{\substack{r_1 s_1 \leqq U_1 \\ r_2 s_2 \leqq U_2}} \widehat{f}(r_1, r_2) \widehat{g}(s_1, s_2) \delta_{r_1, s_1} \delta_{r_2, s_2} N_1 N_2 c_{r_1}(h_1) c_{r_2}(h_2)$$

$$=N_1N_2\sum_{\substack{r_1^2\leqq U_1\\ r_2^2\leqq U_2}}\widehat{f}(r_1,r_2)\widehat{g}(r_1,r_2)c_{r_1}(h_1)c_{r_2}(h_2).$$

Therefore we obtain

$$\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}I_1=\sum_{r_1,r_2=1}^{\infty}\widehat{f}(r_1,r_2)\widehat{g}(r_1,r_2)_1c_{r_1}(h_1)c_{r_2}(h_2).$$

As for I_2 we have

$$\begin{aligned}|I_2|=&|\sum_{\substack{r_1s_1\leqq U_1\\ r_2s_2\leqq U_2}}\widehat{f}(r_1,r_2)\widehat{g}(s_1,s_2)O(r_1s_1\log(r_1s_1))\delta_{r_2,s_2}N_2c_{r_2}(h_2)|\\ \ll&\sum_{\substack{r_1s_1\leqq U_1\\ r_2^2\leqq U_2}}|\widehat{f}(r_1,r_2)\widehat{g}(s_1,r_2)|U_1(\log U_1)N_2|c_{r_2}(h_2)|\\ \ll&(U_1\log U_1)N_2\sum_{r_1,r_2,s_1=1}^{\infty}|\widehat{f}(r_1,r_2)\widehat{g}(s_1,r_2)|\sigma_1(h_2)\\ \ll&(\sqrt{N_1}\log\sqrt{N_1})N_2\sigma_1(h_2)\sum_{r_1,r_2,s_1=1}^{\infty}|\widehat{f}(r_1,r_2)\widehat{g}(s_1,r_2)|\\ =&o(N_1N_2),\end{aligned}$$

where we have used $|c_r(h)|\leqq\sigma_1(h)$ and the assumption of Theorem 1.

Similarly we have $|I_3|=o(N_1N_2)$. As for I_4 we have

$$\begin{aligned}|I_4|=&|\sum_{\substack{r_1s_1\leqq U_1\\ r_2s_2\leqq U_2}}\widehat{f}(r_1,r_2)\widehat{g}(s_1,s_2)O(r_1s_1r_2s_2\log(r_1s_1)\log(r_2s_2))|\\ \ll&(U_1\log U_1)(U_2\log U_2)\sum_{r_1,r_2,s_1,s_2=1}^{\infty}|\widehat{f}(r_1,r_2)\widehat{g}(s_1,s_2)|\\ =&o(N_1N_2).\end{aligned}$$

Next we deal with $\sum_{\substack{r_1s_1\leqq U_1\\ r_2s_2>U_2}}$ in (6). By Lemma 2 we have

$$\begin{aligned}&|\sum_{\substack{r_1s_1\leqq U_1\\ r_2s_2>U_2}}\widehat{f}(r_1,r_2)\widehat{g}(s_1,s_2)\sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}}c_{r_1}(n_1)c_{r_2}(n_2)c_{s_1}(n_1+h_1)c_{s_2}(n_2+h_2)|\\ \ll&\sum_{\substack{r_1s_1\leqq U_1\\ r_2s_2>U_2}}|\widehat{f}(r_1,r_2)\widehat{g}(s_1,s_2)|\sqrt{N_1(N_1+|h_1|)N_2(N_2+|h_2|)r_1r_2s_1s_2}\end{aligned}$$

$$\times \tau(r_1)\tau(r_2)\tau(s_1)\tau(s_2)$$
$$\ll N_1N_2 \sum_{\substack{r_1s_1\leqq U_1\\ r_2s_2>U_2}} |\widehat{f}(r_1,r_2)\widehat{g}(s_1,s_2)|\sqrt{r_1r_2s_1s_2}\tau(r_1)\tau(r_2)\tau(s_1)\tau(s_2),$$

which is $o(N_1N_2)$ by the assumption of Theorem 1.

Similarly we have

$$\sum_{\substack{r_1s_1>U_1\\ r_2s_2\leqq U_2}} \widehat{f}(r_1,r_2)\widehat{g}(s_1,s_2) \sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}} c_{r_1}(n_1)c_{r_2}(n_2)c_{s_1}(n_1+h_1)c_{s_2}(n_2+h_2) = o(N_1N_2),$$

and

$$\sum_{\substack{r_1s_1>U_1\\ r_2s_2>U_2}} \widehat{f}(r_1,r_2)\widehat{g}(s_1,s_2) \sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}} c_{r_1}(n_1)c_{r_2}(n_2)c_{s_1}(n_1+h_1)c_{s_2}(n_2+h_2) = o(N_1N_2).$$

This completes the proof of Theorem 1. $\square$

5. Examples

In this section, we give some examples. The following example is an extension of (2).

Example 1. *If $s,t>0$, then*

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}} \frac{\sigma_s(\gcd(n_1,n_2))}{\gcd(n_1,n_2)^s}\frac{\sigma_t(\gcd(n_1+h_1,n_2+h_2))}{\gcd(n_1+h_1,n_2+h_2)^t}$$
$$= \frac{\zeta(s+2)\zeta(t+2)}{\zeta(s+t+4)}\sigma_{-s-t-4}(\gcd(h_1,h_2)).$$

Proof. We first note that by [7, Example 3.8]

$$\frac{\sigma_s(\gcd(n_1,n_2))}{\gcd(n_1,n_2)^s} = \zeta(s+2)\sum_{r_1,r_2=1}^{\infty} \frac{1}{(\operatorname{lcm}(r_1,r_2))^{s+2}}c_{r_1}(h_1)c_{r_2}(h_2) \tag{7}$$

holds for $s>0$. Moreover, since $s>0$, the right-hand side of (7) is absolutely convergent by Lemma 4. Since $\tau(n)=o(n^{\varepsilon})$ holds for any $\varepsilon>0$ by [1], we have

$$\sum_{r_1,r_2,s_1,s_2=1}^{\infty} \frac{1}{(\operatorname{lcm}(r_1,r_2))^{s+2}}\frac{1}{(\operatorname{lcm}(s_1,s_2))^{t+2}}\sqrt{r_1s_1r_2s_2}\tau(r_1)\tau(r_2)\tau(s_1)\tau(s_2)$$
$$\leqq \sum_{r_1,r_2,s_1,s_2=1}^{\infty} \frac{(r_1r_2s_1s_2)^{\frac{1}{2}+\varepsilon}}{(\operatorname{lcm}(r_1,r_2))^{s+2}(\operatorname{lcm}(s_1,s_2))^{t+2}},$$

which is finite by Lemma 3. Therefore we have by Theorem 1

$$\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}\sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}}\frac{\sigma_s(\gcd(n_1,n_2))}{\gcd(n_1,n_2)^s}\frac{\sigma_t(\gcd(n_1+h_1,n_2+h_2))}{\gcd(n_1+h_1,n_2+h_2)^t}$$
$$=\zeta(s+2)\zeta(t+2)\sum_{r_1,r_2=1}^{\infty}\frac{1}{(\mathrm{lcm}(r_1,r_2))^{s+t+4}}c_{r_1}(h_1)c_{r_2}(h_2).$$

By (7) we see that the above is equal to

$$\zeta(s+2)\zeta(t+2)\frac{1}{\zeta(s+t+4)}\frac{\sigma_{s+t+4}(\gcd(h_1,h_2))}{\gcd(h_1,h_2)^{s+t+4}}$$
$$=\frac{\zeta(s+2)\zeta(t+2)}{\zeta(s+t+4)}\sigma_{-s-t-4}(\gcd(h_1,h_2)),$$

which completes the proof of Example 1. □

The following example is an extension of (1).

Example 2. *If $s,t>0$, then*

$$\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}\sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}}\frac{\varphi_s(\gcd(n_1,n_2))}{\gcd(n_1,n_2)^s}\frac{\varphi_t(\gcd(n_1+h_1,n_2+h_2))}{\gcd(n_1+h_1,n_2+h_2)^t}$$
$$=\prod_{\substack{p\in\mathcal{P}\\ p\nmid h_1 \text{ or } p\nmid h_2}}(1-\frac{1}{p^{s+2}}-\frac{1}{p^{t+2}})\prod_{\substack{p\in\mathcal{P}\\ p|h_1,\ p|h_2}}(1-\frac{1}{p^{s+2}}-\frac{1}{p^{t+2}}+\frac{1}{p^{s+t+2}}).$$

Proof. We first note that by [7, Example 3.11]

$$\frac{\varphi_s(\gcd(n_1,n_2))}{\gcd(n_1,n_2)^s}=\frac{1}{\zeta(s+2)}\sum_{r_1,r_2=1}^{\infty}\frac{\mu(\mathrm{lcm}(r_1,r_2))}{\varphi_{s+2}(\mathrm{lcm}(r_1,r_2))}c_{r_1}(h_1)c_{r_2}(h_2) \tag{8}$$

holds for $s>0$. Since $1/\varphi_{s+2}(n)\ll n^\delta/n^{s+2}$ holds for any $\delta>0$ by [5, Theorem 1, pp. 81], the right-hand side of (8) is absolutely convergent by Lemma 4. Since $\tau(n)=o(n^\varepsilon)$ holds for any $\varepsilon>0$ we have

$$\sum_{r_1,r_2,s_1,s_2=1}^{\infty}\left|\frac{\mu(\mathrm{lcm}(r_1,r_2))\mu(\mathrm{lcm}(s_1,s_2))}{\varphi_{s+2}(\mathrm{lcm}(r_1,r_2))\varphi_{t+2}(\mathrm{lcm}(s_1,s_2))}\right|\sqrt{r_1s_1r_2s_2}\tau(r_1)\tau(r_2)\tau(s_1)\tau(s_2)$$
$$\ll\sum_{r_1,r_2,s_1,s_2=1}^{\infty}\frac{(r_1r_2s_1s_2)^{\frac{1}{2}+\delta+\varepsilon}}{(\mathrm{lcm}(r_1,r_2))^{s+2}(\mathrm{lcm}(s_1,s_2))^{t+2}},$$

which is finite by Lemma 3. Therefore we have by Theorem 1

$$\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}\sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}}\frac{\varphi_s(\gcd(n_1,n_2))}{\gcd(n_1,n_2)^s}\frac{\varphi_t(\gcd(n_1+h_1,n_2+h_2))}{\gcd(n_1+h_1,n_2+h_2)^t}$$
$$=\frac{1}{\zeta(s+2)\zeta(t+2)}\sum_{r_1,r_2=1}^{\infty}\frac{\mu(\mathrm{lcm}(r_1,r_2))^2}{\varphi_{s+2}(\mathrm{lcm}(r_1,r_2))\varphi_{t+2}(\mathrm{lcm}(r_1,r_2))}c_{r_1}(h_1)c_{r_2}(h_2). \quad (9)$$

Here we recall the definition of a multiplicative function of two variables. We say that $f:\mathbb{N}^2\to\mathbb{C}$ is a multiplicative function of two variables if f satisfies

$$f(m_1n_1,m_2n_2)=f(m_1,m_2)\ f(n_1,n_2)$$

for any $m_1,m_2,n_1,n_2\in\ \mathbb{N}$ satisfying $\gcd(m_1m_2,\ n_1n_2)=1$.

Since

$$(r_1,r_2)\mapsto\frac{\mu(\mathrm{lcm}(r_1,r_2))^2}{\varphi_{s+2}(\mathrm{lcm}(r_1,r_2))\varphi_{t+2}(\mathrm{lcm}(r_1,r_2))}c_{r_1}(h_1)c_{r_2}(h_2)$$

is a multiplicative function of two variables for fixed h_1 and h_2, we have

$$\sum_{r_1,r_2=1}^{\infty}\frac{\mu(\mathrm{lcm}(r_1,r_2))^2}{\varphi_{s+2}(\mathrm{lcm}(r_1,r_2))\varphi_{t+2}(\mathrm{lcm}(r_1,r_2))}c_{r_1}(h_1)c_{r_2}(h_2)$$
$$=\prod_{p\in\mathcal{P}}\Big(\sum_{e_1,e_2\geqq0}\frac{\mu(\mathrm{lcm}(p^{e_1},p^{e_2}))^2}{\varphi_{s+2}(\mathrm{lcm}(p^{e_1},p^{e_2}))\varphi_{t+2}(\mathrm{lcm}(p^{e_1},p^{e_2}))}c_{p^{e_1}}(h_1)c_{p^{e_2}}(h_2)\Big)$$
$$=\prod_{p\in\mathcal{P}}\Big(1+\frac{c_p(h_1)+c_p(h_2)}{\varphi_{s+2}(p)\varphi_{t+2}(p)}+\frac{c_p(h_1)c_p(h_2)}{\varphi_{s+2}(p)\varphi_{t+2}(p)}\Big). \quad (10)$$

Since $\varphi_\alpha(p)=p^\alpha-1$ and

$$c_p(h)=\begin{cases}p-1, & \text{if } p\mid h;\\ -1, & \text{otherwise,}\end{cases}$$

we see that (10) is equal to

$$\prod_{p\mid h_1,p\mid h_2}\Big(1+\frac{2(p-1)}{(p^{s+2}-1)(p^{t+2}-1)}+\frac{(p-1)^2}{(p^{s+2}-1)(p^{t+2}-1)}\Big)$$
$$\times\prod_{p\mid h_1,p\nmid h_2}\Big(1+\frac{p-2}{(p^{s+2}-1)(p^{t+2}-1)}-\frac{p-1}{(p^{s+2}-1)(p^{t+2}-1)}\Big)$$
$$\times\prod_{p\nmid h_1,p\mid h_2}\Big(1+\frac{p-2}{(p^{s+2}-1)(p^{t+2}-1)}-\frac{p-1}{(p^{s+2}-1)(p^{t+2}-1)}\Big)$$

$$\times \prod_{p\nmid h_1, p\nmid h_2} \Big(1+\frac{-2}{(p^{s+2}-1)(p^{t+2}-1)}+\frac{1}{(p^{s+2}-1)(p^{t+2}-1)}\Big)$$
$$= \prod_{\substack{p\in\mathcal{P}\\ p\nmid h_1 \text{ or } p\nmid h_2}} \Big(1-\frac{1}{p^{s+2}}-\frac{1}{p^{t+2}}\Big)/\Big((1-\frac{1}{p^{s+2}})(1-\frac{1}{p^{s+2}})\Big)$$
$$\times \prod_{\substack{p\in\mathcal{P}\\ p\mid h_1,\ p\mid h_2}} \Big(1-\frac{1}{p^{s+2}}-\frac{1}{p^{t+2}}+\frac{1}{p^{s+t+2}}\Big)/\Big((1-\frac{1}{p^{s+2}})(1-\frac{1}{p^{s+2}})\Big).$$

Therefore we see that (9) is equal to

$$\frac{1}{\zeta(s+2)\zeta(t+2)} \prod_{\substack{p\in\mathcal{P}\\ p\nmid h_1 \text{ or } p\nmid h_2}} \Big(1-\frac{1}{p^{s+2}}-\frac{1}{p^{t+2}}\Big)/\Big((1-\frac{1}{p^{s+2}})(1-\frac{1}{p^{s+2}})\Big)$$
$$\times \prod_{\substack{p\in\mathcal{P}\\ p\mid h_1,\ p\mid h_2}} \Big(1-\frac{1}{p^{s+2}}-\frac{1}{p^{t+2}}+\frac{1}{p^{s+t+2}}\Big)/\Big((1-\frac{1}{p^{s+2}})(1-\frac{1}{p^{s+2}})\Big)$$
$$= \prod_{\substack{p\in\mathcal{P}\\ p\nmid h_1 \text{ or } p\nmid h_2}} (1-\frac{1}{p^{s+2}}-\frac{1}{p^{t+2}}) \prod_{\substack{p\in\mathcal{P}\\ p\mid h_1,\ p\mid h_2}} (1-\frac{1}{p^{s+2}}-\frac{1}{p^{t+2}}+\frac{1}{p^{s+t+2}}).$$

This completes the proof of Example 2. ☐

It is well known that the natural density that two positive integers are coprime is $1/\zeta(2) = 6/\pi^2 = \prod_{p\in\mathcal{P}}(1-1/p^2)$, namely,

$$\lim_{N\to\infty} \frac{1}{N^2}\#\{(n_1,n_2)\in(\mathbb{N}\cap[1,N])^2 : \gcd(n_1,n_2)=1\} = \prod_{p\in\mathcal{P}}(1-1/p^2).$$

The following example is an extension of this result.

Example 3. *If $h_1, h_2 \in \mathbb{N}$, then*

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2}\#\{(n_1,n_2)\in\mathbb{N}^2 : 1\leqq n_1\leqq N_1, 1\leqq n_2\leqq N_2,$$
$$\gcd(n_1,n_2)=\gcd(n_1+h_1,n_2+h_2)=1\}$$
$$= \prod_{\substack{p\in\mathcal{P}\\ p\nmid h_1 \text{ or } p\nmid h_2}} (1-2/p^2) \prod_{\substack{p\in\mathcal{P}\\ p\mid h_1,\ p\mid h_2}} (1-1/p^2).$$

Proof. We first note that

$$\lim_{s\downarrow 0}\frac{\varphi_s(n)}{n^s} = \lim_{s\downarrow 0}\prod_{p\mid n}(1-1/p^s) = \delta(n)$$

holds where $\delta(n) = 1$ or 0 according to whether $n = 1$ or not.

We let $s, t \downarrow 0$ in Example 2. Since

$$\lim_{s,t\downarrow 0} \lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}} \frac{\varphi_s(\gcd(n_1,n_2))}{\gcd(n_1,n_2)^s} \frac{\varphi_t(\gcd(n_1+h_1,n_2+h_2))}{\gcd(n_1+h_1,n_2+h_2)^t}$$

$$= \lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}} \delta(\gcd(n_1,n_2))\delta(\gcd(n_1+h_1,n_2+h_2))$$

$$= \lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \#\{(n_1,n_2) \in \mathbb{N}^2 : 1 \leqq n_1 \leqq N_1, 1 \leqq n_2 \leqq N_2,$$
$$\gcd(n_1,n_2) = \gcd(n_1+h_1,n_2+h_2) = 1\},$$

and

$$\lim_{s,t\downarrow 0} \prod_{\substack{p\in\mathcal{P}\\ p\nmid h_1 \text{ or } p\nmid h_2}} (1 - \frac{1}{p^{s+2}} - \frac{1}{p^{t+2}}) \prod_{\substack{p\in\mathcal{P}\\ p|h_1,\ p|h_2}} (1 - \frac{1}{p^{s+2}} - \frac{1}{p^{t+2}} + \frac{1}{p^{s+t+2}})$$
$$= \prod_{\substack{p\in\mathcal{P}\\ p\nmid h_1 \text{ or } p\nmid h_2}} (1 - 2/p^2) \prod_{\substack{p\in\mathcal{P}\\ p|h_1,\ p|h_2}} (1 - 1/p^2),$$

we see that Example 3 holds. □

Ushiroya [6, Example 4] proved

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}} \mu^2(\gcd(n_1,n_2)) = \prod_{p\in\mathcal{P}} (1 - \frac{1}{p^4}). \tag{11}$$

The following example is an extension of this result.

Example 4. *Let q be a prime number or $q = 1$. Then*

$$\lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}} \mu^2(\gcd(n_1,n_2))\mu^2(\gcd(n_1+q,n_2+q))$$
$$= \prod_{p\in\mathcal{P}} (1 - \frac{2}{p^4}).$$

Proof. We treat the case when q is a prime number. Let $f(n_1,n_2) = \mu^2(\gcd(n_1,n_2))$. Then we have by (11)

$$M(f) = \lim_{N_1,N_2\to\infty} \frac{1}{N_1N_2} \sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}} \mu^2(\gcd(n_1,n_2)) = \prod_{p\in\mathcal{P}} (1 - \frac{1}{p^4}).$$

By Ushiroya [7, Theorem 2.10] we have the following Ramanujan expansion of f

$$f(n_1,n_2)=\sum_{r_1,r_2=1}^{\infty} M(f)A(\mathrm{lcm}(r_1,r_2))c_{r_1}(n_1)c_{r_2}(n_2),$$

where $A(n)$ is the multiplicative function determined by

$$A(p^e)=\begin{cases}1, & \text{if } e=0;\\ -1/(p^4-1), & \text{if } e=1,2;\\ 0, & \text{if } e\geqq 3.\end{cases}$$

It is easy to see that f satisfies the assumption of Theorem 1. Hence we have

$$\begin{aligned}
&\lim_{N_1,N_2\to\infty}\frac{1}{N_1N_2}\sum_{\substack{n_1\leqq N_1\\ n_2\leqq N_2}}\mu^2(\gcd(n_1,n_2))\mu^2(\gcd(n_1+q,n_2+q))\\
&=\sum_{r_1,r_2=1}^{\infty}\big(M(f)A(\mathrm{lcm}(r_1,r_2))\big)^2 c_{r_1}(q)c_{r_2}(q)\\
&=(M(f))^2\prod_{p\in\mathcal{P}}\Big(1+\sum_{\substack{0\leqq e_1,e_2\leqq 2\\ e_1+e_2\geqq 1}}A(\mathrm{lcm}(p^{e_1},p^{e_2}))^2c_{p^{e_1}}(q)c_{p^{e_2}}(q)\Big)\\
&=(M(f))^2\prod_{p\in\mathcal{P}}\Big(1+\sum_{\substack{0\leqq e_1,e_2\leqq 2\\ e_1+e_2\geqq 1}}(\frac{-1}{p^4-1})^2c_{p^{e_1}}(q)c_{p^{e_2}}(q)\Big)\\
&=(M(f))^2\Big(1+\sum_{\substack{0\leqq e_1,e_2\leqq 2\\ e_1+e_2\geqq 1}}(\frac{1}{q^4-1})^2c_{q^{e_1}}(q)c_{q^{e_2}}(q)\Big)\\
&\qquad\times\prod_{\substack{p\in\mathcal{P}\\ p\neq q}}\Big(1+\sum_{\substack{0\leqq e_1,e_2\leqq 2\\ e_1+e_2\geqq 1}}(\frac{1}{p^4-1})^2c_{p^{e_1}}(q)c_{p^{e_2}}(q)\Big). \qquad (12)
\end{aligned}$$

Since

$$c_1(q)=1,\qquad c_p(q)=\begin{cases}q-1, & \text{if } p=q;\\ -1 & \text{if } p\neq q,\end{cases}\qquad c_{p^2}(q)=\begin{cases}-q, & \text{if } p=q;\\ 0, & \text{if } p\neq q,\end{cases}$$

we have

$$\begin{aligned}
&1+\sum_{\substack{0\leqq e_1,e_2\leqq 2\\ e_1+e_2\geqq 1}}(\frac{1}{q^4-1})^2c_{q^{e_1}}(q)c_{q^{e_2}}(q)\\
=&1+(\frac{1}{q^4-1})^2(c_q(q)c_1(q)+c_1(q)c_q(q)+c_q(q)c_q(q)+c_{q^2}(q)c_1(q)+c_1(q)c_{q^2}(q)
\end{aligned}$$

$$
\begin{aligned}
&+ c_{q^2}(q)c_q(q) + c_q(q)c_{q^2}(q) + c_{q^2}(q)c_{q^2}(q)) \\
=&1 - (\frac{1}{q^4-1})^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&1 + \sum_{\substack{0 \leqq e_1, e_2 \leqq 2 \\ e_1+e_2 \geqq 1}} (\frac{1}{p^4-1})^2 c_{p^{e_1}}(q) c_{p^{e_2}}(q) \\
=&1 + (\frac{1}{p^4-1})^2 (c_p(q)c_1(q) + c_1(q)c_p(q) + c_p(q)c_p(q) + c_{p^2}(q)c_1(q) + c_1(q)c_{p^2}(q) \\
&+ c_{p^2}(q)c_p(q) + c_p(q)c_{p^2}(q) + c_{p^2}(q)c_{p^2}(q)) \\
=&1 - (\frac{1}{p^4-1})^2,
\end{aligned}
$$

for $p \neq q$. Therefor we see that (12) is equal to

$$
\begin{aligned}
&(M(f))^2 \Big(1 - (\frac{1}{q^4-1})^2\Big) \prod_{\substack{p \in \mathcal{P} \\ p \neq q}} \Big(1 - (\frac{1}{p^4-1})^2\Big) \\
&= \Big(\prod_{p \in \mathcal{P}} (1 - \frac{1}{p^4})\Big)^2 \prod_{p \in \mathcal{P}} \Big(1 - (\frac{1}{p^4-1})^2\Big) = \prod_{p \in \mathcal{P}} (1 - \frac{2}{p^4}).
\end{aligned}
$$

The proof for the case in which $q = 1$ is similar. This completes the proof of Example 4. □

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