A RECURRING PATTERN IN NATURAL NUMBERS OF A CERTAIN PROPERTY

Daniel Tsai
Graduate School of Mathematics, Nagoya University, Furucho, Chikusa-ku, Nagoya, Japan
dsai@outlook.jp

Received: 5/16/20, Revised: 11/14/20, Accepted: 3/3/21, Published: 3/23/21

Abstract

Natural numbers satisfying an unusual property are mentioned by the author in a previous note, in which their infinitude is also proved. In this paper, we start with an arbitrary natural number which is not a multiple of 10 and non-palindromic, form numbers by concatenating its decimal digits, and investigate which of them have the unusual property. In particular, the pattern of which of them have the unusual property recurs.

1. Introduction

An unusual property which some natural numbers, e.g., 198, satisfy are defined by the author in [5]. We see that

$$\begin{aligned} 198 &= 2 \cdot 3^2 \cdot 11, \\ 891 &= 3^4 \cdot 11, \end{aligned}$$

and

$$2 + (3 + 2) + 11 = (3 + 4) + 11.$$

That is, the sum of the numbers appearing in the prime factorizations of the two numbers are equal. Notice that the exponents 1 do not appear. In general, the definition is, that a natural number n has this property if $10 \nmid n$, n is non-palindromic, and that the sum of the numbers appearing in the prime factorization of n is equal to that of the number formed by reversing its decimal digits. In [5], the infinitude of such numbers is proved, in particular

$$18, 1818, 181818, \ldots, \tag{1}$$

$$18, 198, 1998, 19998, \ldots \tag{2}$$

all have this property. The first is the sequence of concatenations of 18; the second is the sequence of numbers $19\dots98$, with any number of 9's in between. In this paper, we start with an arbitrary non-palindromic natural number $10 \nmid n$, form, like in the sequence (1), numbers by concatenating its decimal digits, and show that there is a recurring pattern in which of them have this property. More precisely, whether one of them has this property depends only on the number of times the digits of n are concatenated to form it modulo some natural number.

This unusual property, called v-palindromicity in this paper, is defined using the following two concepts.

- Reversing the decimal digits of a natural number n. In this paper we only allow n to not be a multiple of 10, and denote the resulting number by $r(n)$. The reason is that we do not want to have leading digits of 0 after reversing.
- In the prime factorization of a natural number

$$n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}, \tag{3}$$

summing all the numbers that appear, i.e., the prime factors and the exponents, but not including an exponent when it is 1, because they are usually not written. In this paper we denote this sum by $v(n)$, i.e.,

$$v(n) = \sum_{i=1}^{m} (p_i + \iota(e_i)), \tag{4}$$

where $\iota(e) = 0$ if $e = 1$ and $\iota(e) = e$ if $e \geq 2$.

About reversing the decimal digits of a natural number, some investigations have been done by others. In [3], numbers n such that n divides $r(n)$, i.e., $n \mid r(n)$, are mentioned. In particular, all of the numbers in

$$2178, 21978, 219978, 2199978, \dots, \tag{5}$$

i.e., the sequence of numbers $219\dots978$, with any number of 9's in between, satisfy $4n = r(n)$. The resemblance of the sequences (2) and (5) is a bit interesting. While the relation $n \mid r(n)$ is studied in [3], the relation $v(n) = v(r(n))$ is studied in this paper. In [2], non-palindromic prime numbers p such that $r(p)$ is also prime are mentioned. They are called emirps.

About $v(n)$, similar arithmetic functions have been studied. In [1], assuming Equation (3), the arithmetic function

$$A(n) = \sum_{i=1}^{m} p_i e_i \tag{6}$$

is studied. Also, the entries A008474 and A000026 of the OEIS [4] are similar to $v(n)$. The entry A008474 is, assuming Equation (3),

$$v'(n) = \sum_{i=1}^{m} (p_i + e_i), \tag{7}$$

which is almost the same as $v(n)$ except that it has e_i instead of $\iota(e_i)$, i.e., when summing all the numbers that appear on the right-hand-side of Equation (3), also including an exponent when it is 1.

Palindromes are numbers n such that $n = r(n)$. These obviously satisfy $n \mid r(n)$ and also $v(n) = v(r(n))$. Therefore the problem studied in [3], as well as the content of this paper, are more about non-palindromes, rather than palindromes.

2. Definition of the Unusual Property

In this section we will recall the definition in [5] of the unusual property. In the following,

$$\mathbb{N}_{\neq 10} = \{n \in \mathbb{N} : 10 \nmid n\},$$
$$\mathbb{Z}_{\geq 0} = \{z \in \mathbb{Z} : z \geq 0\}.$$

Definition 1. For $n \in \mathbb{N}_{\neq 10}$ with decimal representation $n = d_{k-1} \dots d_1 d_0$, we put

$$r(n) = d_0 d_1 \dots d_{k-1}.$$

That is, $r(n)$ is the number formed by writing the decimal digits of n in reverse order. Hence we have $r\colon \mathbb{N}_{\neq 10} \to \mathbb{N}_{\neq 10}$. We define n to be *palindromic* if $n = r(n)$.

Definition 2. We put

- $v(p) = p$ for p a prime,
- $v(p^e) = p + e$ for p a prime and $e \geq 2$,

and insist that $v\colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$ be an additive arithmetic function. If we put

$$\iota(e) = \begin{cases} 0 & (e = 1) \\ e & (e \geq 2), \end{cases}$$

then we may combine the above two points and just put

$$v(p^e) = p + \iota(e).$$

Let $n \geq 2$ be a natural number with prime factorization

$$n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}. \tag{8}$$

Then

$$v(n) = \sum_{i=1}^{m} v(p_i^{e_i}) = \sum_{i=1}^{m} (p_i + \iota(e_i)).$$

Hence $v(n)$ is the sum of the numbers appearing in the prime factorization of n, not counting exponents which are 1.

We may now define the unusual property, which we call v-palindromicity.

Definition 3. A natural number n is *v-palindromic* if $n \in \mathbb{N}_{\neq 10}$, $n \neq r(n)$, and $v(n) = v(r(n))$.

It is clear that if n is v-palindromic then so is $r(n)$. As noted in the Introduction, 198 and the numbers (1) are v-palindromic numbers. In the next section we shall state our main theorem.

3. Statement of the Main Theorem

In this section we shall define some notations to state our main theorem.

Definition 4. For $c, k \geq 1$, put

$$\rho_{c,k} = \overbrace{1\underbrace{0\ldots0}_{k-1}1\underbrace{0\ldots0}_{k-1}1\ldots1\underbrace{0\ldots0}_{k-1}1}^{c},$$

meaning that 1 appears c times and that between each consecutive pair of them 0 appears $k-1$ times.

It is clear that if n is a k-digit number then the number formed by concatenating its digits c times is just $n\rho_{c,k}$. We may now state our main theorem.

Theorem 1. *Let n be a natural number with k digits and with $n \in \mathbb{N}_{\neq 10}$ and $n \neq r(n)$. Then there exists a natural number $\omega > 0$ such that for every $c \geq 1$, $n\rho_{c,k}$ is v-palindromic if and only if $n\rho_{c+\omega,k}$ is. In other words, whether $n\rho_{c,k}$ is v-palindromic depends only on c modulo ω.*

Remark 1. In fact the main theorem also holds if in defining v-palindromic numbers we used the v' in Equation (7) instead of v. Moreover the proof will be slightly shorter because one does not have to deal with the subtlety caused by not summing an exponent when it is 1.

We make the following definition based on the truth of the above theorem.

Definition 5. A natural number $\omega > 0$ satisfying the condition of the above theorem is called a *period* of n and the smallest one is denoted $\omega(n)$. If there exists a $c \geq 1$ such that $n\rho_{c,k}$ is v-palindromic, the smallest one is called the *order* of n and denoted $c(n)$. If such a c does not exist then we write $c(n) = \infty$.

We have the following.

Theorem 2. *The set of all periods of n is $\{q\omega(n) : q \in \mathbb{N}\}$.*

We prove our main theorem in Section 7. Before that, we need some preparation. In Section 4 we investigate some divisibility properties of the numbers $\rho_{c,k}$. In Section 6 we first consider the case $n = 819$ of the main theorem; the proof of the main theorem is essentially a generalization of this.

4. Divisibility Properties of $\rho_{c,k}$

We consider the divisibility of the numbers $\rho_{c,k}$ by prime powers p^α. Recall that $\mathrm{ord}_p(n)$ is the largest integer β with $p^\beta \mid n$. We have the following lemma.

Lemma 1. *Let p^α be a prime power, with $p \neq 2, 5$. Let $k \geq 1$, let $\beta = \mathrm{ord}_p(10^k - 1)$, and let h be the order of 10^k regarded as an element of $(\mathbb{Z}/p^{\alpha+\beta}\mathbb{Z})^\times$. Then $h > 1$ and for $c \geq 1$, $p^\alpha \mid \rho_{c,k}$ if and only if $h \mid c$.*

Proof. We first show that $h > 1$. That $h = 1$ means that $10^k \equiv 1 \pmod{p^{\alpha+\beta}}$, or equivalently, $p^{\alpha+\beta} \mid 10^k - 1$, or equivalently, $p^{\alpha+\mathrm{ord}_p(10^k-1)} \mid 10^k - 1$. This cannot be, whence $h > 1$. We have

$$(10^k - 1)\rho_{c,k} = (10^k - 1)\sum_{i=0}^{c-1} 10^{ki} = 10^{kc} - 1.$$

As $\beta = \mathrm{ord}_p(10^k - 1)$, $p^\alpha \mid \rho_{c,k}$ if and only if $10^{kc} - 1 \equiv 0 \pmod{p^{\alpha+\beta}}$, or equivalently, $10^{kc} \equiv 1 \pmod{p^{\alpha+\beta}}$, or equivalently, $h \mid c$. The last equivalence is due to the structure of cyclic groups. □

Remark 2. In Lemma 1, if $p = 2, 5$, then 10^k cannot be regarded as an element of $(\mathbb{Z}/p^{\alpha+\beta}\mathbb{Z})^\times$. But obviously for every $c \geq 1$, $p^\alpha \nmid \rho_{c,k}$. Also, let us denote the h in the lemma by $h_{p^\alpha,k}$.

Using Mathematica [6] we can calculate the following values.

p^α	7	7^2	13	13^2	17	17^2
$h_{p^\alpha,3}$	2	14	2	26	16	272

Table 1: Values of $h_{p^\alpha,3}$ for various prime powers p^α.

Regarding divisibility in general, not just for $\rho_{c,k}$, we recall the following facts.

Lemma 2. *Let n be a natural number, let p be a prime, and let $g = \mathrm{ord}_p(n)$. Then*

(i) $g = 0$ *if and only if* $p \nmid n$,

(ii) $g = 1$ *if and only if* $p \mid n$ *and* $p^2 \nmid n$,

(iii) $g \leq 1$ *if and only if* $p^2 \nmid n$,

(iv) $g \geq 1$ *if and only if* $p \mid n$, *and*

(v) $g \geq 2$ *if and only if* $p^2 \mid n$.

We will need this lemma later.

5. The Functions $\varphi_{p,\delta}$

For a fixed prime p, the sequence of powers of p is

$$1, p, p^2, \ldots, p^\alpha, \ldots.$$

Applying v to them yields

$$0, p, p+2, \ldots, p+\alpha, \ldots.$$

Now we take differences of consecutive terms to get

$$p, 2, 1, \ldots, 1, \ldots, \tag{9}$$

with all 1's from the third term onwards. We give notation for the terms of this sequence.

Definition 6. For a prime p and integer $\alpha \geq 0$, put

$$\varphi_{p,1}(\alpha) = v(p^{\alpha+1}) - v(p^\alpha).$$

In this notation then, the sequence (9) is $(\varphi_{p,1}(\alpha))_{\alpha=0}^{\infty}$. More generally we define the following.

Definition 7. For a prime p, an integer $\alpha \geq 0$, and a $\delta \geq 1$, put

$$\varphi_{p,\delta}(\alpha) = v(p^{\alpha+\delta}) - v(p^{\alpha}).$$

In this notation, for instance, the sequence $(\varphi_{p,3}(\alpha))_{\alpha=0}^{\infty}$ is

$$p+3, 4, 3, \ldots, 3, \ldots,$$

with all 3's from the third term onwards. More generally, for $\delta \geq 2$, the sequence $(\varphi_{p,\delta}(\alpha))_{\alpha=0}^{\infty}$ is just

$$p+\delta, \delta+1, \delta, \ldots, \delta, \ldots. \tag{10}$$

We may view, for a prime p and $\delta \geq 1$, $\varphi_{p,\delta} \colon \mathbb{Z}_{\geq 0} \to \mathbb{N}$ as a function of $\alpha \in \mathbb{Z}_{\geq 0}$. Rephrasing the sequences (9) and (10), the values of $\varphi_{p,\delta}$ may be summarized as

$$\varphi_{2,1}(\alpha) = \begin{cases} 2 & (\alpha = 0, 1) \\ 1 & (\alpha \geq 2), \end{cases} \tag{11}$$

and if $p \neq 2$,

$$\varphi_{p,1}(\alpha) = \begin{cases} p & (\alpha = 0) \\ 2 & (\alpha = 1) \\ 1 & (\alpha \geq 2), \end{cases} \tag{12}$$

and if $\delta \geq 2$,

$$\varphi_{p,\delta}(\alpha) = \begin{cases} p+\delta & (\alpha = 0) \\ \delta+1 & (\alpha = 1) \\ \delta & (\alpha \geq 2). \end{cases} \tag{13}$$

We have deliberately distinguished between the cases where the values are distinct. We give a notation for the ranges of $\varphi_{p,\delta}$.

Definition 8. For a prime p and $\delta \geq 1$ put $R_{p,\delta} = \varphi_{p,\delta}(\mathbb{Z}_{\geq 0})$.

Remark 3. In view of Equations (11), (12), and (13), it is clear that $|R_{2,1}| = 2$ and $|R_{p,\delta}| = 3$ otherwise. Also, any nonempty fiber of $\varphi_{p,\delta}$ is one of

$$\{0\}, \quad \{1\}, \quad \{0, 1\}, \quad \mathbb{Z}_{\geq 2} = \{z \in \mathbb{Z} : z \geq 2\}.$$

Following directly from Equations (11), (12), and (13), we have the following lemma.

Lemma 3. *Let p be a prime, $\delta \geq 1$, $u \in R_{p,\delta}$, and $\mu \geq 0$. Then we have the following.*

(i) *In case $\varphi_{p,\delta}^{-1}(u) = \{0\}$, for $g \geq 0$,*

$$\begin{aligned} \varphi_{p,\delta}(\mu+g) = u \quad & \text{if and only if} \quad \mu + g = 0 \\ & \text{if and only if} \quad \begin{cases} g = 0 & (\mu = 0) \\ \text{impossible} & (\mu \geq 1). \end{cases} \end{aligned} \tag{14}$$

(ii) *In case* $\varphi_{p,\delta}^{-1}(u) = \{1\}$, *for* $g \geq 0$,

$$\begin{aligned}\varphi_{p,\delta}(\mu+g) = u \quad &\text{if and only if} \quad \mu+g=1\\ &\text{if and only if} \quad \begin{cases} g = 1-\mu & (\mu = 0,1) \\ \text{impossible} & (\mu \geq 2). \end{cases}\end{aligned} \tag{15}$$

(iii) *In case* $\varphi_{p,\delta}^{-1}(u) = \{0,1\}$, *for* $g \geq 0$,

$$\begin{aligned}\varphi_{p,\delta}(\mu+g) = u \quad &\text{if and only if} \quad \mu+g \in \{0,1\}\\ &\text{if and only if} \quad \begin{cases} g \leq 1 & (\mu = 0) \\ g = 0 & (\mu = 1) \\ \text{impossible} & (\mu \geq 2). \end{cases}\end{aligned} \tag{16}$$

(iv) *In case* $\varphi_{p,\delta}^{-1}(u) = \mathbb{Z}_{\geq 2}$, *for* $g \geq 0$,

$$\begin{aligned}\varphi_{p,\delta}(\mu+g) = u \quad &\text{if and only if} \quad \mu+g \geq 2\\ &\text{if and only if} \quad \begin{cases} g \geq 2-\mu & (\mu = 0,1) \\ \text{always true} & (\mu \geq 2). \end{cases}\end{aligned} \tag{17}$$

Here "impossible" *means that no* $g \geq 0$ *can be found to fulfill* $\varphi_{p,\delta}(\mu+g) = u$, *and* "always true" *means that all* $g \geq 0$ *fulfills* $\varphi_{p,\delta}(\mu+g) = u$.

6. The Case of $n = 819$

We consider the case $n = 819$ of Theorem 1. We have the prime factorizations

$$\begin{aligned}819 &= 3^2 \cdot 7 \cdot 13,\\ 918 &= 2 \cdot 3^3 \cdot 17.\end{aligned}$$

Let the prime factorization of $\rho_{c,3}$ be

$$\rho_{c,3} = 3^{g_1} \cdot 7^{g_2} \cdot 13^{g_3} \cdot 17^{g_4} \cdot b,$$

where $(b, 3 \cdot 7 \cdot 13 \cdot 17) = 1$. The numbers g_1, g_2, g_3, g_4, b obviously depend on c, but we have suppressed the notation for simplicity. Now

$$\begin{aligned}819\rho_{c,3} &= 3^{2+g_1} \cdot 7^{1+g_2} \cdot 13^{1+g_3} \cdot 17^{g_4} \cdot b,\\ r(819\rho_{c,3}) = 918\rho_{c,3} &= 2 \cdot 3^{3+g_1} \cdot 7^{g_2} \cdot 13^{g_3} \cdot 17^{1+g_4} \cdot b.\end{aligned}$$

Applying the additive function v to these equations,

$$v(819\rho_{c,3}) = v(3^{2+g_1}) + v(7^{1+g_2}) + v(13^{1+g_3}) + v(17^{g_4}) + v(b),$$
$$v(r(819\rho_{c,3})) = v(918\rho_{c,3}) = v(2) + v(3^{3+g_1}) + v(7^{g_2}) + v(13^{g_3}) + v(17^{1+g_4}) + v(b).$$

Hence $819\rho_{c,3}$ is a v-palindromic number if and only if the above two quantities are equal, that is, after rearranging,

$$\begin{aligned}(v(7^{1+g_2}) - v(7^{g_2})) &+ (v(13^{1+g_3}) - v(13^{g_3})) \\ &= 2 + (v(3^{3+g_1}) - v(3^{2+g_1})) + (v(17^{1+g_4}) - v(17^{g_4})).\end{aligned}$$

In terms of the functions $\varphi_{p,\delta}$ of Section 5, this becomes

$$\varphi_{7,1}(g_2) + \varphi_{13,1}(g_3) = 2 + \varphi_{3,1}(2 + g_1) + \varphi_{17,1}(g_4). \tag{18}$$

Since $2 + g_1 \geq 2$, by Equation (12), $\varphi_{3,1}(2 + g_1) = 1$, therefore Equation (18) becomes

$$\varphi_{7,1}(g_2) + \varphi_{13,1}(g_3) = 3 + \varphi_{17,1}(g_4). \tag{19}$$

Now consider the equation

$$u_2 + u_3 = 3 + u_4. \tag{20}$$

We want to solve it for $u_2 \in R_{7,1}$, $u_3 \in R_{13,1}$, and $u_4 \in R_{17,1}$. In view of Equation (12),

$$R_{7,1} = \{7, 2, 1\}, \quad R_{13,1} = \{13, 2, 1\}, \quad R_{17,1} = \{17, 2, 1\}.$$

By trying all possibilities we see that the only solutions are $(u_2, u_3, u_4) = (7, 13, 17)$ and $(2, 2, 1)$. Whence Equation (19) is satisfied if and only if

$$(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (7, 13, 17) \quad \text{or} \quad (2, 2, 1).$$

We first consider when $(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (7, 13, 17)$. By Lemma 3 (or more easily just by looking at Equation (12)), Lemma 2, Lemma 1, and Table 1,

$$\begin{aligned}\varphi_{7,1}(g_2) = 7 \quad &\text{if and only if} \quad g_2 = 0 \quad \text{if and only if} \quad 7 \nmid \rho_{c,3} \\ &\text{if and only if} \quad h_{7,3} \nmid c \quad \text{if and only if} \quad 2 \nmid c,\end{aligned} \tag{21}$$

and

$$\begin{aligned}\varphi_{13,1}(g_3) = 13 \quad &\text{if and only if} \quad g_3 = 0 \quad \text{if and only if} \quad 13 \nmid \rho_{c,3} \\ &\text{if and only if} \quad h_{13,3} \nmid c \quad \text{if and only if} \quad 2 \nmid c,\end{aligned} \tag{22}$$

and

$$\begin{aligned}\varphi_{17,1}(g_4) = 17 \quad &\text{if and only if} \quad g_4 = 0 \quad \text{if and only if} \quad 17 \nmid \rho_{c,3} \\ &\text{if and only if} \quad h_{17,3} \nmid c \quad \text{if and only if} \quad 16 \nmid c.\end{aligned} \tag{23}$$

Hence $(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (7, 13, 17)$ simply when c is odd. We next consider when $(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (2, 2, 1)$. Similarly we have

$$\begin{aligned} \varphi_{7,1}(g_2) = 2 \quad &\text{if and only if} \quad g_2 = 1 \quad \text{if and only if} \quad \begin{cases} 7 \mid \rho_{c,3}, \\ 7^2 \nmid \rho_{c,3} \end{cases} \\ &\text{if and only if} \quad \begin{cases} h_{7,3} \mid c, \\ h_{7^2,3} \nmid c \end{cases} \quad \text{if and only if} \quad \begin{cases} 2 \mid c, \\ 14 \nmid c, \end{cases} \end{aligned} \tag{24}$$

and

$$\begin{aligned} \varphi_{13,1}(g_3) = 2 \quad &\text{if and only if} \quad g_3 = 1 \quad \text{if and only if} \quad \begin{cases} 13 \mid \rho_{c,3}, \\ 13^2 \nmid \rho_{c,3} \end{cases} \\ &\text{if and only if} \quad \begin{cases} h_{13,3} \mid c, \\ h_{13^2,3} \nmid c \end{cases} \quad \text{if and only if} \quad \begin{cases} 2 \mid c, \\ 26 \nmid c, \end{cases} \end{aligned} \tag{25}$$

and

$$\begin{aligned} \varphi_{17,1}(g_4) = 1 \quad &\text{if and only if} \quad g_4 \geq 2 \quad \text{if and only if} \quad 17^2 \mid \rho_{c,3} \\ &\text{if and only if} \quad h_{17^2,3} \mid c \quad \text{if and only if} \quad 272 \mid c, \end{aligned} \tag{26}$$

where two divisibility relations to the right of a left brace means that both must hold. Hence $(\varphi_{7,1}(g_2), \varphi_{13,1}(g_3), \varphi_{17,1}(g_4)) = (2, 2, 1)$ precisely when $272 \mid c$ and $(c, 7 \cdot 13) = 1$. Hence we have established the following characterization.

Theorem 3. *For $c \geq 1$, the number $819\rho_{c,3}$ is v-palindromic if and only if c is odd or if $272 \mid c$ and $(c, 7 \cdot 13) = 1$.*

From the above theorem, we immediately see that $c(819) = 1$ (refer to definitions in Definition 5). We see that $819\rho_{c,3}$ is v-palindromic if and only if all 3 conditions (21), (22), and (23) hold, or if all 3 conditions (24), (25), and (26) hold. Now these conditions have the same truth values when c increases by $\mathrm{lcm}(16, 14, 26, 272) = 24752$. Hence $\omega = 24752$ is a period of 819. With some work, it can be shown that actually it is the smallest period, that is, $\omega(819) = 24752$.

7. Proof of the Main Theorem

We now enter the proof of the main theorem and this is essentially writing the discussion about 819 in the previous section in the general setting.

Let the prime factorizations of n and $r(n)$ be

$$\begin{aligned} n &= p_1^{e_1} p_2^{e_2} \dots p_m^{e_m}, \\ r(n) &= p_1^{f_1} p_2^{f_2} \dots p_m^{f_m}, \end{aligned}$$

where we have done the factorization over the set of primes which divide one of n or $r(n)$, setting $e_i = 0$ or $f_i = 0$ if necessary. Since $n \neq r(n)$, we have $e_i \neq f_i$ for some i. Let the set of i such that $e_i \neq f_i$ be

$$i_1 < i_2 < \ldots < i_t. \tag{27}$$

Let the prime factorization of $\rho_{c,k}$ be

$$\rho_{c,k} = p_1^{g_1} p_2^{g_2} \ldots p_m^{g_m} b, \tag{28}$$

where $(b, p_1 p_2 \ldots p_m) = 1$. The $g_1, g_2, \ldots, g_m, b$ obviously depend on c, but we suppress them from our notation for simplicity. Then

$$\begin{aligned} n\rho_{c,k} &= p_1^{e_1+g_1} p_2^{e_2+g_2} \ldots p_m^{e_m+g_m} b, \\ r(n\rho_{c,k}) = r(n)\rho_{c,k} &= p_1^{f_1+g_1} p_2^{f_2+g_2} \ldots p_m^{f_m+g_m} b. \end{aligned}$$

Taking their v, we have

$$\begin{aligned} v(n\rho_{c,k}) &= \sum_{i=1}^{m} v(p_i^{e_i+g_i}) + v(b), \\ v(r(n\rho_{c,k})) &= \sum_{i=1}^{m} v(p_i^{f_i+g_i}) + v(b). \end{aligned}$$

Hence $n\rho_{c,k}$ is v-palindromic, that is, $v(n\rho_{c,k}) = v(r(n\rho_{c,k}))$, if and only if

$$\sum_{i=1}^{m} (v(p_i^{e_i+g_i}) - v(p_i^{f_i+g_i})) = 0. \tag{29}$$

If $e_i = f_i$, of course the term $v(p_i^{e_i+g_i}) - v(p_i^{f_i+g_i}) = 0$, so by (27), Equation (29) is equivalent to

$$\sum_{j=1}^{t} (v(p_{i_j}^{e_{i_j}+g_{i_j}}) - v(p_{i_j}^{f_{i_j}+g_{i_j}})) = 0. \tag{30}$$

This is a cumbersome notation, and we will just write p_{i_j} as p_j, e_{i_j} as e_j, f_{i_j} as f_j, and g_{i_j} as g_j, because we will not refer to the other prime factors or exponents from here on. Consequently, Equation (30) becomes

$$\sum_{j=1}^{t} (v(p_j^{e_j+g_j}) - v(p_j^{f_j+g_j})) = 0. \tag{31}$$

We also write

$$\begin{aligned} \delta_j &= e_j - f_j, \\ \mu_j &= \min(e_j, f_j), \\ \alpha_j &= \mu_j + g_j, \end{aligned}$$

for $1 \le j \le t$. Then it is clear that the left-hand-side of Equation (31) can be rewritten, using the functions $\varphi_{p,\delta}$ of Section 5, as

$$\begin{aligned}&\sum_{j=1}^{t}(v(p_j^{e_j+g_j}) - v(p_j^{f_j+g_j}))\\ &\qquad = \sum_{j=1}^{t}\operatorname{sgn}(\delta_j)(v(p_j^{\alpha_j+|\delta_j|}) - v(p_j^{\alpha_j})) = \sum_{j=1}^{t}\operatorname{sgn}(\delta_j)\varphi_{p_j,|\delta_j|}(\alpha_j),\end{aligned} \tag{32}$$

where sgn is the sign function with $\operatorname{sgn}(\delta_j) = 1$ if $\delta_j > 0$ and $\operatorname{sgn}(\delta_j) = -1$ if $\delta_j < 0$. Now consider the equation

$$\sum_{j=1}^{t}\operatorname{sgn}(\delta_j)u_j = 0. \tag{33}$$

Supposedly we can solve it for

$$(u_1, u_2, \ldots, u_t) \in R_{p_1,|\delta_1|} \times R_{p_2,|\delta_2|} \times \cdots \times R_{p_t,|\delta_t|}.$$

Let the set of all solutions be

$$U = \{u = (u_1, \ldots, u_t)\}.$$

Then we see that

$$\sum_{j=1}^{t}\operatorname{sgn}(\delta_j)\varphi_{p_j,|\delta_j|}(\alpha_j) = 0$$

holds if and only if for some $u \in U$,

$$\varphi_{p_j,|\delta_j|}(\alpha_j) = u_j$$

for all $1 \le j \le t$. Summarizing what we have done up to now, we have shown the following.

Lemma 4. *For $c \ge 1$, the number $n\rho_{c,k}$ is v-palindromic if and only if for some $u \in U$, $\varphi_{p_j,|\delta_j|}(\alpha_j) = u_j$ for all $1 \le j \le t$.*

Now let us just consider any particular condition $\varphi_{p_j,|\delta_j|}(\alpha_j) = \varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j$. If in the statement of Lemma 3, we substitute the p, δ, u, and μ by p_j, $|\delta_j|$, u_j, and μ_j, respectively, we have

$$\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j \quad \text{if and only if} \quad \begin{cases} g_j = 0 & \text{(if (i, 0), or (ii, 1), or (iii, 1))} \\ g_j = 1 & \text{(if (ii, 0))} \\ g_j \le 1 & \text{(if (iii, 0))} \\ g_j \ge 1 & \text{(if (iv, 1))} \\ g_j \ge 2 & \text{(if (iv, 0))} \\ \text{impossible} & \text{(otherwise)} \\ \text{always true} & \text{(if (iv}, \ge 2)), \end{cases} \tag{34}$$

where on the right, a notation like (N, μ), where N is a Roman numeral and $\mu = 0, 1$, denotes the case (N) in Lemma 3 and in addition the case where $\mu_j = \mu$; $(\mathrm{iv}, \geq 2)$ denotes the case (iv) of Lemma 3 and in addition the case where $\mu_j \geq 2$. As the last two cases ("impossible" and "always true") never change as c varies, we exclude them from our consideration. By Lemma 2, we can continue the equivalences in (34) respectively (here we do not write out the cases as in (34)), recalling that $g_j = \mathrm{ord}_{p_j}(\rho_{c,k})$,

$$\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j \quad \text{if and only if} \quad \begin{cases} p_j \nmid \rho_{c,k} \\ p_j \mid \rho_{c,k} \text{ and } p_j^2 \nmid \rho_{c,k} \\ p_j^2 \nmid \rho_{c,k} \\ p_j \mid \rho_{c,k} \\ p_j^2 \mid \rho_{c,k}. \end{cases} \tag{35}$$

In case $p_j \neq 2, 5$, we can apply Lemma 1 to (35) to obtain, respectively,

$$\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j \quad \text{if and only if} \quad \begin{cases} h_{p_j,k} \nmid c \\ h_{p_j,k} \mid c \text{ and } h_{p_j^2,k} \nmid c \\ h_{p_j^2,k} \nmid c \\ h_{p_j,k} \mid c \\ h_{p_j^2,k} \mid c. \end{cases} \tag{36}$$

However, in case $p_j = 2, 5$, by Remark 2, (35) becomes

$$\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j \quad \text{if and only if} \quad \begin{cases} \text{always true} \\ \text{impossible} \\ \text{always true} \\ \text{impossible} \\ \text{impossible.} \end{cases} \tag{37}$$

In general, if $a, b' \geq 1$ are integers with $a \mid b'$, then for integers $b \geq 1$ (not the b introduced in Equation (28)), $a \mid b$ if and only if $a \mid (b + b')$. Hence we see that the truth value of $\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j$ does not change if we increase c by

$$\omega = \mathrm{lcm}\{h_{p_j,k}, h_{p_j^2,k} : p_j \neq 2, 5\}. \tag{38}$$

In view of Lemma 4, whether $n\rho_{c,k}$ is v-palindromic or not depends only on the truth values of the individual $(\varphi_{p_j,|\delta_j|}(\mu_j + g_j) = u_j)$'s. Hence this ω serves as a possible ω as required by the main theorem.

8. Further problems

In the proof of the main theorem, we constructively found a possible ω in Equation (38); let us denote it by $\omega_f(n)$. However, whether or not $\omega_f(n)$ is the smallest period, i.e., $\omega(n)$, is still unclear, although we know by Theorem 2 that $\omega(n) \mid \omega_f(n)$. The following is a table of $\omega_f(n)$, $\omega(n)$, and $c(n)$, for $n \le 56$ with $n < r(n)$, computed using Mathematica [6]. We can assume without loss of generality that $n < r(n)$ because the patterns for n and $r(n)$ are exactly the same, i.e.,

$$\begin{aligned} \omega_f(n) &= \omega_f(r(n)), \\ \omega(n) &= \omega(r(n)), \\ c(n) &= c(r(n)). \end{aligned}$$

n	12	13	14	15	16	17	18	19	23
$\omega_f(n)$	21	6045	4305	136	1830	337960	9	15561	253
$\omega(n)$	1	6045	1	1	1	337960	1	15561	1
$c(n)$	∞	15	∞	∞	∞	280	1	819	∞
n	24	25	26	27	28	29	34	35	36
$\omega_f(n)$	21	39	6045	9	4305	102718	122808	14469	21
$\omega(n)$	1	1	6045	1	1	1	1	1	1
$c(n)$	∞	∞	15	∞	∞	∞	∞	∞	∞
n	37	38	39	45	46	47	48	49	56
$\omega_f(n)$	32412	581913	6045	9	253	119991	21	22701	273
$\omega(n)$	32412	1	6045	1	1	1	21	22701	273
$c(n)$	12	∞	15	∞	∞	∞	3	3243	3

Table 2: Values of $\omega_f(n)$, $\omega(n)$, and $c(n)$, for $n \le 56$ with $n < r(n)$.

From this table, it seems that we always have $\omega(n) = 1$ or $\omega(n) = \omega_f(n)$. Therefore we make the following conjecture.

Conjecture 1. Let n be a natural number with $n \in \mathbb{N}_{\neq 10}$ and $n \neq r(n)$. Then $\omega(n) = 1$ or $\omega(n) = \omega_f(n)$.

For the third rows, i.e., the rows of values of $c(n)$, ∞ means that by concatenating the decimal digits of n any number of times, no v-palindromic number will be reached; otherwise $c(n)$ is the least number of times we have to concatenate the decimal digits of n to reach a v-palindromic number. Therefore we can consider such a problem.

Problem 1. Is there a simple way to determine whether $c(n) = \infty$ or not?

Finally, it seems that for most n, $c(n) = \infty$. In fact, it can be shown that all the numbers in (5) have $c(n) = \infty$, so in particular there are infinitely many such numbers. Hence it is natural to conjecture the following.

Conjecture 2. Let $S = \{n \in \mathbb{N} : 10 \nmid n, n < r(n)\}$ and let $T = \{n \in S : c(n) = \infty\}$. Then the asymptotic density of T in S is 1.

9. Some Sequences

After releasing the manuscript, the author had some correspondences with Michel Marcus. Inspired by the author's manuscript, Michel Marcus created the entries A338038, A338039, A338166, and A338371 of the OEIS [4]. The entry A338038 is the function $v(n)$ and the entry A338039 is the sequence of v-palindromic numbers. The entry A338371 is the sequence of integers $n > 0$ such that $10 \nmid n$, $n \neq r(n)$, and $c(n) < \infty$.

Acknowledgements. The author is grateful for the careful reading by Professor Kohji Matsumoto and Professor Hiroshi Suzuki. The author also wants to thank Michel Marcus for valuable correspondences. The author also appreciates comments from the referee that improved this manuscript.

References

[1] K. Alladi and P. Erdös, On an additive arithmetic function, *Pacific J. Math.* **71(2)** (1977), 275-294.

[2] M. Gardner, *The Magic Numbers of Dr. Matrix*, Prometheus Books, New York, 1985.

[3] L. F. Klosinski and D. C. Smolarski, On the reversing of digits, *Math. Mag.* **42(4)** (1969), 208-210.

[4] The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[5] D. Tsai, Natural numbers satisfying an unusual property, *Sūgaku Seminar* **57(11)** (2018), 35-36 (written in Japanese).

[6] Wolfram Research, Inc., Mathematica, Version 12.0.0.0, Champaign, IL (2020).