A SEXTIC DIOPHANTINE CHAIN AND A RELATED MORDELL CURVE

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Abstract

In this paper we obtain parametric as well as numerical solutions of the sextic diophantine chain $\phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2) = \phi(x_3, y_3, z_3) = k$ where $\phi(x, y, z)$ is a sextic form defined by $\phi(x, y, z) = x^6 + y^6 + z^6 - 2x^3y^3 - 2x^3z^3 - 2y^3z^3$ and k is an integer. Each numerical solution of such a sextic chain yields, in general, nine rational points on the Mordell curve $y^2 = x^3 + k/4$. While all of these nine points are not independent in the group of rational points of the Mordell curve, we have constructed a parametrized family of Mordell curves of generic rank ≥ 6 using the aforementioned parametric solution of the sextic diophantine chain. Similarly, the numerical solutions of the sextic chain yield additional examples of Mordell curves whose rank is ≥ 6.

1. Introduction

Let $\phi(x, y, z)$ be a sextic form, with integer coefficients, in the three variables x, y and z. While a limited number of diophantine equations of the type $\phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2)$ have been solved (see [**?**], [**?**], [**?**]), until now no sextic diophantine chains of the type

$$\phi(x_1, y_1, z_1) = \phi(x_2, y_2, z_2) = \phi(x_3, y_3, z_3), \tag{1}$$

have been published.

In this paper we obtain parametric as well as numerical solutions of the diophantine chain (**??**) where the form $\phi(x, y, z)$ is defined by

$$\phi(x, y, z) = x^6 + y^6 + z^6 - 2x^3y^3 - 2x^3z^3 - 2y^3z^3. \tag{2}$$

It is interesting to observe that if we write $k = \phi(x_1, y_1, z_1)/4$ where x_1, y_1, z_1

are rational numbers, three rational points on the Mordell curve

$$y^2 = x^3 + k, \tag{3}$$

are given by

$$(x_1y_1, \ (x_1^3 + y_1^3 - z_1^3)/2), \quad (y_1z_1, \ (y_1^3 + z_1^3 - x_1^3)/2), \quad (x_1z_1, \ (x_1^3 + z_1^3 - y_1^3)/2).$$

In view of the above, it is clear that any solution of the diophantine chain (**??**) immediately yields 9 rational points (not necessarily distinct) on the Mordell curve (**??**) where $k = \phi(x_1, y_1, z_1)/4$. Thus, apart from their intrinsic interest, solutions of the sextic diophantine chain (**??**) are also expected to yield Mordell curves of high rank.

In this context it is pertinent to note that Kihara [**?**] has obtained a family of Mordell curves over the field $\mathbb{Q}(t)$ of generic rank ≥ 6 and till now, this is the best known result of this type regarding families of Mordell curves defined over the field $\mathbb{Q}(t)$. The parametric solution of the sextic diophantine chain (**??**) obtained in this paper also yields a family of Mordell curves, defined over the field $\mathbb{Q}(m, n)$, of generic rank ≥ 6.

2. Sextic Diophantine Chains

In Section **??** we describe a general method of constructing certain sextic diophantine chains. We will apply this method in Sections **??** and **??** to obtain solutions of the sextic diophantine chain (**??**) where $\phi(x, y, z)$ is defined by (**??**). We also show how more solutions of the sextic chain (**??**) may be obtained.

2.1. A General Method of Constructing Sextic Diophantine Chains

Let an arbitrary sextic form $\phi(x, y, z)$ in the three variables x, y, z be expressible as

$$\phi(x, y, z) = k_1 Q^3(x, y, z) + k_2 C^2(x, y, z), \tag{4}$$

where k_1, k_2 are constants while $Q(x, y, z)$ is a quadratic form and $C(x, y, z)$ is a cubic form in the three variables x, y, z such that

$$Q(x, y, z) = Q(y, x, z) \quad \text{and} \quad C(x, y, z) = C(y, x, z). \tag{5}$$

It is clear from (**??**) that to construct a sextic diophantine chain (**??**), it suffices to construct the simultaneous diophantine chains,

$$Q(x_1, y_1, z_1) = Q(x_2, y_2, z_2) = Q(x_3, y_3, z_3), \tag{6}$$

$$C(x_1, y_1, z_1) = C(x_2, y_2, z_2) = C(x_3, y_3, z_3). \tag{7}$$

We will first obtain a parametric solution of the simultaneous diophantine equations,

$$Q(x_1,\, y_1,\, z_1) = Q(x_2,\, y_2,\, z_2),$$
$$C(x_1,\, y_1,\, z_1) = C(x_2,\, y_2,\, z_2), \tag{8}$$

together with the auxiliary equation,

$$x_1 + y_1 + hz_1 = x_2 + y_2 + hz_2, \tag{9}$$

where h is a rational parameter, and then use this solution to obtain a solution of the simultaneous diophantine chains (??) and (??) together with the auxiliary diophantine chain,

$$x_1 + y_1 + hz_1 = x_2 + y_2 + hz_2 = x_3 + y_3 + hz_3. \tag{10}$$

A parametric solution of the simultaneous Equations (??) and (??) may be obtained either by the general method described in [?] or by any other appropriate method.

Now, let $(x_i,\, y_i,\, z_i) = (\alpha_i,\, \beta_i,\, \gamma_i)$, $i = 1,\, 2$, be a solution of the simultaneous Equations (??) and (??) where $\alpha_i,\, \beta_i,\, \gamma_i$, $i = 1,\, 2$ are given in terms of certain independent parameters. We will solve the simultaneous Equations (??), (??) and (??) by obtaining three distinct solutions of the following three simultaneous equations,

$$x + y + hz = k_1, \tag{11}$$
$$Q(x,\, y,\, z) = k_2, \tag{12}$$
$$C(x,\, y,\, z) = k_3, \tag{13}$$

where

$$k_1 = \alpha_1 + \beta_1 + h\gamma_1 = \alpha_2 + \beta_2 + h\gamma_2,$$
$$k_2 = Q(\alpha_1, \beta_1, \gamma_1) \quad = Q(\alpha_2, \beta_2, \gamma_2),$$
$$k_3 = C(\alpha_1, \beta_1, \gamma_1) \quad = C(\alpha_2, \beta_2, \gamma_2).$$

In fact, we already know two solutions of Equations (??), (??) and (??), namely $(x,\, y,\, z) = (\alpha_1, \beta_1, \gamma_1)$ and $(x,\, y,\, z) = (\alpha_2, \beta_2, \gamma_2)$.

To obtain a third solution of Equations (??), (??) and (??), we eliminate x, y from these three equations when, in view of the relations (??), we get the following cubic equation in z:

$$(z - \gamma_1)(z - \gamma_2)(z - \gamma_3) = 0, \tag{14}$$

where γ_3 is a rational function of the parameters occurring in the parametric solution of the simultaneous Equations (??) and (??). The first two roots of Equation (??), namely $z = \gamma_1$ and $z = \gamma_2$, yield the two known solutions of the simultaneous Equations (??), (??) and (??).

We will use the third solution $z = \gamma_3$ to obtain the diophantine chains (??), (??) and (??). On substituting $z = \gamma_3$ in Equations (??) and (??), and eliminating y from these two equations, we get a quadratic equation in x which will have two rational roots if its discriminant is a perfect square. If we can choose the parameters such that the discriminant is a perfect square, we will get two rational solutions of Equations (??) and (??), and we thus get a solution of the simultaneous diophantine chains (??) and (??), and hence also of the sextic diophantine chain (??).

2.2. A Parametric Solution of the Sextic Chain (??)

When $\phi(x, y, z)$ is the sextic form defined by (??), we have the identity,

$$\phi(x,\, y,\, z) = (x^3 + y^3 - z^3)^2 - 4(xy)^3, \tag{15}$$

from which it follows that a solution of the simultaneous diophantine chains,

$$x_1^3 + y_1^3 - z_1^3 \;=\; x_2^3 + y_2^3 - z_2^3 \;=\; x_3^3 + y_3^3 - z_3^3, \tag{16}$$

$$x_1 y_1 = \qquad x_2 y_2 = \qquad x_3 y_3, \tag{17}$$

will yield a solution of the sextic diophantine chain (??)

Following the method described in Section ??, we will first obtain a solution of the simultaneous diophantine equations,

$$x_1^3 + y_1^3 - z_1^3 = x_2^3 + y_2^3 - z_2^3, \tag{18}$$

$$x_1 y_1 = x_2 y_2, \tag{19}$$

$$x_1 + y_1 + h z_1 = x_2 + y_2 + h z_2, \tag{20}$$

where h is a rational parameter.

The complete solution of Equation (??) is given by

$$x_1 = pu, \quad y_1 = qv, \quad x_2 = pv, \quad y_2 = qu, \tag{21}$$

where p, q, u, v are arbitrary parameters.

With these values of x_i, y_i, $i = 1,\, 2$, Equation (??) may be written as,

$$z_1^3 - z_2^3 - (u - v)(u^2 + uv + v^2)(p - q)(p^2 + pq + q^2) = 0, \tag{22}$$

and on writing,

$$z_1 = (n - m)(pu - qv) - m(pv + q(u + v)),$$
$$z_2 = m(pu - qv) + n(pv + q(u + v)), \tag{23}$$

where m, n are arbitrary parameters, Equation (??) reduces to

$$(u^2 + uv + v^2)(p^2 + pq + q^2)\{(2m^3 p + m^3 q - 3m^2 np + 3mn^2 p - n^3 p$$
$$+ n^3 q + p - q)u + (m^3 p - m^3 q + 3m^2 nq - 3mn^2 q + n^3 p + 2n^3 q - p + q)v\} = 0.$$

Accordingly, we get,

$$u = m^3p - m^3q + 3m^2nq - 3mn^2q + n^3p + 2n^3q - p + q,$$
$$v = -(2m^3p + m^3q - 3m^2np + 3mn^2p - n^3p + n^3q + p - q),$$

and on substituting these values of u and v in (**??**) and (**??**), we get a solution of the simultaneous Equations (**??**) and (**??**) which may be written in terms of arbitrary parameters m, n, p and q as $(x_i, y_i, z_i) = (\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2$, where

$$\begin{aligned}
\alpha_1 &= (m^3 + n^3 - 1)p^2 + (-m^3 + 3m^2n - 3mn^2 + 2n^3 + 1)pq, \\
\beta_1 &= (-2m^3 + 3m^2n - 3mn^2 + n^3 - 1)pq + (-m^3 - n^3 + 1)q^2, \\
\gamma_1 &= (m^4 - 2m^3n + 3m^2n^2 - 2mn^3 + n^4 + 2m - n)p^2 \\
&\quad + (m^4 - 2m^3n + 3m^2n^2 - 2mn^3 + n^4 - m + 2n)pq \\
&\quad + (m^4 - 2m^3n + 3m^2n^2 - 2mn^3 + n^4 - m - n)q^2, \\
\alpha_2 &= (-2m^3 + 3m^2n - 3mn^2 + n^3 - 1)p^2 + (-m^3 - n^3 + 1)pq, \\
\beta_2 &= (m^3 + n^3 - 1)pq + (-m^3 + 3m^2n - 3mn^2 + 2n^3 + 1)q^2, \\
\gamma_2 &= (m^4 - 2m^3n + 3m^2n^2 - 2mn^3 + n^4 - m - n)p^2 \\
&\quad + (m^4 - 2m^3n + 3m^2n^2 - 2mn^3 + n^4 + 2m - n)pq \\
&\quad + (m^4 - 2m^3n + 3m^2n^2 - 2mn^3 + n^4 - m + 2n)q^2.
\end{aligned} \tag{24}$$

We note that the solution (**??**) also satisfies Equation (**??**) when $h = -(m^2 - mn + n^2)$.

We will now obtain a solution of the simultaneous diophantine chains (**??**), (**??**) and the auxiliary diophantine chain (**??**) by obtaining three distinct solutions of the simultaneous diophantine Equations (**??**), (**??**), (**??**) where $h = -(m^2 - mn + n^2)$, $Q(x, y, z) = xy$, $C(x, y, z) = x^3 + y^3 - z^3$, $k_1 = \alpha_1 + \beta_1 + h\gamma_1$, $k_2 = \alpha_1\beta_1$, $k_3 = \alpha_1^3 + \beta_1^3 - \gamma_1^3$, with α_1, β_1, γ_1 being defined by (**??**), so that Equations (**??**), (**??**), (**??**) have already two known solutions $(x, y, z) = (\alpha_1, \beta_1, \gamma_1)$ and $(x, y, z) = (\alpha_2, \beta_2, \gamma_2)$.

To obtain a third solution of the simultaneous diophantine Equations (**??**), (**??**), (**??**), we eliminate x and y from these three equations to get the cubic Equation (**??**) where γ_1, γ_2 are defined by (**??**) and

$$\begin{aligned}
\gamma_3 = &\{(m^{10} - 5m^9n + 15m^8n^2 - 30m^7n^3 + 45m^6n^4 - 51m^5n^5 + 45m^4n^6 \\
&- 30m^3n^7 + 15m^2n^8 - 5mn^9 + n^{10} + 2m^7 - 10m^6n + 24m^5n^2 - 38m^4n^3 \\
&+ 40m^3n^4 - 30m^2n^5 + 14mn^6 - 4n^7 + 5m^4 - 10m^3n + 15m^2n^2 - 10mn^3 \\
&+ 5n^4 + m - 2n)p^2 + (m^{10} - 5m^9n + 15m^8n^2 - 30m^7n^3 + 45m^6n^4 - 51m^5n^5 \\
&+ 45m^4n^6 - 30m^3n^7 + 15m^2n^8 - 5mn^9 + n^{10} + 5m^7 - 19m^6n + 42m^5n^2 \\
&- 59m^4n^3 + 58m^3n^4 - 39m^2n^5 + 17mn^6 - 4n^7 + 2m^4 - 4m^3n + 6m^2n^2 \\
&- 4mn^3 + 2n^4 + m + n)pq + (m^2 - mn + n^2 - 1)(m^4 - 2m^3n + 3m^2n^2 - 2mn^3 \\
&+ n^4 + m^2 - mn + n^2 + 1)(m^4 - 2m^3n + 3m^2n^2 - 2mn^3 + n^4 + 2m - n)q^2\} \\
&\times \{(m^2 - mn + n^2 - 1)((m^2 - mn + n^2)^2 + m^2 - mn + n^2 + 1)\}^{-1}.
\end{aligned}$$

While the first two roots, $z = \gamma_1$ and $z = \gamma_2$, of Equation (??) yield the two known solutions of Equations (??), (??), (??), the third root $z = \gamma_3$ will yield a new solution. We will take $z = \gamma_3$ in Equations (??) and (??), and solve them to get the values of x and y. It follows from (??) that $x + y = k_1 - h\gamma_3$, and hence $(x - y)^2 = (x + y)^2 - 4xy = (k_1 - h\gamma_3)^2 - 4k_2$. Thus, both x, y will be rational if $(k_1 - h\gamma_3)^2 - 4k_2$ is a perfect square. This gives us a quartic function in p and q to be made a perfect square. As this function is too cumbersome to write in full, we restrict ourselves to writing it as follows:

$$\{(m-2n)(m^2-mn+n^2)^4+5(m^2-mn+n^2)^3+2(m-2n)(m^2-mn+n^2)+1\}^2 p^4$$
$$+ \cdots + (m^2 - mn + n^2 - 1)^2(2m^3 - 3m^2n + 3mn^2 - n^3 + 1)^2$$
$$\times \{(m^2 - mn + n^2)^2 + m^2 - mn + n^2 + 1\}^2 q^4. \quad (25)$$

Since the coefficients of p^4 and q^4 in the quartic function (??) are perfect squares, we can readily find infinitely many values of p and q that make the function (??) a perfect square by repeatedly applying a method described by Fermat (as quoted by Dickson [?, p. 639]), and each such solution will lead to a solution of the simultaneous diophantine chains (??), (??) and (??), and hence also of the sextic chain (??), in terms of the rational parameters m and n.

As an example, the quartic function (??) becomes a perfect square if we choose p and q as follows:

$$p = -(m - n)((m + n)t + 2)(t^3 - 1),$$
$$q = ((2m - n)t + 1)((m - n)t^3 + t^2 + m),$$

where $t = m^2 - mn + n^2$. This yields a solution of the sextic chain (??) which may be written, in terms of arbitrary parameters m, n, as follows:

$$x_1 = (m - n)((m + n)t + 2)(t^3 - 1)(3(m - n)t^6 + (2m - n)(m - 2n)t^4$$
$$- 3(m^3 + mn^2 - n^3)t^2 - 3tm^2 + m - 2n),$$
$$y_1 = ((m - 2n)t - 1)((2m - n)t + 1)^2(2t^2 + m - 2n)((m - n)t^3 + t^2 + m),$$
$$z_1 = 3(m - n)^2 t^{11} + 9m(m - n)^2 t^9 + (19m^4 - 68m^3 n$$
$$+ 81m^2 n^2 - 35mn^3 + 4n^4)t^7 + (4m^5 - 67m^4 n + 163m^3 n^2$$
$$- 176m^2 n^3 + 104mn^4 - 26n^5)t^5 - (23m^4 - 22m^3 n - 18m^2 n^2$$
$$+ 32mn^3 - 13n^4)t^4 - (19m^3 - 36m^2 n + 39mn^2 - 14n^3)t^3$$
$$+ t(2m^4 - 7m^3 n - 6m^2 n^2 + 8mn^3 - 4n^4)$$
$$+ (m - 2n)(5m^2 - 5mn + 2n^2);$$

$$x_2 = -(m - n)((m + n)t + 2)((m - 2n)t - 1)((2m - n)t + 1)$$
$$\times (2t^2 + m - 2n)(t^3 - 1),$$

$$y_2 = -((2m - n)t + 1)((m - n)t^3 + t^2 + m)(3(m - n)t^6$$
$$+ (2m - n)(m - 2n)t^4 - 3(m^3 + mn^2 - n^3)t^2 - 3m^2t + m - 2n),$$

$$z_2 = 3(m - n)^2 t^{11} - (14m^4 - 37m^3n + 36m^2n^2 - 22mn^3 + 8n^4)t^7$$
$$- (23m^5 - 98m^4n + 140m^3n^2 - 103m^2n^3 + 37mn^4 - 4n^5)t^5$$
$$+ (m^4 + 58m^3n - 90m^2n^2 + 46mn^3 - 5n^4)t^4 + (23m^3 - 12m^2n$$
$$- 9mn^2 + 8n^3)t^3 + (11m^4 - 28m^3n + 39m^2n^2 - 19mn^3 + 2n^4)t$$
$$- (m - 2n)(m^2 + 2mn - 2n^2); \tag{26}$$

$$x_3 = ((m - 2n)t - 1)((m - n)t^3 + t^2 + m)(3(m - n)t^6$$
$$+ (2m - n)(m - 2n)t^4 - (3(m^3 + mn^2 - n^3))t^2 - 3m^2t + m - 2n),$$

$$y_3 = (m - n)((m + n)t + 2)((2m - n)t + 1)^2(2t^2 + m - 2n)(t^3 - 1),$$

$$z_3 = 3(m - n)^2 t^{11} + 9(m - n)^3 t^9 + (13m^4 - 35m^3n + 36m^2n^2$$
$$- 14mn^3 + n^4)t^7 + (19m^5 - 55m^4n + 79m^3n^2 - 44m^2n^3$$
$$- mn^4 + 7n^5)t^5 + (4m^4 - 17m^3n + 54m^2n^2 - 41mn^3 + 10n^4)t^4$$
$$- 2(11m^3 - 21m^2n + 5n^3)t^3 - t(22m^4 - 68m^3n + 78m^2n^2$$
$$- 47mn^3 + 10n^4) - (m - 2n)(4m^2 - 7mn + 4n^2);$$

where, as before, $t = m^2 - mn + n^2$.

As a numerical example, when $m = 1$, $n = 2$, after removing common factors, we get the following solution of the sextic chain (??):

$$\begin{array}{lll}
x_1 = 100958, & y_1 = 425, & z_1 = 113259, \\
x_2 = -7150, & y_2 = -6001, & z_2 = 75081, \\
x_3 = -60010, & y_3 = -715, & z_3 = 59223.
\end{array} \tag{27}$$

2.3. More Solutions of the Sextic Chain (??)

It is interesting to note that when $\phi(x, y, z)$ is defined by (??), in addition to the identity (??), we also have the identity,

$$\phi(x, y, z) = 4Q^3(x, y, z) - 3C^2(x, y, z),$$

where

$$Q(x, y, z) = x^2 + y^2 + z^2 + xy + yz + zx,$$
$$C(x, y, z) = x^3 + y^3 + z^3 + 2x^2y + 2xy^2 + 2x^2z + 2xz^2 + 2y^2z + 2yz^2 + 2xyz.$$

We can now apply the method described in Section ?? to obtain solutions of the diophantine chain (??).

A parametric solution of the simultaneous diophantine Equations (**??**) may be obtained by a straightforward application of the method described in [**?**]. We accordingly omit the tedious details and simply state below the solution thus obtained.

If we define three functions $f_i(u, v, w)$, $i = 1, 2, 3$, as

$$
\begin{aligned}
f_1(u,\,v,\,w) &= (3u^2 - 2uv - 2uw - v^2 + 2vw - w^2)(u^3 + uv^2 \\
&\quad - 2uvw + uw^2 - 2v^3 + 2v^2 w + 2vw^2 - 2w^3), \\
f_2(u,\,v,\,w) &= -2(v - w)(u + v - w)(u - v - w)(u - v + w) \\
&\quad \times (uv + uw - v^2 - vw - w^2), \\
f_3(u,\,v,\,w) &= u^6 - 2u^5 v - 2u^5 w + 2u^4 v^2 + 2u^4 w^2 - 2u^3 v^3 \\
&\quad + 2u^3 v^2 w + 2u^3 vw^2 - 2u^3 w^3 + 2u^2 v^4 + 2u^2 v^3 w \\
&\quad - 8u^2 v^2 w^2 + 2u^2 vw^3 + 2u^2 w^4 - 2uv^5 + 2uv^3 w^2 \\
&\quad + 2uv^2 w^3 - 2uw^5 + v^6 - 2v^5 w + 2v^4 w^2 - 2v^3 w^3 \\
&\quad + 2v^2 w^4 - 2vw^5 + w^6,
\end{aligned}
$$

the aforesaid solution of the simultaneous Equations (**??**) may be written, in terms of arbitrary parameters p, q, r and m, as $(x_i, y_i, z_i) = (\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2$, where

$$
\begin{aligned}
\alpha_1 &= f_1(p,\,q,\,r)m^2 + f_2(p,\,q,\,r)m + pf_3(p,\,q,\,r), \\
\beta_1 &= f_1(q,\,r,\,p)m^2 + f_2(q,\,r,\,p)m + qf_3(p,\,q,\,r), \\
\gamma_1 &= f_1(r,\,p,\,q)m^2 + f_2(r,\,p,\,q)m + rf_3(p,\,q,\,r), \\
\alpha_2 &= f_1(p,\,q,\,r)m^2 - f_2(p,\,q,\,r)m + pf_3(p,\,q,\,r), \\
\beta_2 &= f_1(q,\,r,\,p)m^2 - f_2(q,\,r,\,p)m + qf_3(p,\,q,\,r), \\
\gamma_2 &= f_1(r,\,p,\,q)m^2 - f_2(r,\,p,\,q)m + rf_3(p,\,q,\,r).
\end{aligned}
\tag{28}
$$

Further, the values of α_i, β_i, γ_i, $i = 1, 2$, given by (**??**) satisfy Equation (**??**) where

$$
h = (p^2 + q^2 - r^2)/(p^2 + pq - pr + q^2 - qr).
\tag{29}
$$

Now with the values of α_i, β_i, γ_i, $i = 1, 2$ and h given by (**??**) and (**??**), we will solve Equations (**??**), (**??**) and (**??**). On eliminating x and y from these three equations, we get Equation (**??**) where

$$
\begin{aligned}
\gamma_3 &= (p^2 + pq - pr + q^2 - qr)\{(3p^7 - 2p^6 q - 2p^6 r + 4p^5 q^2 - 4p^5 qr \\
&\quad - 5p^4 q^3 + 2p^4 q^2 r + 6p^4 qr^2 - 3p^4 r^3 - 5p^3 q^4 + 8p^3 q^3 r - 6p^3 q^2 r^2 \\
&\quad + 8p^3 qr^3 - 5p^3 r^4 + 4p^2 q^5 + 2p^2 q^4 r - 6p^2 q^3 r^2 - 10p^2 q^2 r^3 \\
&\quad + 10p^2 qr^4 - 2pq^6 - 4pq^5 r + 6pq^4 r^2 + 8pq^3 r^3 + 10pq^2 r^4 - 36pqr^5 \\
&\quad + 18pr^6 + 3q^7 - 2q^6 r - 3q^4 r^3 - 5q^3 r^4 + 18qr^6 - 11r^7)m^2 \\
&\quad + (p^3 + q^3 - r^3)f_3(p,\,q,\,r)\}\{p^4 + p^3 q - p^3 r + 3p^2 q^2 \\
&\quad - 3p^2 qr + pq^3 - 3pq^2 r + 3pqr^2 - pr^3 + q^4 - q^3 r - qr^3 + r^4\}^{-1}.
\end{aligned}
$$

With $z = \gamma_3$, we have to solve Equations (**??**) and (**??**). This leads to a quadratic equation in x whose discriminant is to be made a perfect square. As this discriminant is too cumbersome to write, we will take specific numerical values of the parameters that yield the desired sextic diophantine chains (**??**). We take for simplicity $q = 0$, and now the condition that the discriminant be a perfect square reduces to finding rational solutions of the following equation:

$$
\begin{aligned}
Y^2 = r(&36p^{11} + 96p^{10}r + 220p^9r^2 + 357p^8r^3 + 522p^7r^4 + 541p^6r^5 \\
&+ 462p^5r^6 + 228p^4r^7 + 22p^3r^8 - 99p^2r^9 - 54pr^{10} - 27r^{11})m^4 \\
&+ 2(8p^{10} + 28p^9r + 12p^8r^2 + 12p^7r^3 - 9p^6r^4 + 12p^5r^5 - 8p^4r^6 \\
&+ 14p^3r^7 - 12p^2r^8 - 9r^{10})(p^2 + pr + r^2)^2m^2 \\
&+ r(4p^3 - 3r^3)(p^2 - pr + r^2)^2(p^2 + pr + r^2)^4. \quad (30)
\end{aligned}
$$

When $p = 3$, $r = 4$, Equation (**??**) reduces to the quartic equation,

$$
Y^2 = 4916053296m^4 - 16574603472m^2 - 106422358224, \quad (31)
$$

which represents a quartic model of an elliptic curve. On making the birational transformation defined by the relations,

$$
\begin{aligned}
m =\ &37(521u - 11v + 318899904)/\{3(42871u - 11v - 2751064896)\}, \\
Y =\ &958300(1331u^3 - 289453824u^2 - 68536946349036u + 86313500544v \quad (32) \\
&+ 3183632918644552704)/(42871u - 11v - 2751064896)^2
\end{aligned}
$$

and

$$
\begin{aligned}
u =\ &(136052568m^2 + 1925Y - 24886644m - 96169512)/\{2(3m - 37)^2\}, \\
v =\ &175(9012764376m^3 + 128613Ym - 1231896204m^2 - 19277Y \quad (33) \\
&- 15193386516m - 15819539736)/\{2(3m - 37)^3\},
\end{aligned}
$$

Equation (**??**) reduces to the Weierstrass form of the elliptic curve given by

$$
v^2 = u^3 + u^2 + 51492677220u - 3062315437673472. \quad (34)
$$

The rank of the elliptic curve (**??**), as determined by the software SAGE, is 3, with the three generators of the Mordell-Weil group being

$$
\begin{gathered}
(101376,\ 56565600), \quad (3761676,\ -7308840000), \\
\text{and} \quad (498157004/529,\ 11417003301600/12167).
\end{gathered}
$$

We can now find infinitely many rational points on the elliptic curve (**??**) using the group law, and then find infinitely many rational points on the quartic curve (**??**) using the relations (**??**). These rational points on the curve (**??**) yield infinitely many numerical examples of the sextic chain (**??**).

In order to find rational points of small height on the curve (??), we used Stoll's program 'ratpoints' [?] and readily obtained the following four values of m for which the right-hand side of Equation (??) becomes a perfect square:

$$37/3, \quad 481/87, \quad 14911/4695, \quad 135679/50151.$$

The first two values of m do not lead to nontrivial sextic chains but the next two values yield the following two solutions of the sextic chain (??):

$$
\begin{aligned}
(x_1,\, y_1,\, z_1) &= (14900543,\, -2461462,\, 15194895), \\
(x_2,\, y_2,\, z_2) &= (12571823,\, 2923703,\, 13884990), \\
(x_3,\, y_3,\, z_3) &= (4528874,\, 11547071,\, 13636239),
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
(x_1,\, y_1,\, z_1) &= (17217348683,\, -3153451318,\, 17759190363), \\
(x_2,\, y_2,\, z_2) &= (14274889211,\, 3650986211,\, 16104056910), \\
(x_3,\, y_3,\, z_3) &= (11570059211,\, 7442013386,\, 15638543835).
\end{aligned}
\tag{36}
$$

3. Mordell Curves Related to Sextic Diophantine Chains

With every solution of the sextic diophantine chain (??), we may associate a Mordell curve (??) where we take

$$4k = \phi(x_1,\, y_1,\, z_1) = \phi(x_2,\, y_2,\, z_2) = \phi(x_3,\, y_3,\, z_3), \tag{37}$$

and, as noted in the Introduction, there will, in general, be 9 rational points on the curve (??) whose coordinates are immediately obtained from the sextic diophantine chain.

We now consider the family of Mordell curves related to the parametric solution (??) of the sextic diophantine chain (??). Using the relations (??) and (??), we can compute the value of k in terms of the arbitrary parameters m and n. We thus get a Mordell curve defined over the field $\mathbb{Q}(m,\, n)$ on which we can readily find 9 rational points. The value of k is too cumbersome to write and we do not give it explicitly.

We note that in view of the relations (??), only 7 of the 9 known rational points on our Mordell curve are actually distinct. We will now apply a theorem of Silverman [?, Theorem 11.4, p. 271] to show that 6 of these 7 points are linearly independent in the group of rational points of the Mordell curve. For this, we must find specific numerical values of m and n such that 6 of the 7 points are linearly independent on the related Mordell curve over $\mathbb{Q}$.

When $m = 1$, $n = 2$, the numerical solution (??) of the sextic chain (??) is related to the Mordell curve,

$$y^2 = x^3 + 449068256221150549783528528 41, \tag{38}$$

on which we get 7 rational points whose coordinates are given below:

$$(42907150, -211912492824721), \quad (48135075, 211912569590346),$$
$$(11434402122, 1240928701242633), \quad (-450561081, 211696384806720),$$
$$(-536829150, 211546966949721), \quad (-42344445, 211912127298846),$$
$$(-3553972230, -4195525176279).$$

The regulator of the first six of these points, as computed by SAGE, is 10390179.16. As this is nonzero, it follows from a well-known theorem [**?**, Theorem 8.1, p. 242] that these 6 points are linearly independent. It follows that the generic rank of the family of Mordell curves related to the sextic chain given by the parametric solution (**??**) is at least 6.

We could not find any numerical values of m and n such that all the 7 known points are linearly independent in the group of rational points of the Mordell curve.

Next we consider the Mordell curves related to the two numerical solutions (**??**) and (**??**) of the sextic chain (**??**).

The Mordell curve related to the solution (**??**) is

$$y^2 = x^3 + 608811416028729407262237319171150516833400, \tag{39}$$

on which we get 9 distinct rational points P_1, P_2, $\ldots$, P_9 whose coordinates are as follows:

$$P_1 = (-36677120373866, 107436818637424863748),$$
$$P_2 = (226412186327985, 3415747486107335266755),$$
$$P_3 = (-37401656636490, 92523324542363200620),$$
$$P_4 = (36756276620569, 332475185129096665153),$$
$$P_5 = (174559636636770, 2319461032255991683920),$$
$$P_6 = (40595586917970, 357467113076423815080),$$
$$P_7 = (52295229628054, 451550293014200834692),$$
$$P_8 = (61756808264886, 544440667643008046316),$$
$$P_9 = (157458619905969, 1991177257485343473603).$$

The regulator of the 6 points P_1, P_2, P_3, P_4, P_5 and P_8 is 11390832.16. Since this is nonzero, these 6 points are independent, and the rank of the Mordell curve (**??**) is at least 6.

Similarly, the solution (**??**) of the sextic chain (**??**) yields 9 distinct rational points on the Mordell curve,

$$y^2 = x^3 + 2929940522529734819579573131655813436044395771052383353305868$$
$$91185336. \tag{40}$$

Again we found that only 6 of the 9 points are independent, and so the rank of the Mordell curve (**??**) is at least 6.

We could not determine the precise rank of any of the three Mordell curves (**??**), (**??**) or (**??**) as the value of k for each of these three curves is very large.

4. An Open Problem

It would be of interest to solve the sextic diophantine chain,

$$\phi(x_1,\, y_1,\, z_1) = \phi(x_2,\, y_2,\, z_2) = \cdots = \phi(x_n,\, y_n,\, z_n), \qquad (41)$$

when $\phi(x,\, y,\, z)$ is defined by (**??**) and $n > 3$. While the existence of such diophantine chains when $n = 4$ and $n = 5$ is not inconceivable, it certainly seems that there must be an upper bound for n for the solvability of the diophantine chain (**??**). It would be of interest to determine the largest integer n for which the diophantine chain (**??**) is solvable.

Any solution of the diophantine chain (**??**) will immediately yield $3n$ rational points on the Mordell curve (**??**) where $k = \phi(x_1,\, y_1,\, z_1)/4$, and may therefore yield Mordell curves of rank higher than the examples already known in the literature.

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