



THE NUMBER OF PARTS IN THE (DISTINCT) PARTITIONS WITH PARTS FROM A SET

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Abstract

The number of parts in the partitions (resp., distinct partitions) of n with parts from a set were considered. Its generating functions were obtained. Consequently, we derive several recurrence identities for the following functions: the number of prime divisors of n , p -adic valuation of n , the number of Carlitz-binary compositions of n and the Hamming weight function. Finally, we obtain an asymptotic estimate for the number of parts in the partitions of n with parts from a finite set of relatively prime integers.

1. Motivation and Basic Definitions

One can find a list of papers [2, 4, 8, 9, 10, 12] in the literature concerning the distribution of a given type of parts in a given class of partitions. In this article, we define the *number of parts function* in two major classes of partitions, namely, in the partitions with parts from a given set and in the distinct partitions with parts from a given set. We derive their generating functions. All the results of this paper are obtained as the consequence of these generating functions. In Section 2, we present several recurrence identities for the following list of functions: the number of Carlitz-binary compositions of n , the number of prime divisors of n , p -adic valuation of n and the Hamming weight function. In Section 3, we obtain an asymptotic estimate for the number of parts in the partitions of n with parts from a finite set of relatively prime positive integers.

The main classes of partitions and compositions of this paper are given in the following definition.

Definition 1. Let n be a positive integer. By a partition (resp. composition) of n , we mean an unordered (resp. ordered) sequence of positive integers whose sum equals n . Each summand in the sum is called a part.

1. A partition of n is said to be a *distinct* partition of n if each part of it is different from the other parts.

2. A partition (resp. composition) of n is said to be a *binary* partition (resp. composition) of n if each part of it is of the form 2^k with integer $k \geq 0$.
3. A composition of n is said to be a *Carlitz* composition of n if each part of it is different from its adjacent parts.
4. A composition of n is said to be a *Carlitz-binary* composition of n if it is both Carlitz and binary.

We introduce the *number of parts function* which will be used for deriving several interesting results.

Definition 2. Let n be a positive integer and let A be a set of positive integers.

1. The function $N_A^p(n)$ is defined to be the number of parts in the partitions of n with parts from the set A .
2. The function $N_A^q(n)$ is defined to be the number of parts in the distinct partitions of n with parts from the set A .

The list of partition functions and divisor-sum functions involved in the recurrence identities of this paper are given in the following definition.

Definition 3. Let n be a positive integer and let A be a set of positive integers.

1. The function $p_A(n)$ (resp. $q_A(n)$) is defined to be the number of partitions (resp. distinct partitions) of n with parts from the set A .
2. The function $p_A^e(n)$ (resp. $p_A^o(n)$) is defined to be the number of partitions of n with even (resp. odd) number of parts from the set A .
3. The function $q_A^e(n)$ (resp. $q_A^o(n)$) is defined to be the number of distinct partitions of n with even (resp. odd) number of parts from the set A .
4. The function $\tau_A(n)$ is defined by

$$\tau_A(n) = \sum_{\substack{a|n \\ a \in A}} 1.$$

5. The function $\tau_A^s(n)$ is defined by

$$\tau_A^s(n) = \sum_{\substack{a|n \\ a \in A}} (-1)^{\frac{n}{a}-1}.$$

6. The function $\sigma_A(n)$ is defined by

$$\sigma_A(n) = \sum_{\substack{a|n \\ a \in A}} a.$$

7. The function $\sigma_A^s(n)$ is defined by

$$\sigma_A^s(n) = \sum_{\substack{a|n \\ a \in A}} (-1)^{\frac{n}{a}-1} a.$$

8. The Hamming weight function, denoted $h(n)$, is defined to be the number of ones in the binary representation of n .

9. Let p be a prime number and let q be a rational number. The p -adic valuation of q , denoted $\nu_p(q)$, is defined by $\nu_p(q) = r$, where $q = p^r \frac{a}{b}$ with $\gcd(a, b) = 1$.

Throughout this article, we assume x to be a real variable with $|x| < 1$. It is well-known and easy to prove that

$$\sum_{n=0}^{\infty} p_A(n)x^n = \prod_{a \in A} (1 - x^a)^{-1} \tag{1}$$

with $p_A(0) = 1$,

$$\sum_{n=0}^{\infty} q_A(n)x^n = \prod_{a \in A} (1 + x^a) \tag{2}$$

with $q_A(0) = 1$,

$$\sum_{n=0}^{\infty} (p_A^e(n) - p_A^o(n))x^n = \prod_{a \in A} (1 + x^a)^{-1} \tag{3}$$

with $p_A^e(0) = 1$ and $p_A^o(n) = 0$

and

$$\sum_{n=0}^{\infty} (q_A^e(n) - q_A^o(n))x^n = \prod_{a \in A} (1 - x^a) \tag{4}$$

with $q_A^e(0) = 1$ and $q_A^o(0) = 0$.

From the Lambert series expansion one can readily get the following two generating functions:

$$\sum_{n=1}^{\infty} \tau_A(n)x^n = \sum_{a \in A} \frac{x^a}{1 - x^a}, \tag{5}$$

$$\sum_{n=1}^{\infty} \sigma_A(n)x^n = \sum_{a \in A} \frac{ax^a}{1 - x^a}. \tag{6}$$

Since $|x| < 1$, we have

$$\frac{x^a}{1 + x^a} = x^a - x^{2a} + x^{3a} - \dots$$

This gives

$$\sum_{a \in A} \frac{x^a}{1 + x^a} = \sum_{a \in A} (x^a - x^{2a} + x^{3a} - \dots).$$

In the right side sum, every exponent of x is of the form $n = ka$ for some $a \in A$. Thus, the coefficient of x^n in this sum is

$$\sum_k (-1)^{k-1} = \sum_{\substack{a \in A \\ a|n}} (-1)^{\frac{n}{a}-1} = \tau_A^s(n).$$

Consequently, we have

$$\sum_{n=1}^{\infty} \tau_A^s(n) x^n = \sum_{a \in A} \frac{x^a}{1 + x^a}. \tag{7}$$

Similar argument gives

$$\sum_{n=1}^{\infty} \sigma_A^s(n) x^n = \sum_{a \in A} \frac{ax^a}{1 + x^a}. \tag{8}$$

2. Generating Functions for $N_A^p(n)$ and $N_A^q(n)$, and Six ways to Convolution and Recurrence Identities

2.1. Derivation with a Direct Application

In this subsection, we derive generating functions for $N_A^p(n)$ and $N_A^q(n)$ by defining appropriate bijective maps.

Theorem 1. *Let n be a positive integer and let A be a set of positive integers. We have*

(a)

$$\sum_{n=1}^{\infty} N_A^p(n) x^n = \left(\prod_{a \in A} \frac{1}{1 - x^a} \right) \left(\sum_{b \in A} \frac{x^b}{1 - x^b} \right), \tag{9}$$

(b)

$$N_A^p(n) = \sum_{k=0}^{n-1} p_A(k) \tau_A(n - k), \tag{10}$$

(c)

$$\sum_{n=1}^{\infty} N_A^q(n) x^n = \left(\prod_{a \in A} (1 + x^a) \right) \left(\sum_{b \in A} \frac{x^b}{1 + x^b} \right), \tag{11}$$

(d)

$$N_A^q(n) = \sum_{k=0}^{n-1} q_A(k) \tau_A^s(n-k). \tag{12}$$

Proof. Let $b \in A$. Let P_A^n be the set of partitions of n with parts from the set A . Let $N_b^p(n)$ be the number of times b occurs in P_A^n . Consider the mapping

$$(b_1, b_2, \dots, b_s) \rightarrow (b_1, b_2, \dots, b_s, b, b, \dots, (k \text{ times}) b)$$

with $b_i \in A \setminus \{b\}$ and $b_1 + b_2 + \dots + b_s + kb = n$.

We observe that this mapping establishes a one-to-one correspondence between $P_{A \setminus \{b\}}^{n-kb}$ and the set of partitions of n with part b occurring exactly k times and parts from the set A . This gives

$$N_b^p(n) = p_{A \setminus \{b\}}(n-b) + 2p_{A \setminus \{b\}}(n-2b) + 3p_{A \setminus \{b\}}(n-3b) + \dots$$

Now, we have

$$\begin{aligned} \sum_{n=1}^{\infty} N_b^p(n)x^n &= \sum_{n=1}^{\infty} \left(\sum_{k \geq 1} kp_{A \setminus \{b\}}(n-kb) \right) x^n \\ &= (x^b + 2x^{2b} + \dots) \left(\sum_{n=0}^{\infty} p_{A \setminus \{b\}}(n)x^n \right) \\ &= \frac{x^b}{(1-x^b)^2} \prod_{a \in A \setminus \{b\}} \frac{1}{1-x^a} \\ &= \frac{x^b}{1-x^b} \prod_{a \in A} \frac{1}{1-x^a}. \end{aligned}$$

Since $N_A^p(n) = \sum_{b \in A} N_b^p(n)$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} N_A^p(n)x^n &= \sum_{n=1}^{\infty} \left(\sum_{b \in A} N_b^p(n) \right) x^n \\ &= \sum_{b \in A} \sum_{n=1}^{\infty} N_b^p(n)x^n \\ &= \sum_{b \in A} \left(\frac{x^b}{1-x^b} \prod_{a \in A} \frac{1}{1-x^a} \right) \\ &= \left(\prod_{a \in A} \frac{1}{1-x^a} \right) \left(\sum_{b \in A} \frac{x^b}{1-x^b} \right) \end{aligned}$$

as expected in (a).

Substituting (1) and (5) in (a), and equating the coefficients of x^n on both sides gives (b).

Let $b \in A$. Let Q_A^n be the set of all distinct partitions of n with parts from the set A . Let $N_b^q(n)$ be the number of times b occurs in Q_A^n . Consider the mapping

$$(b_1, b_2, \dots, b_s) \rightarrow (b_1, b_2, \dots, b_s, b)$$

with $b_i \in A \setminus \{b\}$ and $b_1 + b_2 + \dots + b_s + b = n$. This mapping establishes a one-to-one correspondence between $Q_{A \setminus \{b\}}^{n-b}$ and the set of distinct partitions of n in Q_A^n having b as a part. This gives $N_b^q(n) = |Q_{A \setminus \{b\}}^{n-b}| = q_{A \setminus \{b\}}(n - b)$. Now, we have

$$\begin{aligned} \sum_{n=1}^{\infty} N_b^q(n)x^n &= \sum_{n=1}^{\infty} q_{A \setminus \{b\}}(n - b)x^n \\ &= x^b \sum_{n=0}^{\infty} q_{A \setminus \{b\}}(n)x^n \\ &= x^b \prod_{a \in A \setminus \{b\}} (1 + x^a) \\ &= \frac{x^b}{1 + x^b} \prod_{a \in A} (1 + x^a). \end{aligned}$$

Since $N_A^q(n) = \sum_{b \in A} N_b^q(n)$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} N_A^q(n)x^n &= \sum_{n=1}^{\infty} \left(\sum_{b \in A} N_b^q(n) \right) x^n \\ &= \sum_{b \in A} \sum_{n=1}^{\infty} N_b^q(n)x^n \\ &= \sum_{b \in A} \left(\frac{x^b}{1 + x^b} \prod_{a \in A} (1 + x^a) \right) \\ &= \left(\prod_{a \in A} (1 + x^a) \right) \left(\sum_{b \in A} \frac{x^b}{1 + x^b} \right), \end{aligned}$$

as expected in (c).

Substituting (2) and (7) in (c), and equating the coefficients of x^n on both sides gives (d). □

As special cases of (b) of Theorem 1 we have the following three results. First, we have a convolution-sum expression for the number of parts in the (distinct) partitions of n .

Corollary 1. *Let $N^p(n)$ (resp. $N^q(n)$) be the number of parts in the partitions (resp. distinct partitions) of n and let $p(m)$ (resp. $q(m)$) be the number of partitions (resp. distinct partitions) of m . We have*

(a)

$$N^p(n) = \sum_{k=0}^{n-1} p(k)\tau(n-k), \tag{13}$$

where $\tau(m) = \tau_{\mathbb{N}}(m)$,

(b)

$$N^q(n) = \sum_{k=0}^{n-1} q(k)\tau^s(n-k), \tag{14}$$

where $\tau^s(m) = \tau_{\mathbb{N}}^s(m)$.

As the second consequence of (b) of Theorem 1, we have a recurrence identity for the number of prime divisors of n .

Corollary 2. *Let $A = \{2, 3, 5, 7, 11, \dots\}$ be the set of all prime numbers. Then we have*

$$N_A^p(n) = \sum_{k=0}^{n-1} p_A(k)\Omega(n-k), \tag{15}$$

where $\Omega(m)$ denotes the number of prime divisors of m .

As the third consequence of (b) of Theorem 1, we have a recurrence identity for the p -adic valuation of n for prime p .

Corollary 3. *Let p be a prime number and let $A = \{1, p, p^2, p^3, \dots\}$. We have*

$$N_A^p(n) = \sum_{k=0}^{n-1} p_A(k) (\vartheta_p(n-k) + 1). \tag{16}$$

As a consequence of (d) of Theorem 1, we have an elegant recurrence relation for the Hamming weight function.

Corollary 4. *For integer $n \geq 2$, we have*

$$h(n) = h(n-1) + 1 - \vartheta_2(n). \tag{17}$$

Proof. Set $A = \{1, 2, 2^2, 2^3, \dots\}$. Then by basis representation theorem, we have $q_A(n) = 1$ for every $n \geq 1$. By the definition: $q_A(0) = 1$. Also we have $\tau_A^s(n) = 1 - \vartheta_2(n)$ and $N_A^q(n) = h(n)$. Substituting these values in (d) of Theorem 1 gives

$$\begin{aligned} h(n) &= \sum_{k=1}^n (1 - \vartheta_2(k)) \\ &= h(n-1) + 1 - \vartheta_2(n) \end{aligned}$$

as expected. □

Note 1. Corollary 4 can be written as $h(n) = \vartheta_2\left(\frac{2^n}{n!}\right)$.

2.2. Inversion

The following result is an inversion formula for (b) and (d) of Theorem 1.

Theorem 2. Let n be a positive integer and let A be a set of positive integers. We have

$$(a) \quad \sum_{k=1}^n N_A^p(k) (q_A^e(n-k) - q_A^o(n-k)) = \tau_A(n), \tag{18}$$

$$(b) \quad \sum_{k=1}^n N_A^q(k) (p_A^e(n-k) - p_A^o(n-k)) = \tau_A^s(n). \tag{19}$$

Proof. From (a) of Theorem 1, we have

$$\left(\sum_{n=1}^{\infty} N_A^p(n)x^n\right) \left(\prod_{a \in A} (1 - x^a)\right) = \sum_{b \in A} \frac{x^b}{1 - x^b}. \tag{20}$$

Substituting (4) and (5) in (20), and equating the coefficients of x^n on both sides gives (a).

From (c) of Theorem 1, we have

$$\left(\sum_{n=1}^{\infty} N_A^q(n)x^n\right) \left(\prod_{a \in A} \frac{1}{1 + x^a}\right) = \sum_{b \in A} \frac{x^b}{1 + x^b}. \tag{21}$$

Substituting (3) and (7) in (21), and equating the coefficients of x^n on both sides gives (b). □

A recurrence identity of the form $g(n) = \sum_{k=0}^n \omega(k)f(n-k)$ with

$$\omega(m) = \begin{cases} 1 & \text{if } m = 0; \\ (-1)^k & \text{if } m = \frac{3k^2 \pm k}{2} \end{cases}$$

is called an Euler-type recurrence identity. As a consequence of Theorem 2 we have an Euler-type recurrence identity for the number of parts in the partitions of n .

Corollary 5. We have

$$(a) \quad \sum_{k=1}^n N^p(k)\omega(n-k) = \tau(n), \tag{22}$$

(b)

$$\sum_{k=1}^n (-1)^{n-k} N^q(k) o(n-k) = \tau^s(n), \tag{23}$$

where $o(m)$ denotes the number of distinct partitions of m with odd parts, $\tau^s(m) = \tau_{\mathbb{N}}^s(m)$ and $\tau(m) = \tau_{\mathbb{N}}(m)$.

Proof. Set $A = \mathbb{N}$. Then by Euler’s pentagonal number theorem [6] we have

$$q_A^e(n-k) - q_A^o(n-k) = \omega(n-k).$$

Substituting this value in (a) of Theorem 2, and equating the coefficients of x^n on both sides gives (a).

From (3) we have

$$\prod_{n=1}^{\infty} \frac{1}{1+x^n} = \sum_{m=0}^{\infty} (p_{\mathbb{N}}^e(m) - p_{\mathbb{N}}^o(m)) x^m.$$

In view of Euler’s partition theorem [5], we have

$$\prod_{n=1}^{\infty} \frac{1}{1+x^n} = \prod_{n=1}^{\infty} (1-x^{2n-1}) = \sum_{m=0}^{\infty} (-1)^m o(m) x^m. \tag{24}$$

Substituting these values in (b) of Theorem 2, and equating the coefficients of x^n on both sides gives (b). □

As another application of Theorem 2 we express the number of prime divisors of n as the convolution sum of $N_A^p(k)$ and $q_A^e(k) - q_A^o(k)$ with A being the set of prime numbers.

Corollary 6. *Let $A = \{2, 3, 5, 7, 11, \dots\}$ be the set of all prime numbers. We have*

$$\sum_{k=1}^n N_A^p(k) (q_A^e(n-k) - q_A^o(n-k)) = \Omega(n). \tag{25}$$

A result similar to the above one is achieved for the p -adic valuation of n .

Corollary 7. *Let p be a prime number and let $A = \{1, p, p^2, p^3, \dots\}$. We have*

$$\sum_{k=1}^n N_A^p(k) (q_A^e(n-k) - q_A^o(n-k)) = \vartheta_p(n) + 1. \tag{26}$$

As a special case of Corollary 7 we have a relation between Hamming weight function, the number of parts in the binary partitions of n and the 2-adic valuation of n .

Corollary 8. Let $N_{bin}^p(m)$ be the number of parts in the binary partitions of m . Define $s(0) = 1$ and $s(m) = (-1)^{h(m)}$ for $m \geq 1$. Then we have

$$\sum_{k=1}^n N_{bin}^p(k)s(n-k) = \vartheta_2(n) + 1. \tag{27}$$

2.3. Product with $\prod_{n=1}^\infty(1-x^n)$ and $\prod_{n=1}^\infty(1+x^n)^{-1}$

Multiplying the terms $\prod_{n=1}^\infty(1-x^n)$ and $\prod_{n=1}^\infty(1+x^n)^{-1}$, respectively, with the generating functions (a) and (c) of Theorem 1 gives recurrence identities (separately) for $N_A^p(m)$ and $N_A^q(m)$ when $A \neq \mathbb{N}$, of which one is Euler-type.

Theorem 3. We have

(a)

$$\sum_{k=1}^n N_A^p(k)\omega(n-k) = \sum_{k=1}^n \tau_A(k) (q_{\mathbb{N}-A}^e(n-k) - q_{\mathbb{N}-A}^o(n-k)), \tag{28}$$

(b)

$$\sum_{k=1}^n (-1)^{n-k} N_A^q(k)o(n-k) = \sum_{k=1}^n \tau_A^s(k) (p_{\mathbb{N}-A}^e(n-k) - p_{\mathbb{N}-A}^o(n-k)). \tag{29}$$

Proof. From (a) of Theorem 1, we have

$$\sum_{n=1}^\infty N_A^p(n)x^n = \left(\prod_{a \in A} \frac{1}{1-x^a} \right) \left(\sum_{b \in A} \frac{x^b}{1-x^b} \right).$$

Multiplying both sides with $\prod_{m=1}^\infty(1-x^m)$ gives

$$\left(\sum_{n=1}^\infty N_A^p(n)x^n \right) \left(\prod_{m=1}^\infty (1-x^m) \right) = \left(\prod_{a \in \mathbb{N} \setminus A} (1-x^a) \right) \left(\sum_{b \in A} \frac{x^b}{1-x^b} \right).$$

Now in view of Euler’s pentagonal number theorem [6], (4) and (5), we get (a).

From (c) of Theorem 1, we have

$$\sum_{n=1}^\infty N_A^q(n)x^n = \left(\prod_{a \in A} (1+x^a) \right) \left(\sum_{b \in A} \frac{x^b}{1+x^b} \right).$$

Multiplying both sides by $\prod_{m=1}^\infty \frac{1}{1+x^m}$ gives

$$\left(\sum_{n=1}^\infty N_A^q(n)x^n \right) \left(\prod_{m=1}^\infty \frac{1}{1+x^m} \right) = \left(\prod_{a \in \mathbb{N} \setminus A} \frac{1}{1+x^a} \right) \left(\sum_{b \in A} \frac{x^b}{1+x^b} \right). \tag{30}$$

Now substituting (24), (3) and (7) in the equation above, and equating the coefficients of like powers of x on both sides gives (b). □

2.4. Logarithmic Differentiation

Logarithmic differentiation of the generating function of $p(n)$ gives the following relation:

$$np(n) = \sum_{k=0}^{n-1} \sigma(k)p(n-k). \tag{31}$$

We wield the same technique for $N_A^p(n)$ and $N_A^q(n)$ to get results of similar kind.

Theorem 4. *We have*

(a)

$$nN_A^p(n) = \sum_{k=1}^{n-1} N_A^p(k)\sigma_A(n-k) + \sum_{t=1}^n t\tau_A(t)p_A(n-t), \tag{32}$$

(b)

$$nN_A^q(n) = \sum_{k=1}^{n-1} N_A^q(k)\sigma_A^s(n-k) + \sum_{t=1}^n t\tau_A^s(t)q_A(n-t). \tag{33}$$

Proof. Define the following:

$$N_A^p(x) = \left(\prod_{a \in A} \frac{1}{1-x^a} \right) \left(\sum_{b \in A} \frac{x^b}{1-x^b} \right);$$

$$N_A^q(x) = \left(\prod_{a \in A} (1+x^a) \right) \left(\sum_{b \in A} \frac{x^b}{1+x^b} \right);$$

$$p_A(x) = \prod_{a \in A} \frac{1}{(1-x^a)};$$

$$q_A(x) = \prod_{a \in A} (1+x^a);$$

$$\tau_A(x) = \sum_{b \in A} \frac{x^b}{1-x^b};$$

$$\tau_A^s(x) = \sum_{b \in A} \frac{x^b}{1+x^b}.$$

Now taking the logarithm of $N_A^p(x)$ and then differentiating gives

$$\frac{N_A^{p'}(x)}{N_A^p(x)} = \sum_{a \in A} \frac{ax^{a-1}}{1-x^a} + \frac{\tau_A'(x)}{\tau_A(x)}.$$

Multiplying both sides by x and rearranging gives

$$\begin{aligned} xN_A^{p'}(x) &= N_A^p(x) \left(\sum_{a \in A} \frac{ax^a}{1-x^a} + \frac{x\tau'_A(x)}{\tau_A(x)} \right) \\ &= N_A^p(x) \left(\sum_{a \in A} \frac{ax^a}{1-x^a} + \frac{x\tau'_A(x)}{N_A^p(x)} p_A(x) \right) \\ &= N_A^p(x) \left(\sum_{a \in A} \frac{ax^a}{1-x^a} \right) + x\tau'_A(x)p_A(x). \end{aligned}$$

Now equating the coefficients of x^n on extreme terms of the above chain of equalities gives (a).

Taking the logarithm of $N_A^q(x)$ and differentiating gives

$$\frac{N_A^{q'}(x)}{N_A^q(x)} = \sum_{a \in A} \frac{ax^{a-1}}{1+x^a} + \frac{\tau_A^{s'}(x)}{\tau_A^s(x)}.$$

Multiplying both sides by x gives

$$\begin{aligned} xN_A^{q'}(x) &= N_A^q(x) \left(\sum_{a \in A} \frac{ax^a}{1+x^a} + \frac{x\tau_A^{s'}(x)}{\tau_A^s(x)} \right) \\ &= N_A^q(x) \left(\sum_{a \in A} \frac{ax^a}{1+x^a} + \frac{x\tau_A^{s'}(x)}{N_A^q(x)} q_A(x) \right) \\ &= N_A^q(x) \left(\sum_{a \in A} \frac{ax^a}{1+x^a} \right) + x\tau_A^{s'}(x)q_A(x). \end{aligned}$$

Now equating the coefficients of x^n on extreme terms of the above chain of equalities gives (b). □

Corollary 9. *We have*

(a)

$$nN^p(n) = \sum_{k=1}^{n-1} N^p(k)\sigma(n-k) + \sum_{t=1}^n t\tau(t)p(n-t), \tag{34}$$

(b)

$$nN^q(n) = \sum_{k=1}^{n-1} N^q(k)\sigma^s(n-k) + \sum_{t=1}^n t\tau^s(t)q(n-t), \tag{35}$$

(c)

$$nN_{bin}^p(n) = \sum_{k=1}^{n-1} N_{bin}^p(k) \left(2^{\vartheta_2(n-k)+1} - 1 \right) + \sum_{t=1}^n t(\vartheta_2(t) + 1)b(n-t). \tag{36}$$

2.5. Recurrence Identities over A

In this section, we derive recurrence identities for $N_A^p(n)$ and $N_A^q(n)$ over the set A .

Theorem 5. *Let n be a positive integer and let A be a set of positive integers. For $s \in A$, we have*

$$N_A^p(n) - N_A^p(n - s) = p_A(n - s) + N_{A \setminus \{s\}}^p(n) \tag{37}$$

with $p_A(0) = 1$.

Proof. By Theorem 1,

$$\sum_{n=1}^{\infty} N_A^p(n)x^n = \left(\prod_{a \in A} \frac{1}{1 - x^a} \right) \left(\sum_{b \in A} \frac{x^b}{1 - x^b} \right). \tag{38}$$

This gives

$$\begin{aligned} \left(\sum_{n=1}^{\infty} N_A^p(n)x^n \right) (1 - x^s) &= \left(\prod_{a \in A \setminus \{s\}} \frac{1}{1 - x^a} \right) \left(\sum_{b \in A} \frac{x^b}{1 - x^b} \right) \\ &= \left(\prod_{a \in A \setminus \{s\}} \frac{1}{1 - x^a} \right) \left(\frac{x^s}{1 - x^s} + \sum_{b \in A \setminus \{s\}} \frac{x^b}{1 - x^b} \right) \\ &= x^s \prod_{a \in A} \frac{1}{1 - x^a} + \left(\prod_{a \in A \setminus \{s\}} \frac{1}{1 - x^a} \right) \left(\sum_{b \in A \setminus \{s\}} \frac{x^b}{1 - x^b} \right). \end{aligned}$$

Now writing the products in the right side as infinite sums in accordance with (1) and (a) of Theorem 1, and equating the coefficients of x^n on both sides gives the expected end. \square

Theorem 6. *Let n be a positive integer and let A be a set of positive integers. Let $s \in A$. We have*

$$N_A^q(n) - N_A^q(n - s) + N_A^q(n - 2s) + \dots = N_{A \setminus \{s\}}^q(n) + q_{A \setminus \{s\}}(n - s) - q_{A \setminus \{s\}}(n - 2s) + \dots \tag{39}$$

Proof. By Theorem 1,

$$\begin{aligned} \frac{1}{1 + x^s} \sum_{n=1}^{\infty} N_A^q(n)x^n &= \left(\prod_{a \in A \setminus \{s\}} (1 + x^a) \right) \left(\sum_{b \in A \setminus \{s\}} \frac{x^b}{1 + x^b} \right) \\ &\quad + \frac{x^s}{1 + x^s} \left(\prod_{a \in A \setminus \{s\}} (1 + x^a) \right). \end{aligned}$$

Now writing the products in the right side as infinite sums in accordance with (2) and (c) of Theorem 1, and equating the coefficients of x^n on both sides gives the expected end. \square

2.6. Interplay with Carlitz Compositions

In this subsection we find some connections between the number of Carlitz’s compositions, 2-adic valuation and the number of parts in the distinct partitions in terms of finite discrete convolutions.

Theorem 7. *Let $cl_A(m)$ be the number of Carlitz compositions of m . We have*

(a)

$$cl_A(n) = \sum_{k=0}^{n-1} cl_A(k)\tau_A^s(n-k), \tag{40}$$

(b)

$$\sum_{k=0}^n cl_A(k)q_A(n-k) - \sum_{t=0}^{n-1} cl_A(t)N_A^q(n-t) = q_A(n) \tag{41}$$

with $cl_A(0) = 1$.

Proof. Heubach and Mansour [7] derived the following generating function for the number of Carlitz compositions of n with parts from the set A :

$$\sum_{n=0}^{\infty} cl_A(n)x^n = \frac{1}{1 - \sum_{a \in A} \frac{x^a}{1+x^a}}, \tag{42}$$

which can be written as

$$\left(\sum_{n=0}^{\infty} cl_A(n)x^n \right) \left(1 - \sum_{a \in A} \frac{x^a}{1+x^a} \right) = 1.$$

The coefficient of x^n on the left side term is

$$cl_A(n) - \sum_{k=0}^{n-1} cl_A(k)\tau_A^s(n-k).$$

Since the coefficient of x^n on the right side is zero, we have (a).

Multiplying both sides of (42) by $\frac{1}{\prod_{a \in A} (1+x^a)}$ gives

$$\left(\sum_{n=0}^{\infty} cl_A(n)x^n \right) \left(\frac{1}{\prod_{a \in A} (1+x^a)} \right) = \frac{1}{\prod_{a \in A} (1+x^a) - (\prod_{a \in A} (1+x^a)) (\sum_{a \in A} \frac{x^a}{1+x^a})}.$$

This can be arranged as

$$\left(\sum_{n=0}^{\infty} cl_A(n)x^n\right) \left[\left(\prod_{a \in A} (1+x^a)\right) - \left(\prod_{a \in A} (1+x^a)\right) \left(\sum_{a \in A} \frac{x^a}{1+x^a}\right)\right] = \prod_{a \in A} (1+x^a).$$

Then in view of (c) of Theorem 1 and (2) we can write

$$\left(\sum_{n=0}^{\infty} cl_A(n)x^n\right) \left[\sum_{l=0}^{\infty} q_A(l)x^l - \sum_{m=1}^{\infty} N_A^q(m)x^m\right] = \sum_{k=0}^{\infty} q_A(k)x^k.$$

Now equating the coefficients of like powers of x on both sides gives (b). □

Corollary 10. *let $cl_b(m)$ be the number of Carlitz-binary compositions of m . We have*

(a)

$$cl_b(n) = \sum_{k=0}^{n-1} cl_b(k)(1 - \vartheta_2(n - k)), \tag{43}$$

(b)

$$\sum_{k=0}^n cl_b(k)(1 - h(n - k)) = 1 \tag{44}$$

with $h(0) = 0$.

Proof. Let $A = \{1, 2, 2^2, \dots\}$. Then in view of basis representation theorem we have $q_A(m) = 1$ and $N_A^q(m) = h(m)$ for every positive integer m . Moreover, we have $\tau_A^s(m) = 1 - \vartheta_2(m)$. The result is immediate while we substitute these values in Theorem 7. □

3. Asymptotic Estimate of $N_A^p(n)$ when A is a Finite Set

Throughout this section we assume $A = \{a_1, a_2, \dots, a_k\}$, a finite set of positive integers with $\gcd(A) = 1$. Based on this assumption we find an asymptotic estimate for $N_A^p(n)$. To that end, we take cue from an earlier paper of the author [3], where the function $p_A(n)$ was considered. It was shown there that $p_A(a_1 a_2 \dots a_k l + r)$ is a polynomial in l of degree k for each $r \in \{0, 1, 2, \dots, a_1 a_2 \dots a_k - 1\}$ with the identical leading coefficient $\frac{(a_1 a_2 \dots a_k)^{k-2}}{(k-1)!}$. This fact was used in [3] to arrive at the estimate of Netto [11]:

$$p_A(n) \sim \frac{n^{k-1}}{(a_1 a_2 \dots a_k)(k-1)!}.$$

A quasi-polynomial representation of $p_A(n)$ found in [3] is recalled in the following lemma.

Lemma 1. *Let l be a non-negative integer and let $A = \{a_1, a_2, \dots, a_k\}$ be a set of positive integers such that $\gcd(A) = 1$. For each $r \in \{0, 1, \dots, a_1 a_2 \dots a_k - 1\}$, the term $p_A(a_1 a_2 \dots a_k l + r)$ is a polynomial in l of degree $k - 1$ with the leading coefficient $\frac{(a_1 a_2 \dots a_k)^{k-2}}{(k-1)!}$.*

The main result of this section is given below.

Theorem 8. *Let $A = \{a_1, a_2, \dots, a_k\}$ be a set of positive integers such that $\gcd(A) = 1$. Then we have*

$$N_A^p(n) \sim \frac{1}{k!} \frac{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_k}}{a_1 a_2 \dots a_k} n^k. \tag{45}$$

Proof. The crux of this proof is to show that $N_A^p(a_1 a_2 \dots a_k l + r)$ is a polynomial in l of degree k with the leading coefficient

$$\frac{\sum_{i=1}^k \left(a_i^{k-2} \prod_{\substack{j \neq i \\ 0 \leq j \leq k}} a_j^{k-1} \right)}{k!}$$

for each $r \in \{0, 1, \dots, a_1 a_2 \dots a_k - 1\}$. Once this is established, then we will have

$$\lim_{l \rightarrow \infty} \frac{N_A^p(a_1 a_2 \dots a_k l + r)}{(a_1 a_2 \dots a_k l + r)^k} = \frac{1}{k!} \frac{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_k}}{a_1 a_2 \dots a_k}$$

for each $r \in \{0, 1, \dots, a_1 a_2 \dots a_k - 1\}$, which is equivalent to the asymptotic estimate (45).

We will establish this main claim using induction over $|A| = k$. Since the term a_i^{k-2} is involved in the leading coefficient, we take $k = 3$ as the initial case for induction. Nevertheless the cases $k = 1, 2$ are needed to realise the case $k = 3$.

When $|A| = 1$ the only way out is $a_1 = 1$. In this case we have $N_A^p(n) = n$, and the targeted estimate follows. To proceed further, we need the following observation:

$$N_{\{a\}}^p(n) = \begin{cases} \frac{n}{a} & \text{if } a \mid n; \\ 0 & \text{otherwise.} \end{cases}$$

Assume $|A| = 2$. That is, $A = \{a_1, a_2\}$ with $\gcd(a_1, a_2) = 1$. Fix $r \in \{1, 2, \dots, a_1\}$. Applying Theorem 5 a_1 times, we get

$$\begin{aligned} N_A^p(a_1 a_2 l + r) - N_A^p(a_1 a_2 (l - 1) + r) &= \sum_{m=1}^{a_1} p_A(a_1 a_2 l + r - m a_2) \\ &\quad + \sum_{t=0}^{a_1-1} N_{\{a_1\}}^p(a_1 a_2 l + r - t a_2). \end{aligned}$$

Since $\gcd(a_1, a_2) = 1$, the congruence equation $a_2t \equiv r \pmod{a_1}$ has a unique solution modulo a_1 , say t^* . This gives

$$\begin{aligned} \sum_{t=0}^{a_1-1} N_{\{a_1\}}^p(a_1a_2l + r - ta_2) &= \frac{a_1a_2l + r - t^*a_2}{a_1} \\ &= a_2l + \frac{r - t^*a_2}{a_1} \\ &= a_2l + k_1 \end{aligned}$$

for each $r \in \{0, 1, \dots, a_1 - 1\}$, where k_1 is an integer constant.

Next, we analyze the sum $\sum_{m=1}^{a_1} p_A(a_1a_2l + r - ma_2)$. Let $m \in \{1, 2, \dots, a_1\}$. Case i. Assume $r - ma_2 < 0$. In this case, consider the term $p_A(a_1a_2l + r - ma_2)$, which can be written as $p_A(a_1a_2(l - 1) + a_1a_2 + r - ma_2)$. Here we note that $0 \leq a_1a_2 + r - ma_2 \leq a_1a_2 - 1$. Now, in view of Lemma 1, we have

$$p_A(a_1a_2(l - 1) + a_1a_2 + r - ma_2) = (l - 1) + c_m,$$

where c_m is an integer constant.

Case ii. Assume $r - ma_2 \geq 0$. Then by Lemma 1, we have $p_A(a_1a_2l + r - ma_2) = l + d_m$, where d_m is an integer constant.

In both the cases, $p_A(a_1a_2l + r - ma_2) = l + b_m$ for some constant b_m . This gives

$$\begin{aligned} \sum_{m=1}^{a_1} p_A(a_1a_2l + r - ma_2) &= \sum_{m=1}^{a_1} (l + b_m) \\ &= a_1l + k_2, \end{aligned}$$

where $k_2 = \sum_{m=1}^{a_1} b_m$ is an integer constant. Consequently,

$$N_A^p(a_1a_2l + r) - N_A^p(a_1a_2(l - 1) + r) = (a_1 + a_2)l + c,$$

where $c = k_1 + k_2$ is an integer constant.

This gives

$$N_A^p(a_1a_2l + r) = (a_1 + a_2) \left(\frac{l^2 + l}{2} \right) + cl + N_A^p(r).$$

Subsequently, we have

$$\lim_{l \rightarrow \infty} \frac{N_A^p(a_1a_2l + r)}{(a_1a_2l + r)^2} = \frac{\frac{1}{a_1} + \frac{1}{a_2}}{2!(a_1a_2)}.$$

Since this is true for each $r \in \{0, 1, \dots, a_1a_2 - 1\}$, the targeted estimate follows for the case $k = 2$.

Now we have the initial case of the induction, that is, $k = 3$. Assume $A = \{a_1, a_2, a_3\}$ with $\gcd(a_1, a_2, a_3) = 1$. Fix $r \in \{0, 1, \dots, a_1 a_2 a_3 - 1\}$.

Let $d = \gcd(a_1, a_2)$. Applying Theorem 5 $a_1 a_2$ times, we get

$$\begin{aligned}
 & N_A^p(a_1 a_2 a_3 l + r) - N_A^p(a_1 a_2 a_3(l - 1) + r) \\
 &= \sum_{t=1}^{a_1 a_2} p_A(a_1 a_2 a_3 l + r - t a_3) + \sum_{m=0}^{a_1 a_2 - 1} N_{A \setminus \{a_3\}}^p(a_1 a_2 a_3 l + r - m a_3) \\
 &= \sum_{t=1}^{a_1 a_2} p_A(a_1 a_2 a_3 l + r - t a_3) + \sum_{\substack{0 \leq m \leq a_1 a_2 - 1 \\ d | r - m a_3}} N_{\{a_1, a_2\}}^p\left(\frac{a_1 a_2 a_3 l}{d} + \frac{r - m a_3}{d}\right) \\
 &= \sum_{t=1}^{a_1 a_2} p_A(a_1 a_2 a_3 l + r - t a_3) + \sum_{\substack{0 \leq m \leq a_1 a_2 - 1 \\ d | r - m a_3}} N_{\{\frac{a_1}{d}, \frac{a_2}{d}\}}^p\left(\frac{a_1}{d} \frac{a_2}{d} \frac{a_3 d^2}{d} l + \frac{r - m a_3}{d}\right).
 \end{aligned} \tag{46}$$

If $r - t a_3 < 0$, then by Lemma 1, $p_A(a_1 a_2 a_3(l - 1) + a_1 a_2 a_3 + r - t a_3)$ is a polynomial in $l - 1$ of degree 2 with the leading coefficient $\frac{a_1 a_2 a_3}{2!}$. If $r - t a_3 \geq 0$, then again by Lemma 1, $p_A(a_1 a_2 a_3 l + r - t a_3)$ is a polynomial in l of degree 2 with the leading coefficient $\frac{a_1 a_2 a_3}{2!}$. We observe that, in both the cases, $p_A(a_1 a_2 a_3 l + r - t a_3)$ is a polynomial in l of degree 2 with the leading coefficient $\frac{a_1 a_2 a_3}{2!}$. This in turn gives $\sum_{t=1}^{a_1 a_2} p_A(a_1 a_2 a_3 l + r - t a_3)$ is a polynomial in l of degree 2 with the leading coefficient $\frac{a_1^2 a_2^2 a_3}{2!}$.

Since $\gcd(d, a_3) = 1$, the congruence equation $a_3 m \equiv r \pmod{d}$ has a unique solution modulo d . Consequently, there exists a unique integer $m_i \in \{id + 0, id + 1, \dots, id + (d - 1)\}$ such that $a_3 m_i \equiv r \pmod{d}$ for each $0 \leq i \leq \frac{a_1 a_2}{d} - 1$. Based on these observations, we can write

$$\begin{aligned}
 & \sum_{\substack{0 \leq m \leq a_1 a_2 - 1 \\ d | r - m a_3}} N_{\{\frac{a_1}{d}, \frac{a_2}{d}\}}^p\left(\frac{a_1}{d} \frac{a_2}{d} \frac{a_3 d^2}{d} l + \frac{r - m a_3}{d}\right) \\
 &= \sum_{i=0}^{\frac{a_1 a_2}{d} - 1} N_{\{\frac{a_1}{d}, \frac{a_2}{d}\}}^p\left(\frac{a_1}{d} \frac{a_2}{d} \frac{a_3 d^2}{d} l + \frac{r - m_i a_3}{d}\right) \\
 &= \sum_{i=0}^{\frac{a_1 a_2}{d} - 1} N_{\{\frac{a_1}{d}, \frac{a_2}{d}\}}^p\left(\frac{a_1}{d} \frac{a_2}{d} (a_3 d l + q_i) + r_i\right),
 \end{aligned}$$

where q_i and r_i were uniquely determined (in view of division algorithm) satisfying the relation: $\frac{r - m_i a_3}{d} = \frac{a_1}{d} \frac{a_2}{d} q_i + r_i$. Since $\gcd(\frac{a_1}{d}, \frac{a_2}{d}) = 1$, from the case $|A| = 2$, we

have that

$$N_{\{\frac{a_1}{d}, \frac{a_2}{d}\}}^p \left(\frac{a_1}{d} \frac{a_2}{d} (a_3dl + q_i) + r_i \right)$$

is a polynomial in $a_3dl + q_i$ of degree 2 with the leading coefficient $\frac{\frac{a_1}{d} + \frac{a_2}{d}}{2!}$. This in turn gives that

$$N_{\{\frac{a_1}{d}, \frac{a_2}{d}\}}^p \left(\frac{a_1}{d} \frac{a_2}{d} (a_3dl + q_i) + r_i \right)$$

is a polynomial in l of degree 2 with the leading coefficient

$$\frac{\frac{a_1}{d} + \frac{a_2}{d}}{2!} a_3^2 d^2.$$

Consequently,

$$\sum_{i=0}^{\frac{a_1 a_2}{d} - 1} N_{\{\frac{a_1}{d}, \frac{a_2}{d}\}}^p \left(\frac{a_1}{d} \frac{a_2}{d} (a_3dl + q_i) + r_i \right)$$

is a polynomial in l of degree 2 with the leading coefficient

$$\frac{a_1 a_2}{d} \frac{\frac{a_1}{d} + \frac{a_2}{d}}{2!} a_3^2 d^2 = \frac{a_1^2 a_2 a_3^2 + a_1 a_2^2 a_3^2}{2!}.$$

Now in view of Equation (46), we have that: $N_A^p(a_1 a_2 a_3 l + r) - N_A^p(a_1 a_2 a_3(l - 1) + r)$ is a polynomial in l of degree 2 with the leading coefficient

$$\frac{a_1^2 a_2^2 a_3 + a_1^2 a_2 a_3^2 + a_1 a_2^2 a_3^2}{2!}.$$

From this we infer that $N_A^p(a_1 a_2 a_3 l + r)$ is a polynomial in l of degree 3. Now if one assumes c_3 to be the leading coefficient of $N_A^p(a_1 a_2 a_3 l + r)$, then in accordance with the previous observations, we have

$$3c_3 = \frac{a_1^2 a_2^2 a_3 + a_1^2 a_2 a_3^2 + a_1 a_2^2 a_3^2}{2!}.$$

This gives

$$c_3 = \frac{a_1^2 a_2^2 a_3 + a_1^2 a_2 a_3^2 + a_1 a_2^2 a_3^2}{3!}.$$

Hence, the assertion is true for $k = 3$.

Assume that the assertion is true up to some $k - 1 \geq 3$. Now we shall prove the assertion for k .

Let $A = \{a_1, a_2, \dots, a_k\}$ be a set of positive integers such that $\gcd(a_1, a_2, \dots, a_k) = 1$. Let $d = \gcd(a_1, a_2, \dots, a_{k-1})$. Fix $r \in \{0, 1, \dots, a_1 a_2 \dots a_k - 1\}$. Repeated application of Theorem 5 $a_1 a_2 \dots a_{k-1}$ times gives

$$\begin{aligned}
 & N_A^p(a_1 a_2 \cdots a_k l + r) - N_A^p(a_1 a_2 \cdots a_k (l - 1) + r) \\
 &= \sum_{i=1}^{a_1 a_2 \cdots a_{k-1}} p_A(a_1 a_2 \cdots a_k l + r - i a_k) \\
 &\quad + \sum_{\substack{0 \leq m \leq a_1 a_2 \cdots a_{k-1} - 1 \\ d | r - m a_k}} N_{A \setminus \{a_k\}}^p(a_1 a_2 \cdots a_k l + r - m a_k) \\
 &= \sum_{i=1}^{a_1 a_2 \cdots a_{k-1}} p_A(a_1 a_2 \cdots a_k l + r - i a_k) \\
 &\quad + \sum_{\substack{0 \leq m \leq a_1 a_2 \cdots a_{k-1} - 1 \\ d | r - m a_k}} N_{\{\frac{a_1}{d}, \frac{a_2}{d}, \dots, \frac{a_{k-1}}{d}\}}^p \left(\frac{a_1 a_2 \cdots a_k l}{d} + \frac{r - m a_k}{d} \right) \\
 &= \sum_{i=1}^{a_1 a_2 \cdots a_{k-1}} p_A(a_1 a_2 \cdots a_k l + r - i a_k) \\
 &\quad + \sum_{\substack{0 \leq m \leq a_1 a_2 \cdots a_{k-1} - 1 \\ d | r - m a_k}} N_{\{\frac{a_1}{d}, \frac{a_2}{d}, \dots, \frac{a_{k-1}}{d}\}}^p \left(\frac{a_1}{d} \frac{a_2}{d} \cdots \frac{a_{k-1}}{d} \frac{d^{k-1} a_k l}{d} + \frac{r - m a_k}{d} \right).
 \end{aligned} \tag{47}$$

By Lemma 1, $p_A(a_1 a_2 \cdots a_k l + r - i a_k)$ is a polynomial in l or $l - 1$ (depending respectively on the bounds: $r - i a_k \geq 0$ or $r - i a_k < 0$) of degree $k - 1$ with the leading coefficient $\frac{(a_1 a_2 \cdots a_k)^{k-2}}{(k-1)!}$. In both the cases, $p_A(a_1 a_2 \cdots a_k l + r - i a_k)$ is a polynomial in l of degree $k - 1$ with the leading coefficient $\frac{(a_1 a_2 \cdots a_k)^{k-2}}{(k-1)!}$. Consequently, $\sum_{i=1}^{a_1 a_2 \cdots a_{k-1}} p_A(a_1 a_2 \cdots a_k l + r - i a_k)$ is a polynomial in l of degree $k - 1$ with the leading coefficient $(a_1 a_2 \cdots a_{k-1}) \frac{(a_1 a_2 \cdots a_k)^{k-2}}{(k-1)!} = \frac{(a_1 a_2 \cdots a_{k-1})^{k-1} a_k^{k-2}}{(k-1)!}$.

Since $\gcd(d, a_k) = 1$, the equation $a_k m \equiv r \pmod{d}$ has a unique solution modulo d . Consequently, there exists a unique integer $m_t \in \{td + 0, td + 1, \dots, td + (d - 1)\}$ such that $a_k m_t \equiv r \pmod{d}$ for each $t \in \{0, 1, \dots, a_1 a_2 \cdots a_{k-1} - 1\}$. In view of division algorithm, one can find a unique pair of integers (q_t, r_t) such that $\frac{r - m_t a_k}{d} = \frac{a_1 a_2 \cdots a_{k-1}}{d^{k-1}} q_t + r_t$. Based on these observations, we can write

$$\begin{aligned}
 & \sum_{\substack{0 \leq m \leq a_1 a_2 \cdots a_{k-1} - 1 \\ d | r - m a_k}} N_{\{\frac{a_1}{d}, \frac{a_2}{d}, \dots, \frac{a_{k-1}}{d}\}}^p \left(\frac{a_1}{d} \frac{a_2}{d} \cdots \frac{a_{k-1}}{d} \frac{d^{k-1} a_k l}{d} + \frac{r - m a_k}{d} \right) \\
 &= \sum_{t=0}^{\frac{a_1 a_2 \cdots a_{k-1} - 1}{d}} N_{\{\frac{a_1}{d}, \frac{a_2}{d}, \dots, \frac{a_{k-1}}{d}\}}^p \left(\frac{a_1}{d} \frac{a_2}{d} \cdots \frac{a_{k-1}}{d} (d^{k-2} a_k l + q_t) + r_t \right).
 \end{aligned}$$

Since $\gcd\left(\frac{a_1}{d}, \frac{a_2}{d}, \dots, \frac{a_{k-1}}{d}\right) = 1$, by induction assumption, we have that

$$N_{\left\{\frac{a_1}{d}, \frac{a_2}{d}, \dots, \frac{a_{k-1}}{d}\right\}}^p \left(\frac{a_1}{d} \frac{a_2}{d} \dots \frac{a_{k-1}}{d} (d^{k-2} a_k l + q_t) + r_t \right)$$

is a polynomial in $d^{k-2} a_k l + q_t$ of degree $k - 1$ with the leading coefficient

$$\frac{\sum_{s=1}^{k-1} \left(\frac{a_s}{d}\right)^{k-3} \prod_{\substack{j \neq s \\ 0 \leq j \leq k-1}} \left(\frac{a_j}{d}\right)^{k-2}}{(k-1)!}.$$

Consequently, $N_{\left\{\frac{a_1}{d}, \frac{a_2}{d}, \dots, \frac{a_{k-1}}{d}\right\}}^p \left(\frac{a_1}{d} \frac{a_2}{d} \dots \frac{a_{k-1}}{d} (d^{k-2} a_k l + q_t) + r_t \right)$ is a polynomial in l of degree $k - 1$ with the leading coefficient

$$\frac{(a_k d^{k-2})^{k-1} \sum_{s=1}^{k-1} a_s^{k-3} \prod_{\substack{j \neq s \\ 0 \leq j \leq k-1}} a_j^{k-2}}{d^{k-3} d^{(k-2)(k-2)} (k-1)!}.$$

This gives that

$$\sum_{t=0}^{\frac{a_1 a_2 \dots a_{k-1} - 1}{d}} N_{\left\{\frac{a_1}{d}, \frac{a_2}{d}, \dots, \frac{a_{k-1}}{d}\right\}}^p \left(\frac{a_1}{d} \frac{a_2}{d} \dots \frac{a_{k-1}}{d} (d^{k-2} a_k l + q_i) + r_i \right)$$

is a polynomial in l of degree $k - 1$ with the leading coefficient

$$\begin{aligned} \frac{a_1 a_2 \dots a_{k-1}}{d} \frac{(a_k d^{k-2})^{k-1}}{d^{k-3} d^{(k-2)(k-2)}} \frac{\sum_{s=1}^{k-1} a_s^{k-3} \prod_{\substack{j \neq s \\ 0 \leq j \leq k-1}} a_j^{k-2}}{(k-1)!} &= \frac{a_k^{k-1} \sum_{s=1}^{k-1} a_s^{k-2} \prod_{\substack{j \neq s \\ 0 \leq j \leq k-1}} a_j^{k-1}}{(k-1)!} \\ &= \frac{\sum_{s=1}^{k-1} a_s^{k-2} \prod_{\substack{j \neq s \\ 0 \leq j \leq k}} a_j^{k-1}}{(k-1)!}. \end{aligned}$$

Now in view of (47), we get $N_A^p(a_1 a_2 \dots a_k l + r) - N_A^p(a_1 a_2 \dots a_k (l - 1) + r)$ as a polynomial in l of degree $k - 1$ with the leading coefficient

$$\frac{\sum_{s=1}^{k-1} a_s^{k-2} \prod_{\substack{j \neq s \\ 0 \leq j \leq k}} a_j^{k-1}}{(k-1)!} + \frac{a_k^{k-2} (a_1 a_2 \dots a_{k-1})^{k-1}}{(k-1)!} = \frac{\sum_{i=1}^k a_i^{k-2} \prod_{\substack{j \neq i \\ 0 \leq j \leq k}} a_j^{k-1}}{(k-1)!}.$$

Let c_k be the leading coefficient of $N_A^p(a_1 a_2 \dots a_k l + r)$. Then from the above observations, we have

$$k c_k = \frac{\sum_{i=1}^k a_i^{k-2} \prod_{\substack{j \neq i \\ 0 \leq j \leq k}} a_j^{k-1}}{(k-1)!},$$

which gives

$$c_k = \frac{\sum_{i=1}^k a_i^{k-2} \prod_{\substack{j \neq i \\ 0 \leq j \leq k}} a_j^{k-1}}{k!}.$$

Hence, the main assertion is true for k . Then the targeted estimate follows immediately while adhering to the discussion at the starting part of this proof. \square

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