A NEW PROPERTY OF APPELL SEQUENCES AND ITS APPLICATION

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Abstract

It is the main purpose of this paper to prove a new property of Appell sequences by extending the Appell identity to a more general form of identity. Subsequently, by applying this property to Bernoulli, Euler, and Hermite polynomials we deduce various kinds of shortened recurrence formulas for them, in which some of the preceding polynomials are entirely missing. Lastly, we prove a certain concise identity that connects two sums for Appell polynomials with different variables.

1. Introduction

Appell sequences play an important role in many branches of mathematics and theoretical physics, such as number theory, distribution theory, numerical analysis, p-adic analysis, theory of modular forms, quantum field theory, string theory, etc.

An Appell sequence $A(x) := \{A_n(x)\}_{n\geq 0}$ with $A_0(x)$ a non-zero constant is a family of Appell polynomials $A_n(x)$ satisfying the differential relation

$$\frac{d}{dx}A_n(x) = nA_{n-1}(x) \quad \text{for all} \quad n \geq 1. \tag{1.1}$$

As is clear from this definition, we see that $A_n(x)$ is a polynomial of degree n.

Appell sequences have been studied extensively in parallel with Sheffer sequences and various kinds of properties have been developed over the years by many authors (see, e.g., [1, 11, 16, 17, 21] and the references therein). We now pick out below some important and indispensable characteristic properties of Appell polynomials.

As is easily seen, (1.1) is tantamount to each of the following conditions.

(a) There exists a scalar sequence $A := \{a_n\}_{n\geq 0}$ with $a_0 \neq 0$ satisfying

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}. \tag{1.2}$$

(b) For the same sequence A as the above, it follows that

$$A_n(x) = \left(\sum_{k=0}^{\infty} \frac{a_k}{k!} D^k\right) x^n, \quad D := \frac{d}{dx} \quad \text{(the differential operator)}. \tag{1.3}$$

(c) For an integer $n \geq 0$, it follows that

$$A_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} A_k(x) y^{n-k} \quad \text{(the so-called Appell identity)}. \tag{1.4}$$

These polynomials may be alternatively defined by the generating function

$$\mathcal{A}(t,x) := A(t)e^{xt} = \sum_{n=0}^{\infty} \frac{A_n(x)}{n!} t^n, \tag{1.5}$$

where, with the same scalar sequence A as used in (a),

$$A(t) := \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \quad (a_0 \neq 0), \tag{1.6}$$

which is an analytic function at $t = 0$. Indeed, noting that

$$\frac{d}{dx}\mathcal{A}(t,x) = tA(t)e^{xt} = \sum_{n=0}^{\infty} \left(\frac{d}{dx} A_n(x)\right) \frac{t^n}{n!},$$

we compare the coefficients of $t^n/n!$ on both sides. Then we can confirm from (1.5) that the polynomials $A_n(x)$, $n = 1, 2, 3, \ldots$, satisfy (1.1).

In what follows we use the following symbolic (umbral) notation for convenience sake. Given any $c, d \in \mathbb{C}$ (or $\mathbb{C}[x]$) and a sequence $\{T_n\}_{n\geq 0}$ with $T_n \in \mathbb{C}$ (or $\mathbb{C}[x]$), we denote for integers $m, k \geq 0$,

$$(cT_k + d)^m := \sum_{j=0}^{m} \binom{m}{j} c^j T_{k+j} d^{m-j}. \tag{1.7}$$

In other words, we expand the left-hand side in full based on the binomial theorem and then replace $(T_k)^j$ with T_{k+j} for $j = 0, 1, \ldots, m$. This is just a modified version of Lucas' notation, $T^k(cT + d)^m$, used in [15, p. 252]. For example, one can write (1.2) simply as $A_n(x) = (a_0 + x)^n$ using the sequence A as defined in (a).

In the family of Appell polynomials, besides the monomials $\{x^n\}_{n\geq 0}$, the most typical examples are the Bernoulli and Euler polynomials. Denoting these polynomials by $B_n(x)$ and $E_n(x)$, they are defined by the generating functions

$$\begin{aligned} &\text{(i)} \qquad \mathcal{B}(t,x) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \quad (|t| < 2\pi); \\ &\text{(ii)} \qquad \mathcal{E}(t,x) := \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n \quad (|t| < \pi), \end{aligned} \tag{1.8}$$

respectively. Indeed, taking $A(t) = t/(e^t - 1)$ and $2/(e^t + 1)$ in (1.5), it is easy to confirm that these polynomials form an Appell sequence.

In the author's earlier works, it has been proved that these polynomials satisfy the following bivariate relations (see [2, (3.1)]). For integers $k, m \geq 0$ we have

$$\begin{aligned} &\text{(i)} \quad (B_k(x) + y)^m = (B_m(x + y) - y)^k\,; \\ &\text{(ii)} \quad (E_k(x) + y)^m = (E_m(x + y) - y)^k\,. \end{aligned} \tag{1.9}$$

Furthermore, we observed that several other polynomials also satisfy very similar types of relations to (1.9), as long as they are Appell-like polynomials (cf. [3–5]).

Through these observations, we presumed that exactly the same form of relation as (1.9) might always hold for 'any' Appell sequence. Based on such a motivation, we studied some more general properties of Appell sequences, and as a result, we confirmed that our presumption is certainly correct, as stated in the following main theorem of this paper:

Theorem 1.1. *With the above notation, it can be stated that $\{A_n(x)\}_{n\geq 0}$ forms an Appell sequence if and only if the bivariate relation*

$$(A_k(x) + y)^m = (A_m(x + y) - y)^k \tag{1.10}$$

holds for integers $m, k \geq 0$.

The most significant feature of (1.10) is that the first $\min\{k-1, m-1\}$ polynomial terms are excluded from the equation. Further, one can choose freely non-negative integers k and m without any restriction; thus, looking at from another perspective, we may regard it as a bivariate equation with two integer parameters.

This paper is organized as follows. In Section 2 we concentrate on a proof of Theorem 1.1 by using a higher-order differential property that generalizes (1.1). In Section 3, to confirm the usefulness of (1.10), by applying it to Bernoulli, Euler, and Hermite polynomials, we derive various kinds of *shortened* recurrence formulas for them, in which some of the preceding polynomial terms are entirely missing. In the final Section 4, as additional results, we first prove a certain practical identity that connects two sums for Appell polynomials with different variables, and then applying this to the monomials, we discuss some combinatorial identities.

2. Proof of Theorem 1.1

We begin with presenting a simple combinatorial formula for an alternating sum of products of binomial coefficients that will be needed in a proof of Theorem 1.1. That is, for any integers $n, m, k \geq 0$ we have

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{m+i}{n+m-k} = \binom{m}{k}, \tag{2.1}$$

where $\binom{x}{y} = 0$ if $y < 0$ or $x < y$ by convention. Note that this formula is not new and the proof is quite easy (for instance, see [13, 14]). A rather short and concise proof can be found in [5, Lemma 3.1].

Proof of Theorem 1.1. At first, let us assume that $A(x)$ is an Appell polynomial. Then, by repeatedly using (1.1) we find the higher-order differential property

$$\frac{d^k}{dx^k} A_n(x) = k!\binom{n}{k} A_{n-k}(x) \quad \text{for any} \quad k = 1, 2, \ldots, n. \tag{2.2}$$

Denoting the left and right-hand sides of (1.10) by $P(x, y)$ and $Q(x, y)$, respectively, we regard them as polynomials in y of degree $k + m$ with polynomial coefficients $p_r(x), q_r(x) \in \mathbb{C}[x]$ for $r \in I := \{0, 1, \ldots .k + m\}$, namely

$$P(x, y) = \sum_{r=0}^{k+m} p_r(x) y^r \quad \text{and} \quad Q(x, y) = \sum_{r=0}^{k+m} q_r(x) y^r.$$

For (1.10), it suffices to prove that $p_r(x) = q_r(x)$ holds for all $r \in I$. Expanding the left-hand side of (1.10) based on the binomial theorem, we obtain immediately

$$p_r(x) = \binom{m}{r} A_{k+m-r}(x) \quad \text{for} \quad r \in I. \tag{2.3}$$

On the other hand, by partially differentiating the right-hand side of (1.10) r times with respect to y based on (2.2) we obtain, according to Leibniz's rule,

$$\begin{aligned}
\frac{\partial^r}{\partial y^r} Q(x, y) &= \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^r}{\partial y^r} \left(A_{m+i}(x + y)(-y)^{k-i}\right) \\
&= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \sum_{a=0}^{r} \binom{r}{a} \frac{\partial^{r-a}}{\partial y^{r-a}} A_{m+i}(x + y) \frac{\partial^a}{\partial y^a} y^{k-i} \\
&= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \sum_{\substack{0 \le a \le r \\ a \le k-i}} \binom{r}{a} \left\{ \frac{(m + i)!}{(m + i - r + a)!} A_{m+i-(r-a)}(x + y) \right\} \\
&\qquad \times \left\{ \frac{(k - i)!}{(k - i - a)!} y^{k-i-a} \right\}.
\end{aligned}$$

Set here $y = 0$. Since $0^{k-i-a} = 0$ unless $a = k - i$, we notice that all the terms in the last inner sum vanish except for the only term $\alpha_{k,m,r}(i) A_{k+m-r}(x)$ with

$$\alpha_{k,m,r}(i) := \binom{r}{k-i} \frac{(m + i)!(k - i)!}{(k + m - r)!0!} = r! \binom{m + i}{k + m - r}.$$

Therefore, by using (2.1) we find that

$$q_r(x) = \frac{1}{r!}\frac{\partial^r}{\partial y^r}Q(x,y)\Big|_{y=0} = \left\{\frac{1}{r!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\alpha_{k,m,r}(i)\right\}A_{k+m-r}(x)$$
$$= \binom{m}{r}A_{k+m-r}(x) \quad (r \geq 0),$$

which verifies in view of (2.3) that $p_r(x) = q_r(x)$ for all $r \in I$. Conversely, assuming that (1.10) holds and taking $k = 0$ and $m = n \geq 0$, we get

$$\sum_{i=0}^{n}\binom{n}{i}A_i(x)y^{n-i} = (A_0(x)+y)^n = (A_n(x+y)-y)^0 = A_n(x+y),$$

which is just the Appell identity in (1.4), and thereby $A(x)$ is an Appell polynomial. This completes the proof of Theorem 1.1. □

3. Application

In this section we demonstrate how (1.10) is useful in deducing shortened recurrence formulas for Appell-type polynomials. As notable examples, we deal with below the Bernoulli, Euler, and Hermite polynomials for these demonstrations.

3.1. Bernoulli and Euler Polynomials

Since both the Bernoulli and Euler polynomials are Appell, we recall (1.9) and set $y = a$ a positive integer, that is,

$$\begin{aligned} &\text{(i)} \quad (B_k(x)+a)^m = (B_m(x+a)-a)^k\,; \\ &\text{(ii)} \quad (E_k(x)+a)^m = (E_m(x+a)-a)^k\,. \end{aligned} \tag{3.1}$$

To begin with, we want to show that the right-hand sides of these identities can be expressed as linear combinations of $B_n(x)$ and $E_n(x)$ for $n = m, m+1, \ldots, m+k$, respectively. Using the generating functions defined in (1.8), let us consider the functional equations

$$\mathcal{B}(t,x)(e^{at}-1) = \mathcal{B}(t,x+a) - \mathcal{B}(t,x) = t\sum_{j=0}^{a-1}e^{(x+j)t};$$
$$\mathcal{E}(t,x)(e^{at}-(-1)^a) = \mathcal{E}(t,x+a) - (-1)^a\mathcal{E}(t,x) = 2\sum_{j=0}^{a-1}(-1)^{a-1-j}e^{(x+j)t},$$

which are easily shown to be held after the example of the identity

$$\frac{X^a + (-1)^{a-1}Y^a}{X+Y} = \sum_{j=0}^{a-1}(-Y)^j X^{a-1-j} \quad (X \neq -Y).$$

Equating individually the coefficients of $t^n/n!$ on both sides of the above equations after expanding them into the Taylor series, we obtain for all $n \geq 0$,

$$\begin{aligned} &\text{(i)} \quad B_n(x+a) = B_n(x) + n\sum_{j=0}^{a-1}(x+j)^{n-1}; \\ &\text{(ii)} \quad E_n(x+a) = (-1)^a E_n(x) + 2\sum_{j=0}^{a-1}(-1)^{a-1-j}(x+j)^n. \end{aligned} \tag{3.2}$$

Thus, by using (3.2) (i) the right-hand side of (3.1) (i) can be written as

$$\begin{aligned} (B_m(x+a)-a)^k &= \sum_{i=0}^{k}\binom{k}{i}B_{m+i}(x+a)(-a)^{k-i} \\ &= \sum_{i=0}^{k}\binom{k}{i}\Big\{B_{m+i}(x) + (m+i)\sum_{j=0}^{a-1}(x+j)^{m+i-1}\Big\}(-a)^{k-i} \\ &= \sum_{i=0}^{k}\binom{k}{i}B_{m+i}(x)(-a)^{k-i} + \sum_{i=0}^{k}\binom{k}{i}(m+i)\sum_{j=0}^{a-1}(x+j)^{m+i-1}(-a)^{k-i} \\ &= (B_m(x)-a)^k + \sum_{j=0}^{a-1}(x+j)^{m-1}\sum_{i=0}^{k}\binom{k}{i}(m+i)(x+j)^i(-a)^{k-i}. \end{aligned}$$

Furthermore, using the easy identity $\binom{k}{i}i = k\binom{k-1}{i-1}$ $(0 \leq i \leq k)$, the last double sum part can be calculated as follows:

$$\begin{aligned} &m\sum_{j=0}^{a-1}(x+j)^{m-1}\sum_{i=0}^{k}\binom{k}{i}(x+j)^i(-a)^{k-i} \\ &\qquad + k\sum_{j=0}^{a-1}(x+j)^m\sum_{i=0}^{k-1}\binom{k-1}{i}(x+j)^i(-a)^{k-1-i} \\ &= m\sum_{j=0}^{a-1}(x+j)^{m-1}(x+j-a)^k + k\sum_{j=0}^{a-1}(x+j)^m(x+j-a)^{k-1} \\ &= \sum_{j=0}^{a-1}\{(m+k)(x+j)-ma\}(x+j)^{m-1}(x+j-a)^{k-1}. \end{aligned}$$

So substituting this into the above, we get

$$\begin{aligned}&(B_m(x+a)-a)^k\\&=(B_m(x)-a)^k+\sum_{j=0}^{a-1}\{(m+k)(x+j)-ma\}(x+j)^{m-1}(x+j-a)^{k-1}.\end{aligned}$$

Similarly, using (3.2) (ii), we next rewrite the right-hand side of (3.1) (ii) as

$$\begin{aligned}&(E_m(x+a)-a)^k=\sum_{i=0}^{k}\binom{k}{i}E_{m+i}(x+a)(-a)^{k-i}\\&=\sum_{i=0}^{k}\binom{k}{i}\Big\{(-1)^aE_{m+i}(x)+2\sum_{j=0}^{a-1}(-1)^{a-1-j}(x+j)^{m+i}\Big\}(-a)^{k-i}\\&=(-1)^a\sum_{i=0}^{k}\binom{k}{i}E_{m+i}(x)(-a)^{k-i}\\&\qquad+2\sum_{i=0}^{k}\binom{k}{i}\sum_{j=0}^{a-1}(-1)^{a-1-j}(x+j)^{m+i}(-a)^{k-i}\\&=(-1)^a(E_m(x)-a)^k+2\sum_{j=0}^{a-1}(-1)^{a-1-j}(x+j)^m\sum_{i=0}^{k}\binom{k}{i}(x+j)^i(-a)^{k-i}\\&=(-1)^a(E_m(x)-a)^k+2\sum_{j=0}^{a-1}(-1)^{a-1-j}(x+j)^m(x+j-a)^k.\end{aligned}$$

Finally, substituting these identities into the right-hand sides of (3.1) (i) and (ii), we obtain the following shortened recurrence formulas.

Corollary 3.1. *For any integers $k, m \geq 0$ and $a \geq 1$ we have*

$$\begin{aligned}&\text{(i)}\quad (B_k(x)+a)^m-(B_m(x)-a)^k\\&\qquad=\sum_{j=0}^{a-1}\{(k+m)(x+j)-ma\}(x+j)^{m-1}(x+j-a)^{k-1};\\&\text{(ii)}\quad (E_k(x)+a)^m-(-1)^a(E_m(x)-a)^k\\&\qquad=2\sum_{j=0}^{a-1}(-1)^{a-1-j}(x+j)^m(x+j-a)^k.\end{aligned}\tag{3.3}$$

Here, it should be mentioned that (3.3) (i) can be also derived from the following formula (cf. [2,3]). That is, denoting $\beta_n(x) := B_n(x)/n$ for $n \geq 1$,

$$\begin{aligned}&(\beta_{k+1}(x)+a)^m - (\beta_{m+1}(x)-a)^k \\ &= \sum_{j=0}^{a-1}(x+j)^m(x+j-a)^k + \frac{(-1)^{k+1}}{m+k+1}\binom{m+k}{m}^{-1} a^{m+k+1}.\end{aligned} \tag{3.4}$$

Indeed, to obtain (3.3) (i) we have only to differentiate both sides of (3.4) with respect to x using the Appell property $\frac{d}{dx}\beta_n(x) = B_{n-1}(x)$ $(n \geq 1)$. Note that (3.4) is a polynomial analogue of Saalschütz-Gelfand's formula for Bernoulli numbers proved in [12, 18] (see also [7] for a brief historical overview), namely

$$(\beta_{k+1}+1)^m + (-1)^{m+k}(\beta_{m+1}+1)^k = \frac{(-1)^{k+1}}{m+k+1}\binom{m+k}{m}^{-1},$$

where $\beta_n := B_n/n$ for $n \geq 1$ and B_n, $n = 0, 1, 2, \ldots$, are the Bernoulli numbers defined by the generating function $\mathcal{B}(t) := \mathcal{B}(t, 0) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$ with $|t| < 2\pi$.

Special cases of (3.3) (i) and (ii) provide the following identities, which are more practical than the identities in (3.3) to compute recursively these polynomials.

Corollary 3.2. *Let $c(m,k) := \binom{m}{k} + \binom{m+1}{k}$. For Bernoulli polynomials we have*

$$\begin{aligned}&\text{(i)} \quad 2\sum_{j=0}^{\lfloor m/2\rfloor}\binom{m+1}{2j+1}B_{2m+1-2j}(x) = (m+1)(2x-1)(x^2-x)^m; \\ &\text{(ii)} \quad 2\sum_{j=0}^{\lfloor m/2\rfloor} c(m,2j+1)B_{2m-2j}(x) \\ &\qquad\qquad = \{2(2m+1)(x^2-x)+m\}(x^2-x)^{m-1}.\end{aligned} \tag{3.5}$$

On the other hand, for Euler polynomials we have

$$\begin{aligned}&\text{(i)} \quad \sum_{j=0}^{\lfloor m/2\rfloor}\binom{m}{2j}E_{2m-2j}(x) = (x^2-x)^m; \\ &\text{(ii)} \quad \sum_{j=0}^{\lfloor (m+1)/2\rfloor} c(m,2j)E_{2m+1-2j}(x) = (2x-1)(x^2-x)^m.\end{aligned} \tag{3.6}$$

Proof. We consider the special cases of (3.3) (i) and (ii) for $a = 1$, i.e.,

$$\begin{aligned}&\text{(i)}_{k,m} \quad (B_k(x)+1)^m - (B_m(x)-1)^k \\ &\qquad\qquad = ((k+m)x-m)x^{m-1}(x-1)^{k-1}; \\ &\text{(ii)}_{k,m} \quad (E_k(x)+1)^m + (E_m(x)-1)^k = 2x^m(x-1)^k.\end{aligned} \tag{3.7}$$

Setting here $k = m$, we have

$$\sum_{i=0}^{m} \left(1-(-1)^{m-i}\right)\binom{m}{i} B_{m+i}(x) = m(2x-1)(x^2-x)^{m-1};$$
$$\sum_{i=0}^{m} \left(1+(-1)^{m-i}\right)\binom{m}{i} E_{m+i}(x) = 2(x^2-x)^{m}.$$

As is obviously seen, they involve only the odd-index Bernoulli and even-index Euler polynomials, respectively. So, by reindexing the above summations with particular attention to the parity of m we are able to obtain (3.5) (i) and (3.6) (i). On the other hand, (3.5) (ii) and (3.6) (ii) can be obtained by combining the identities in (3.7) as "$(\mathrm{i})_{m+1,m} + (\mathrm{i})_{m,m+1}$" and "$(\mathrm{ii})_{m+1,m} + (\mathrm{ii})_{m,m+1}$", respectively. □

At the end of this subsection, we wish to confirm the fact that all the shortened recurrence formulas as stated above can be derived as special cases of a certain formula extended (3.4) to the generalized Bernoulli polynomials $B_{n,\chi}(x)$ attached to a primitive Dirichlet character χ with conductor $f = f_\chi$, which are defined by the generating function

$$\mathcal{B}_\chi(t,x) := \frac{1}{e^{ft}-1}\sum_{r=1}^{f} \chi(r) t e^{(x+r)t} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}(x)}{n!} t^n \quad (|t| < 2\pi/f).$$

In fact, letting χ^0 be the principal character and χ' be the unique primitive character with $f_{\chi'} = 4$, one can find the functional relations $\mathcal{B}_{\chi^0}(t,x) = te^{xt} + \mathcal{B}(t,x)$ and $\mathcal{B}_{\chi'}(t, 2x-1) = -\frac{t}{2}\mathcal{E}(2t,x)$. As is clearly seen, these relations imply that

$$\text{(i)} \quad B_{n,\chi^0}(x) = \begin{cases} 1 & \text{for } n = 0; \\ B_n(x) + nx^{n-1} & \text{for } n \geq 1; \end{cases}$$

$$\text{(ii)} \quad B_{n,\chi'}(2x-1) = \begin{cases} 0 & \text{for } n = 0; \\ -2^{n-2} n E_{n-1}(x) & \text{for } n \geq 1, \end{cases}$$

respectively. Therefore, all the formulas for $B_{n,\chi}(x)$ can be reduced to those for ordinary Bernoulli and Euler polynomials by considering the special cases $\chi = \chi^0$ and χ', respectively. It should be noted that the Saalschütz-Gelfand type formula for generalized Bernoulli polynomials corresponding to (3.4) and other related matters have been studied in details in [3] with many application examples.

3.2. Hermite Polynomials

Let H_n and $H_n(x)$, $n = 0, 1, 2, \ldots$, be the Hermite numbers and polynomials defined by the generating functions

$$\mathcal{H}(t) := e^{-t^2} = \sum_{n=0}^{\infty} \frac{H_n}{n!} t^n \quad (|t| < \infty);$$

$$\mathcal{H}(t,x) := e^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \quad (|t| < \infty),$$

respectively. It is easy to see that $H_n = H_n(0)$ for $n \geq 0$ and $H_n = 0$ if $n \geq 1$ is odd since $\mathcal{H}(t)$ is an even function. Further, since $\frac{d}{dt}\mathcal{H}(t) = -2t\mathcal{H}(t)$, we have

$$H_{n+2} = -2(n+1)H_n \quad (n \geq 0). \tag{3.8}$$

The first few Hermite polynomials are given by

$$\begin{aligned} &H_0(x) = 1;\ H_1(x) = 2x;\ \ H_2(x) = 4x^2 - 2;\ \ H_3(x) = 8x^3 - 12x;\\ &H_4(x) = 16x^4 - 48x^2 + 12;\ \ H_5(x) = 32x^5 - 160x^3 + 120x;\\ &H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120. \end{aligned}$$

Many more examples can be found in the OEIS [20, Sequence A06082].

In general, from the obvious relation $\mathcal{H}(t,x) = e^{2xt}\mathcal{H}(t)$ we see that $H_n(x)$ can be expressed in terms of Hermite numbers as follows:

$$H_n(x) = \sum_{i=0}^{n} \binom{n}{i} (2x)^{n-i} H_i = (H_0 + 2x)^n \quad (n \geq 0). \tag{3.9}$$

We now consider the Taylor series expansions of the functions e^{-t^2} and e^{2xt}. Writing $\mathcal{H}(t,x)$ as the product of these series and then comparing the coefficients of $t^n/n!$ on both sides, we get a closed-form expression for $H_n(x)$ such that

$$H_n(x) = n! \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(-1)^i}{i!(n-2i)!} (2x)^{n-2i}.$$

Further, as is well-known, $H_n(x)$ can be represented by the contour integral

$$H_n(x) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{e^{-z^2+2xz}}{z^{n+1}} dz,$$

where the contour traverses around the origin in a counter-clockwise direction. For more details the reader is refer to, e.g., [19] and [8, §13.1].

It should also be added that the family of Hermite polynomials forms a complete orthogonal system on the interval $-\infty < x < +\infty$ with respect to the weighting function $\varphi(x) := e^{-x^2}$ in the sense of that

$$\int_{-\infty}^{+\infty} \varphi(x) H_m(x) H_n(x) dx = \begin{cases} 0 & \text{if } \ m \neq n; \\ 2^n n! \sqrt{\pi} & \text{if } \ m = n. \end{cases}$$

Let $\widetilde{H}_n(x)$ be the monic Hermite polynomial corresponding to $H_n(x)$, that is, $\widetilde{H}_n(x) := H_n(x)/2^n$, where 2^n is the leading coefficient of $H_n(x)$. From the facts that $\widetilde{H}_0(x) = 1$ and $\frac{d}{dx}\widetilde{H}_n(x) = n\widetilde{H}_{n-1}(x)$ for all $n \geq 1$ we see that $\{\widetilde{H}_n(x)\}_{n\geq 0}$ forms an Appell sequence. Therefore, in view of (1.10) they must satisfy

$$(\widetilde{H}_k(x) + y)^m = (\widetilde{H}_m(x + y) - y)^k \tag{3.10}$$

for integers $k, m \geq 0$, or equivalently, after multiplying both sides by 2^{k+m},

$$(H_k(x) + 2y)^m = (H_m(x + y) - 2y)^k. \tag{3.11}$$

In particular, if $k = 0$, then (3.10) reduces to the Appell identity

$$(\widetilde{H}_0(x) + y)^m = \sum_{i=0}^{m} \binom{m}{i} \widetilde{H}_{m-i}(x) y^i = \widetilde{H}_m(x + y).$$

First, we wish to recover a well-known recurrence formula with three terms by observing a special case of (3.11). From this formula we are able to generate all the Hermite polynomials with $H_0(x) = 1$ and $H_1(x) = 2x$ as initial conditions.

Corollary 3.3. *For an integer $n \geq 1$ we have*

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{3.12}$$

Proof. Setting $y = -x$, $m = 1$, and $k = n \geq 1$ in (3.11), we have

$$(H_n(x) - 2x)^1 = H_{n+1}(x) - 2xH_n(x) = (H_1 + 2x)^n.$$

Therefore, for (3.12) it suffices to prove that $(H_1 + 2x)^n = -2nH_{n-1}(x)$. Noting that $H_1 = 0$ and $H_{i+1} = -2iH_{i-1}$ $(i \geq 1)$ in view of (3.8), we obtain from (3.9),

$$\begin{aligned}(H_1 + 2x)^n &= \sum_{i=0}^{n} \binom{n}{i} (2x)^{n-i} H_{i+1} = -2 \sum_{i=1}^{n} \binom{n}{i} i (2x)^{n-i} H_{i-1} \\ &= -2n \sum_{i=1}^{n} \binom{n-1}{i-1} (2x)^{n-i} H_{i-1} = -2n \sum_{j=0}^{n-1} \binom{n-1}{j} (2x)^{n-1-j} H_j \\ &= -2n(H_0 + 2x)^{n-1} = -2nH_{n-1}(x),\end{aligned}$$

from which (3.12) follows. □

It is also possible to prove (3.12) using a more general identity given from (3.11) with $m = 1$ and $k = n$, namely

$$(H_n(x) + 2y)^1 = H_{n+1}(x) + 2yH_n(x) = (H_1(x + y) - 2y)^n. \tag{3.13}$$

Indeed, by differentiating the whole of the functional identity

$$\left(\frac{d}{dt}\mathcal{H}(t, x + y)\right) e^{-2yt} = -2(t - x - y)\mathcal{H}(t, x)$$

n times with respect to t and then setting $t = 0$ we have

$$(H_1(x+y) - 2y)^n = 2(x+y)H_n(x) - 2nH_{n-1}(x).$$

Therefore, we see that (3.12) is nothing but an immediate consequence of (3.13).

In addition, it is worth mentioning that (3.12) can be derived from (3.8). In fact, writing the left-hand side of (3.12) in the polynomial form $f(x) = \sum_{r=0}^{n+1} a_r x^r$, we observe that $a_{n+1} = 2^{n+1}H_0\left(\binom{n+1}{0} - \binom{n}{0}\right) = 0$ and $a_n = 2^n H_1\left(\binom{n+1}{1} - \binom{n}{1}\right) = 0$ since $H_1 = 0$. Next assuming that $0 \le r \le n-1$ and using the easy identities $\binom{n+1}{r} - \binom{n}{r-1} = \binom{n}{r}$ and $n\binom{n-1}{r} = (n-r)\binom{n}{r}$, we obtain

$$\begin{aligned}\frac{a_r}{2^r} &= \binom{n+1}{r}H_{n+1-r} - \binom{n}{r-1}H_{n+1-r} + 2n\binom{n-1}{r}H_{n-1-r}\\ &= \binom{n}{r}H_{n+1-r} + 2n\binom{n-1}{r}H_{n-1-r}\\ &= \binom{n}{r}\left(H_{n+1-r} + 2(n-r)H_{n-1-r}\right),\end{aligned}$$

which shows that $a_r = 0$ if and only if (3.8) holds. So we may state that (3.8) leads to $f(x) \equiv 0$, and hence to (3.12), as desired.

We next prove a somewhat lengthy recurrence formula by applying (3.11), which seems to be quite new and original to the best of our knowledge.

Corollary 3.4. *Let $c(n,i)$ be as defined in Corollary 3.2. Then we have*

$$\sum_{i=0}^{n+1} c(n,i)(-4x)^i H_{2n+1-i}(x) = 0 \quad (n \ge 0). \tag{3.14}$$

Proof. We now set $y = -2x$ in (3.11), i.e.,

$$(H_k(x) - 4x)^m = (H_m(-x) + 4x)^k. \tag{3.15}$$

Using the symmetry property $H_n(x) = (-1)^n H_n(-x)$, the right-hand side of (3.15) can be written as

$$\begin{aligned}(H_m(-x) + 4x)^k &= \sum_{i=0}^{k}\binom{k}{i}(4x)^i H_{m+k-i}(-x)\\ &= (-1)^{m+k}\sum_{i=0}^{k}\binom{k}{i}(-4x)^i H_{m+k-i}(x)\\ &= (-1)^{m+k}(H_m(x) - 4x)^k.\end{aligned}$$

Hence, we see that (3.15) is equivalent to

$$(H_k(x) - 4x)^m - (-1)^{m+k}(H_m(x) - 4x)^k = 0. \tag{3.16}$$

This formula is, however, useless in the case when $k = m$, because the left-hand side identically vanishes. Avoiding such a situation, we take $k = n + 1$ and $m = n$ in (3.16). Then it can be shown that

$$\begin{aligned}&(H_{n+1}(x) - 4x)^n + (H_n(x) - 4x)^{n+1}\\ =&\sum_{i=0}^{n}\binom{n}{i}(-4x)^i H_{2n+1-i}(x) + \sum_{i=0}^{n+1}\binom{n+1}{i}(-4x)^i H_{2n+1-i}(x)\\ =&\sum_{i=0}^{n+1} c(n,i)(-4x)^i H_{2n+1-i}(x) = 0,\end{aligned}$$

as desired in (3.14). □

Since $H_n = 0$ if $n \geq 1$ is odd, we notice that each term on the left-hand side of (3.14) is an odd polynomial function of x of degree $2n + 1$. Based on this fact, we can deduce an unusual but interesting binomial identity as a by-product of (3.14). Actually, since the sum S_r of coefficients of x^r (r odd, $1 \leq r \leq 2n + 1$) gathered from all the terms in (3.14) must equal zero, after dividing S_r by $2^r H_{2n+1-r}$ (which is a non-zero common factor of each term in S_r), we obtain the identity

$$\sum_{i=0}^{n+1}(-2)^i c(n,i)\binom{2n+1-i}{s} = 0 \quad (n \geq 0), \tag{3.17}$$

valid for all even $s \in \{0, 2, \ldots, 2n\}$. Here we assumed that $\binom{x}{y} = 0$ if $y > x$ by convention. So the number of terms involved in the sum on the left-hand side of (3.17) is equal to $n + 2$ if $0 \leq s \leq n$ or $2n + 2 - s$ if $n + 1 \leq s \leq 2n$.

Through the above observations on Bernoulli, Euler, and Hermite polynomials, we have confirmed that (1.10) was very useful in deducing various kinds of shortened recurrence formulas. Other than these typical polynomials, there are many other examples known to be Appell (see, e.g., [9] and [10, Vol. 3]) and we are, of course, able to use (1.10) for all of them to explore new possible formulas.

4. Additional Results

In our recent study of bivariate Miki-type identities for Bernoulli polynomials, which involve two kinds of sums (precisely, an ordinary sum and a binomial sum), we found, as a by-product, a new concise and practical identity that connects two sums for these polynomials with different variables (see [6, Corollary 4.2]). Later, we confirmed that exactly the same type of identity as this can be applied also to any Appell polynomial, as shown in the following theorem.

Theorem 4.1. *Let $\{A_n(x)\}_{n\ge 0}$ be an Appell sequence. Then for an integer $n \ge 1$ it follows that*

$$\sum_{i=1}^{n} \binom{n+1}{i+1} (y-x)^{i-1} A_{n-i}(x) = \sum_{i=1}^{n} i \binom{n+1}{i+1} (x-y)^{i-1} A_{n-i}(y). \tag{4.1}$$

Proof. Since (4.1) holds trivially if $x = y$, assume that $x \ne y$. Multiplying both sides of (4.1) by $(x-y)^2$, we shall prove the equivalent identity

$$\sum_{i=1}^{n} \binom{n+1}{i+1} (y-x)^{i+1} A_{n-i}(x) = \sum_{i=1}^{n} i \binom{n+1}{i+1} (x-y)^{i+1} A_{n-i}(y). \tag{4.2}$$

Replacing y with $y-x$ in (1.10) and setting $k = 0$, we have

$$(A_0(x) + (y-x))^m = (A_m(y) + (x-y))^0 = A_m(y) \quad (m \ge 0), \tag{4.3}$$

which is equivalent to the Appell identity (1.4), and thus to (1.1). Let us denote by $U(n)$ and $V(n)$ the left and the right-hand sides of (4.2), respectively. Changing the summation range and using (4.3), we now rewrite $U(n)$ as

$$\begin{aligned} U(n) &= \sum_{i=0}^{n+1} \binom{n+1}{i} (y-x)^i A_{n+1-i}(x) - A_{n+1}(x) - (n+1)(y-x)A_n(x) \\ &= (A_0(x) + (y-x))^{n+1} - A_{n+1}(x) - (n+1)(y-x)A_n(x) \\ &= A_{n+1}(y) - A_{n+1}(x) - (n+1)(y-x)A_n(x). \end{aligned}$$

Similar to the above, using again (4.3) with x and y interchanged, we have

$$\begin{aligned} V(n) &= \sum_{i=1}^{n} (i+1) \binom{n+1}{i+1} (x-y)^{i+1} A_{n-i}(y) - \sum_{i=1}^{n} \binom{n+1}{i+1} (x-y)^{i+1} A_{n-i}(y) \\ &= (n+1) \sum_{i=0}^{n} \binom{n}{i} (x-y)^{i+1} A_{n-i}(y) - (n+1)(x-y)A_n(y) \\ &\quad - \sum_{i=0}^{n+1} \binom{n+1}{i} (x-y)^i A_{n+1-i}(y) + A_{n+1}(y) + (n+1)(x-y)A_n(y) \\ &= (n+1)(x-y)\left(A_0(y) + (x-y)\right)^n - (A_0(y) + (x-y))^{n+1} + A_{n+1}(y) \\ &= (n+1)(x-y)A_n(x) - A_{n+1}(x) + A_{n+1}(y), \end{aligned}$$

which coincides with the above $U(n)$; thus it was shown that $U(n) = V(n)$. □

In particular, applying (4.2) to the monomials $\{x^n\}_{n\ge 0}$, we get

$$\begin{aligned} \sum_{i=1}^{n} \binom{n+1}{i+1} (y-x)^{i+1} x^{n-i} &= \sum_{i=1}^{n} i \binom{n+1}{i+1} (x-y)^{i+1} y^{n-i} \\ &= (n+1)(x-y)x^n - x^{n+1} + y^{n+1} \quad (n \ge 1). \end{aligned} \tag{4.4}$$

By taking various non-zero values of x and y, this identity provides us with many interesting (known or original) formulas. For example, if we take

$$(x,y)=\begin{cases}(-2,-1);\\(\rho,1)\ \text{ with } \rho \ \text{ a primitive } n\text{th root of unity;}\\(a,b)\ \text{ with } \ a,b\in\mathbb{Z}\backslash\{0\} \text{ satisfying } p\nmid a,b\ \ (p \ \text{ an odd prime}),\end{cases}$$

then it can be deduced that

$$\text{(i)}\quad (-1)^{n+1}\sum_{i=1}^{n}\binom{n+1}{i+1}(-2)^{n-i}=\sum_{i=1}^{n}i\binom{n+1}{i+1}=(n-1)2^n+1;$$

$$\text{(ii)}\quad \sum_{i=1}^{n}\binom{n+1}{i+1}(1-\rho)^{i+1}\rho^{n-i}=\sum_{i=1}^{n}i\binom{n+1}{i+1}(\rho-1)^{i+1}=n(\rho-1);$$

$$\text{(iii)}\quad \sum_{i=1}^{p-2}\left(1-\frac{b}{a}\right)^{i+1}\equiv\sum_{i=1}^{p-2}i\left(1-\frac{a}{b}\right)^{i+1}\equiv\frac{b}{a}-1\pmod{p}.$$

Identities (i) and (ii) are easily shown by direct calculation. The last (iii) can be proved by putting $n=p-2$ in (4.4) and using the congruence $\binom{p-1}{r}\equiv(-1)^r \pmod{p}$ (valid for $r\in\{0,1,\dots,p-1\}$) and Fermat's little theorem.

In the monomial case, the Appell property (1.10) yields

$$\sum_{i=0}^{m}\binom{m}{i}x^{k+i}y^{m-i}=\sum_{j=0}^{k}\binom{k}{j}(x+y)^{m+j}(-y)^{k-j}\quad(k,m\geq 0). \tag{4.5}$$

This can be also obtained directly from $x^k(x+y)^m=((x+y)-y)^k(x+y)^m$ by expanding both sides in a different way based on the binomial theorem. As well as (4.4), the above (4.5) offers us many interesting combinatorial identities by taking various non-zero values of x and y, which we leave for the reader to explore.

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