FROM A HUNGARY-ISRAEL CONTEST PROBLEM

Fang Chen
Oxford College of Emory University, Oxford, Georgia
fchen2@emory.edu

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Abstract

Let $k \geq 2$ be a given integer. For any arbitrary positive integer n, we determine the least integer $\ell_k(n)$ such that every positive integer sequence with terms not exceeding n, sum kn and length at least $\ell_k(n)$ can be separated into k subsequences each with sum n. The methods are elementary.

1. Introduction

The origin of the question discussed in this article is the following problem from the 12th Hungary–Israel Binational Mathematical Competition (2001):

> 32 positive integers, which sum up to 120 and none of which is greater than 60 are given. Prove that they can be divided into two disjoint subsets that have equal sum.

A similar conclusion holds in a general setting without specific numerical values. Indeed the solution in the book [1] is for any n (such as 60) divisible by 6. The proof uses a notion called *universal sequence* which also appeared as *behaving sequence* [2] and other names in papers about sequence sums. In this article we discuss a generalization of the problem. For us the question emerged from investigating zero-sum problems in finite abelian groups. We tried to translate the setting into the language of positive integer sequences, hoping that such a point of view may be useful in answering some zero-sum questions. While our hopes turned out ungrounded, the problem stated below and its solution are interesting in their own right.

All sequences considered in this paper have positive integer terms, so we sometimes call them just sequences for brevity. Sequences may have repeated terms. For convenience we use multiplicative notation where term multiplicities are indicated by exponents, e.g., $u^p v^q$ is the sequence with p terms u and q terms v. The *length* of a (finite) sequence is the number of its terms. Let $k \geq 2$ and n be positive integers. A positive integer sequence with terms not exceeding n and sum kn is called *k-separable* if it can be divided into k parts each with sum n, otherwise it is called

k-inseparable. Note that the restriction of terms not exceeding n is to avoid trivialities, as the presence of a term greater than n would make the sequence trivially k-inseparable. It is clear that for $n = 1, 2$, any positive integer sequence with terms not exceeding n and sum kn is k-separable for any $k \geq 2$. For brevity sometimes we say *separable* or *inseparable* whenever it is not important to be specific or the value of k is clear from the context. Here is the objective.

> Let $k \geq 2$ be a given integer. For an arbitrary integer $n \geq 3$, determine the least integer $\ell_k(n)$ such that each positive integer sequence with terms not exceeding n, sum kn and length at least $\ell_k(n)$ is k-separable.

In other words, restricted to sequences with positive integer terms not exceeding n and sum kn, all with length at least $\ell_k(n)$ must be k-separable, and at least one with length $\ell_k(n) - 1$ must be k-inseparable.

We will present a systematic approach to solve the problem completely. The techniques are elementary.

2. The Trivial Algorithm

For convenience, we will say "collection of weights with integer masses" and "positive integer sequences" interchangeably. Let us describe a procedure for dividing an arbitrary collection of weights into k groups. Because of its simplicity, we call it the *trivial algorithm* (or just the *algorithm*, for brevity). We will see that the trivial algorithm is the basis of our solution, and whenever a length condition guarantees separability, a procedure based on the algorithm separates the sequences.

Fix an integer $k \geq 2$ and let n be an arbitrary positive integer. Let α be a collection of weights with integer masses not exceeding n of total mass kn. Let there be k boxes each with capacity n. Start placing the weights in the boxes, one at a time, in decreasing order of their masses. Each time the current weight w is to be placed in the box with the smallest total mass (select one arbitrarily if there is a tie), provided that after putting w this box has mass not exceeding n. In other words, the capacity n of each box is not to be exceeded at any step.

Suppose the latter condition cannot be satisfied at a certain moment, that is, placing the current weight would overload the lightest box. The first time this occurs, with a weight t, the algorithm stops and we say that it fails. If so, we call t the *critical weight* or the *critical term* of α. Observe that since all terms are at most n, the trivial algorithm cannot fail at the first k steps. Clearly, if all weights can be placed according to the rules, the procedure yields a partition of the system into k groups of mass n. Trivially for $n = 1, 2$ the algorithm always succeeds.

Here is a rough idea to prove that a certain sequence is separable. First we apply the trivial algorithm and immediately conclude separability if the procedure terminates successfully. If not, we focus on the reasons for the failure. It turns out that substantial information can be obtained by looking at them, in which the critical term of the sequence plays a crucial role. Based on the conclusions, we then find a way to apply the trivial algorithm again. Let us make the following observations as the result of analyzing the situation when the trivial algorithm fails.

Observations. *Fix an integer $k \geq 2$ and let n be an arbitrary positive integer. Suppose the trivial algorithm fails for a positive integer sequence α with terms not exceeding n, of length ℓ and sum kn, with a critical term t. Then:*

(i) The sum of the terms after t in the decreasing arrangement of α does not exceed $(k-1)t-k$. In particular, t is never equal to 1.

(ii) The sequence α contains at least $k+1$ terms, each of which is at least t. In particular, $\ell \geq k+1$.

(iii) Let $g_k(x) = (k-1)x^2 - (\ell+2k-1)x + (kn+k)$, then t being a critical term implies $g_k(t) \geq 0$.

Proof. Let A be the sum of the weights after t; then the sum of the weights before t is $kn - t - A$, and it is at least $k(n+1-t)$ since t is a critical term. Hence $kn - t - A \geq k(n+1-t)$, which implies $A \leq (k-1)t - k$. As $A \geq 0$, it follows that $t \geq \frac{k}{k-1}$ and $t \geq 2$ (t is an integer). In particular, $t \neq 1$. Part (i) follows. Part (ii) is clear as no term exceeds n. For Part (iii) we estimate the length of such a sequence,

$$\ell \leq \frac{kn-t-A}{t} + 1 + A = \frac{kn + A(t-1)}{t} \leq \frac{kn + ((k-1)t-k)(t-1)}{t},$$

as $t > 1$ and $A \leq (k-1)t - k$. After rearranging the terms, it follows that $(k-1)t^2 - (\ell+2k-1)t + (kn+k) \geq 0$. □

3. A First Bound and the n Odd Case

Considering the quadratic function $g_k(x)$ from Observation (iii), a length condition comes from $g_k(2) < 0$: $g_k(2) = -2\ell + kn + k - 2 < 0$ implies $\ell > \frac{k(n+1)}{2} - 1$. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. The following proposition confirms that $\ell \geq \lfloor \frac{k(n+1)}{2} \rfloor$ guarantees separability.

Proposition 1. *Fix an integer $k \geq 2$ and let $n \geq 3$ be an arbitrary integer. The trivial algorithm terminates successfully for each positive integer sequence with terms not exceeding n, sum kn and length $\ell \geq \lfloor \frac{k(n+1)}{2} \rfloor$. In other words, such a sequence can be separated into k parts each with sum n by the trivial algorithm.*

Proof. Suppose on the contrary that the algorithm fails for one such sequence α, denote by t the critical term and consider the quadratic function $g_k(x)$ from Observation (iii). The length condition $\ell \geq \lfloor \frac{k(n+1)}{2} \rfloor > \frac{k(n+1)}{2} - 1$ implies $g_k(2) < 0$.

Now g_k has a positive leading coefficient (equal to $k-1$), so its graph is a parabola open upwards. The condition $g_k(2) < 0$ implies that g_k has two distinct real roots r_1 and r_2, and $r_1 < 2 < r_2$. The graph is symmetric with respect to the vertical line $x = \frac{\ell+2k-1}{2(k-1)}$. Note that $\frac{\ell+2k-1}{2(k-1)} \geq 2$, i.e., $\ell \geq 2k-3$, which holds under the length condition and $n \geq 3$. Then $g_k\left(\frac{\ell+2k-1}{k-1} - x\right) = g_k(x)$ for all real x. In particular, $g_k\left(\frac{\ell+2k-1}{k-1} - 2\right) = g_k(2)$, i.e., $g_k\left(\frac{\ell+1}{k-1}\right) = g_k(2) < 0$. Since $\frac{\ell+1}{k-1} \geq 2$, we have $r_1 < 2 \leq \frac{\ell+1}{k-1} < r_2$. On the other hand, by Observation (iii), the critical term t satisfies $g_k(t) \geq 0$. Therefore $t \leq r_1$ or $t \geq r_2$. It follows that $t < 2$ or $t > \frac{\ell+1}{k-1}$.

Since t is a positive integer, $t < 2$ is impossible as $t \neq 1$ by Observation (i). Suppose $t > \frac{\ell+1}{k-1}$. By Observation (ii), in α there are at least $k+1$ terms, each of which is at least t. Since each of the other terms is at least 1, the sum of α would satisfy $kn \geq (k+1)t + (\ell - k - 1)$, which would lead to $k(n-1) + 2 < 0$. A contradiction is reached, and hence the trivial algorithm terminates successfully. $\square$

To construct a sequence with sum kn, length close to $B_1 = \lfloor \frac{k(n+1)}{2} \rfloor$ that is k-inseparable, we need the average of terms to be about 2 (a little less). If n is odd, a trivial reason to be inseparable is due to parity: even terms (such as 2) cannot make an odd sum. Here is an example for any $k \geq 2$; its length is $\frac{k(n+1)}{2} - 1$ and k-inseparable when n is odd:

$$\text{for odd } n \geq 3, \quad \alpha = 2^{\frac{k(n-1)+2}{2}} 1^{k-2}. \tag{1}$$

Therefore,

$$\text{for any } k \geq 2, \quad \ell_k(n) = \left\lfloor \frac{k(n+1)}{2} \right\rfloor \quad \text{if } n \text{ is odd and } n \geq 3.$$

4. Examples and Bounds

For even n, examples similar to (1) with length close to $B_1 = \lfloor \frac{k(n+1)}{2} \rfloor$ cannot be found and evidence suggests that $\ell_k(n)$ is significantly smaller. Before proceeding further, let us construct some examples and investigate possible bounds for n even.

Here is another reason for being inseparable: there are too many large terms. We may have at most k terms, each of which is greater than $n/2$, since each box has capacity n and can only contain one such term. If we restrict attention to the case when n is even, then for any $k \geq 2$ the following is k-inseparable with length $\frac{n}{2}(k-1) + 1$:

$$\text{for even } n \geq 4, \quad \alpha = (n/2+1)^k (n/2) 1^{\frac{n}{2}(k-1)-k}. \tag{2}$$

We will prove that for $k \geq 4$ and even $n \geq 4$, length at least $B_2 = \frac{n}{2}(k-1)+2$ guarantees separability, and hence $\ell_k(n) = \frac{n}{2}(k-1)+2$ (except when $n=8$). One may ask whether limiting the number of terms greater than $n/2$, that is removing the trivial reason for being inseparable, can shorten the length to guarantee separability. The answer is negative due to the following examples for any $k \geq 2$:

$$\text{for } n \equiv 0 \pmod 4, \qquad \alpha = (n/2+1)^2 2^{\frac{n}{2}(k-1)-1}; \tag{3}$$

$$\text{for } n \equiv 2 \pmod 4, \quad \alpha = (n/2+2)(n/2)2^{\frac{n}{2}(k-1)-1}. \tag{4}$$

Example (3) is k-inseparable because each of the two terms $n/2+1$ must be in a different box, and the remaining capacity in these two boxes is $n/2-1$, which is odd hence cannot be filled up by 2's. For a similar reason Example (4) is k-inseparable.

If $k=3$, we will see that for n even, length at least $B_2 = n+2$ is almost enough to guarantee separability except in one special case.

Finally when $k=2$, $B_2 = n/2+2$. We will see that B_2 is enough to guarantee separability only in the case when n is even and divisible by 3, i.e., divisible by 6 (such as 60 in the Hungary-Israel contest problem). Informally this is because there is another reason for inseparability. Example (1) is based on the parity of n. We may look at n modulo other numbers, say 3. Suppose the average of the terms is about 3 and the majority of them are 3. Let us suppose $n \not\equiv 0 \pmod 3$. To fill up each box with capacity n we need terms not divisible by 3, so a reason for being inseparable could be that there are not enough such terms to be distributed to the k boxes. A rough bound is $B_3 = \frac{k(n+2)}{3}$. We will show that when $k=2$, for n even and not divisible by 3, $\ell_2(n)$ is approximately B_3.

Precise results will be presented later and we will see some differences due to the values of k. Instead of being approach-dependent, there seem to be inherent reasons for such differences. One way to understand it is to compare B_2 and B_3: $B_2 - B_3 = \frac{n}{2}(k-1)+2-\frac{k(n+2)}{3} = \frac{(n-4)(k-3)}{6}$. We may assume $n \geq 4$ and notice the difference depending on k: if $k<3$, i.e., $k=2$, then $B_3 > B_2$; if $k \geq 4$ then $B_3 < B_2$; and when $k=3$, $B_2 = B_3$.

5. The n Even Case

Here is a general approach to handle the n even case. Apply the trivial algorithm to α and suppose it fails with a critical term t. Use the length condition and other considerations to restrict possibilities of t. Since the sum kn of α is even, it contains an even number of odd terms. Form a new sequence β by leaving the even terms of α untouched and replacing every pair of odd terms by its sum. Then try the trivial algorithm for the new sequence. Details need to be worked out, especially in the case $k=2$ where n modulo 3 also makes a difference. In the following we treat the cases $k = 2, 3$ and $k \geq 4$ separately. While there are proofs of the cases $k = 3$ and $k \geq 4$ using results on previous values of k ($k = 2, 3$), it seems clearer to handle different k values independently.

5.1. The Case $k = 2$ and n Even

We start with a lemma about the possible values of a critical term when the trivial algorithm fails.

Lemma 1. *Let n be a positive integer. If the trivial algorithm fails for a positive integer sequence of terms not exceeding n, with sum $2n$ and length $\ell \geq 2n/3 + 1$, with a critical term t, then $t = 2$.*

Proof. Let α be a positive integer sequence of terms not exceeding n, with sum $2n$ and length satisfying $\ell \geq 2n/3 + 1$, apply the trivial algorithm and suppose it fails with a critical term t. By Observation (iii) with $k = 2$, $g_2(x) = x^2-(\ell+3)x+(2n+2)$. Using the length condition $\ell \geq 2n/3 + 1$, we have $g_2(3) = -3\ell + 2n + 2 < 0$. It implies that g_2 has distinct real roots r_1 and r_2, and that $r_1 < 3 < r_2$. We know $g_2(x) = g_2(\ell + 3 - x)$ for all x, by symmetry with respect to the line $x = (\ell + 3)/2$. Hence $g_2(\ell) = g_2(3) < 0$, and since $3 \leq (\ell + 3)/2$ (this is equivalent to $\ell \geq 3$ which holds by Observation (ii)), we have $r_1 < 3 \leq \ell < r_2$. On the other hand $g_2(t) \geq 0$ by Observation (iii). Therefore $t \leq 2$ or $t \geq \ell + 1$ (as t is an integer). Notice that $t \geq \ell+1$ is impossible. Indeed by Observation (ii) α has at least three terms, each of which is at least t, we would obtain that its sum satisfies $2n \geq 3t \geq 3\ell+3 \geq 2n+6$. Hence $t \leq 2$. As $t \neq 1$ by Observation (i), it follows that $t = 2$. □

Proposition 2. *Let n be an even positive integer. Each positive integer sequence of terms not exceeding n, with sum $2n$ and length $\ell \geq 2n/3 + 1$ is 2-separable.*

Proof. Denote the sequence by α and apply the trivial algorithm. The task is complete if it is successful. If not, Lemma 1 implies that the critical term t of α is 2. Notice that having a critical term $t = 2$ implies that there is no 1 in the sequence. This is because by Observation (i) with $k = 2$, the sum of the terms after t is at most $t - 2$. If the critical term is $t = 2$, then it is the last term in the sequence of decreasing order. In particular, there is no term equal to 1 in the sequence.

The sum $2n$ of α is even, so it contains an even number of odd terms. Let us form a new sequence β by leaving the even terms of α untouched, grouping all the odd terms into pairs (in an arbitrary way), and then replacing each pair by the sum of the two terms in it. The new sequence β has even terms and the same sum $2n$. Observe that β has at least one term 2, for instance the critical term $t = 2$ of α which is also a term of β. We will show that the trivial algorithm separates β.

Divide the terms of β by 2 to obtain a new sequence β', with the same length as β and with sum n. Notice that β' contains a 1 as β contains a 2. Let $n' = n/2$. This is an integer as n is even. Then β' has sum $2n'$. Apparently the trivial algorithm separates β if and only if it separates β'.

In view of the even term 2 present in α, the length ℓ' of β' satisfies the inequality $\ell' \geq 1 + (\ell - 1)/2 = (\ell + 1)/2$. Now the assumption $\ell \geq 2n/3 + 1$ implies that the analogous inequality holds for n' and ℓ':

$$\ell' \geq \ell/2 + 1/2 \geq 1/2(2n/3 + 1) + 1/2 = 2n'/3 + 1.$$

Finally, the terms of β' are all at most $n/2 = n'$. If not, then some two odd terms in α have a sum greater than n, hence at least $n + 2$ (because n is even). So the remaining $\ell - 2$ terms have a sum at most $n - 2$. But each one of them is at least 2 as α does not contain a 1, implying that the same sum is at least $2(\ell - 2) \geq 2(2n/3 - 1) = 4n/3 - 2$. Thus $n - 2 \geq 4n/3 - 2$ which is false.

We are almost done. Suppose the trivial algorithm fails for β'. Then Lemma 1 applies, saying that the critical term of β' is $t' = 2$. However this is impossible, because β' contains a term 1. The contradiction obtained proves that the trivial algorithm separates β' and hence β. It follows that α is 2-separable. $\square$

Let $\lceil x \rceil$ denote the least integer greater than or equal to x. By Proposition 2, length $\ell \geq \lceil 2n/3 \rceil + 1$ ensures 2-separability for n even. The following examples show that for $n \geq 3$ and $n \not\equiv 0 \pmod 3$, there exists at least one 2-inseparable sequence with sum $2n$ and length $\lceil 2n/3 \rceil$. For $n \equiv 1 \pmod 3$, take the sequence consisting of $(2n - 2)/3$ terms 3 and one term 2. For $n \equiv 2 \pmod 3$, take the sequence consisting of $(2n - 1)/3$ terms 3 and one term 1. So far we have obtained $\ell_2(n) = \lceil 2n/3 \rceil + 1$ for all even $n \geq 4$ that are not divisible by 3.

Let us handle the remaining case: n is even and 0 modulo 3, i.e., $n \equiv 0 \pmod 6$. We believe that in this case $\ell \geq n/2+2$ guarantees separability and that $\ell = n/2+2$ is the least possible such length. The latter will follow if there exists at least one 2-inseparable sequence with length $n/2 + 1$. Example (2) with $k = 2$ is such a sequence: consisting of two terms of $n/2 + 1$, one $n/2$ and $n/2 - 2$ ones. It remains to prove the following statements, analogous to the approach before.

Lemma 2. *Let n be a positive integer. If the trivial algorithm fails for a positive integer sequence of terms not exceeding n, with sum $2n$ and length $\ell \geq n/2+2$ (that is $4\ell \geq 2n + 8$), with a critical term t, then $t = 2$ or $t = 3$.*

Proof. We know that $n \geq 3$ as the trivial algorithm always works for $n = 1, 2$. For $3 \leq n \leq 5$, Observation (ii) implies that the sum of the sequence satisfies $2n \geq 3t + \ell - 3$, which leads to $t \leq n/2 + 1/3 < 3$, hence $t = 2$.

For $n \geq 6$, $\ell \geq n/2 + 2 \geq 5$. Note that $g_2(4) = -4\ell + 2n + 6 < 0$. Analyzing the parabola g_2 and noticing that $g_2(4) = g_2(\ell - 1) < 0$ ($\ell \geq 5$ is used here), yield that

$g_2(t) \geq 0$ implies $t \leq 3$ or $t \geq \ell$. Like before, the second alternative is impossible. Otherwise since there are at least 3 terms, each of which is at least t, we would have $2n \geq 3t + \ell - 3 \geq 3\ell + \ell - 3 = 4\ell - 3 \geq 2n + 5$. Hence the first alternative $t \leq 3$ remains, i.e., $t = 2$ or $t = 3$. □

Proposition 3. *Let n be a positive integer divisible by 6. Each positive integer sequence of terms not exceeding n, with sum $2n$ and length $\ell \geq n/2 + 2$ (that is, $4\ell \geq 2n + 8$) is 2-separable.*

Proof. Notice that for $n = 6$, $n/2 + 2 = 2n/3 + 1 = 5$, and it is included in Proposition 2. In the following we may assume $n \geq 12$.

Denote the sequence by α and apply the trivial algorithm. The task is complete if it succeeds; so let the algorithm fail. By Lemma 2 the critical term t of α is $t = 2$ or $t = 3$. Now we divide the argument into two cases: a) α has at least two 2's; b) α has at most one 2 and at least two 3's. One of them must hold. Indeed, if there is at most one 2 (i. e., a) fails), then look at the critical term t. For $t = 2$, there are no 1's in α, and there is at most one 2 by assumption. If there were at most one 3, the sum of α would satisfy $2n \geq 2 + 3 + 4(\ell - 2) = 4\ell - 3 \geq 2n + 5$. Similarly, for $t = 3$ there are no 2's in α, and at most one 1 (by Observation (i)). If there is at most one 3, then $2n \geq 1 + 3 + 4(\ell - 2) = 4\ell - 4 \geq 2n + 4$.

a) α has at least two 2's. Let us pair up the odd terms in α and proceed as in the proof of Proposition 2 to obtain β'. Let $n' = n/2$; this is an integer as n is even. Then β' has sum $2n'$. Because of the two 2's in α, β' contains at least two 1's, and the length ℓ' of β' satisfies $\ell' \geq 2 + (\ell - 2)/2 = (\ell + 2)/2$. Hence $4\ell' \geq 2\ell + 4 \geq (n + 4) + 4 = n + 8 = 2n' + 8$.

Note that each term of β' is at most $n' = n/2$, since every two terms of α add up to a sum not exceeding n. Otherwise if there are two terms of α add up to a sum greater than n, then the remaining $\ell - 2$ terms of α will have a sum less than n. On the other hand, we have $t = 2$ in case a) ($t = 3$ would imply no 2's), so that 1's are not to be found in α. It turns out that $2(\ell - 2) < n$ which contradicts $4\ell \geq 2n + 8$.

It remains to show that the trivial algorithm works for β'. Suppose not, and let t' be the critical term. Lemma 2 implies $t' = 2$ or $t' = 3$; however both are impossible as β' has two 1's.

b) α has at most one 2 and at least two 3's. The sum $2n$ of α is divisible by 3 because so is n. So the terms not divisible by 3 can be partitioned into groups of size 2 or 3, with the sum of each group a multiple of 3. Replace the terms in each group by their sum, then divide by 3 the terms of the sequence obtained. Set $n' = n/3$ which is an integer. The result is a positive integer sequence α' with sum $2n' = 2n/3$. Due to the two 3's in α, there are at least two 1's in α', and the length ℓ' of α' satisfies $\ell' \geq 2 + (\ell - 2)/3 = (\ell + 4)/3$. It follows that $4\ell' \geq (4\ell + 16)/3 \geq (2n + 24)/3 = 2n' + 8$.

Observe that no terms of α' exceed $n' = n/3$. Otherwise, suppose some term of α' is greater than $n/3$, then the corresponding two or three terms of α add up to a sum greater than n. The remaining at least $\ell-3$ terms will have a sum less than n. On the other hand, there is at most one 2 in α (by the assumption in b)), and at most one 1 (no 1's at all if $t=2$, and at most one 1 if $t=3$). Hence the remaining at least $\ell-3$ terms have a sum at least $1+2+3(\ell-3-2)=3\ell-12\geq 3n/2-6\geq n$ as $n\geq 12$, and a contradiction is reached.

It remains to show that the trivial algorithm works for α'. Suppose not; by Lemma 2 the critical term t' of α' would be 2 or 3. Both are impossible as α' has two 1's. This completes b) and the main proof. □

Summarizing the results for $k=2$ and n even: $\ell_2(n)=\lceil 2n/3\rceil+1$ for all even $n\geq 4$ that are not divisible by 3, and $\ell_2(n)=n/2+2$ for all n divisible by 6.

5.2. The Case $k = 3$ and n Even

Essentially the same approach works for $k=3$ with some modifications.

Lemma 3. *Let n be a positive integer. Suppose the trivial algorithm fails for a positive integer sequence α of terms not exceeding n, with sum $3n$ and length $\ell\geq n+2$, with a critical term t. Then $t=2$ except in two cases: if $n\equiv 2 \pmod 3$ and $\alpha=\alpha_1=3^{n-1}1^3$, then $t=3$; and if n is odd and $\alpha=\alpha_2=\left(\frac{n+1}{2}\right)^4 1^{n-2}$, then $t=\frac{n+1}{2}$.*

Proof. We know that $n\geq 3$ as the trivial algorithm always works for $n=1,2$. If the sequence is $\alpha_2=\left(\frac{n+1}{2}\right)^4 1^{n-2}$ (n must be odd), then clearly $t=\frac{n+1}{2}$. It is also easy to see that if the sequence is $\alpha_1=3^{n-1}1^3$ and $n\equiv 2 \pmod 3$, then $t=3$. Notice that if $n\equiv 0,1 \pmod 3$, the trivial algorithm terminates successfully for $\alpha_1=3^{n-1}1^3$. In the following we assume that α is different from α_1 and α_2, and show that $t=2$. For $3\leq n\leq 4$, Observation (ii) implies that the sum of α satisfies $3n\geq 4t+\ell-4$ which gives $t\leq \frac{n+1}{2}<3$, hence $t=2$. So we may assume $n\geq 5$.

By Observation (iii) the quadratic function is $g_3(x)=2x^2-(\ell+5)x+(3n+3)$. As $n\geq 5$ and $\ell\geq n+2\geq 7$, we have $\frac{\ell+5}{4}\geq 3$ and by symmetry, $g_3\left(\frac{\ell+5}{2}-3\right)=g_3(3)$. Compute $g_3(3)=-3\ell+3n+6\leq 0$. Hence $g_3\left(\frac{\ell-1}{2}\right)=g_3(3)\leq 0$. Then $g_3(t)\geq 0$ implies that $t\leq 3$ or $t\geq\frac{\ell-1}{2}$. We will show that $t\geq\frac{\ell-1}{2}$ is possible only if the sequence is α_2, and $t=3$ is possible only if the sequence is α_1. As both are excluded and $t=1$ is impossible by Observation (i), it follows that $t=2$.

Suppose $t\geq\frac{\ell-1}{2}\geq\frac{n+1}{2}$. By Observation (ii), the sum of the sequence satisfies $3n\geq 4t+(\ell-4)\geq 4(\frac{n+1}{2})+(n+2)-4=3n$. This would lead to a contradiction except in the case when equality holds throughout, implying that the sequence is $\alpha_2=\left(\frac{n+1}{2}\right)^4 1^{n-2}$.

Suppose the critical term is $t=3$. Let x be the number of terms after $t=3$; then the estimate of the sum $3n\geq 3(\ell-x)+x=3\ell-2x\geq 3(n+2)-2x$ leads to $x\geq 3$, i.e., there are at least three terms after $t=3$. On the other hand, by

Observation (i) the sum of the terms afterwards is at most $2t-3=3$. It follows that there are exactly three terms after $t=3$, and each term is 1. By the estimate of the sum again $3n \geq 3(\ell-4)+3+1+1+1 = 3\ell-6 \geq 3(n+2)-6 = 3n$, we see that it is only possible if equality holds throughout, implying $\ell = n+2$ and each term before $t=3$ is 3. In other words, the sequence must be $\alpha_1 = 3^{\ell-3}1^3 = 3^{n-1}1^3$. $\square$

Proposition 4. *Let n be an even positive integer. Each positive integer sequence of terms not exceeding n, with sum $3n$ and length $\ell \geq n+2$, is 3-separable with one exception: the sequence $3^{n-1}1^3$ is 3-inseparable when $n \equiv 2 \pmod 3$.*

Proof. Denote the sequence by α. It is clear that if $\alpha = \alpha_1 = 3^{n-1}1^3$, then it is 3-separable when $n \equiv 0, 1 \pmod 3$ and 3-inseparable when $n \equiv 2 \pmod 3$. In the following, assume α is different from α_1. If the trivial algorithm terminates successfully, then α is 3-separable. Suppose not; by Lemma 3 the critical term is $t=2$ since $\alpha \neq \alpha_1$ by assumption and $\alpha \neq \alpha_2$ as n is even.

By Observation (i) the sum of the terms after $t=2$ does not exceed $2t-3=1$. Hence α has at most one 1. Let x_1 be the number of 1's and x_2 be the number of 2's in α. The sum of α satisfies $3n \geq x_1 + 2x_2 + 3(\ell - x_1 - x_2)$, which gives $2x_1 + x_2 \geq 3\ell - 3n \geq 6$. As $x_1 \leq 1$, it follows that $x_2 \geq 4$.

Since n is even, the sum $3n$ is even, so α contains an even number of odd terms. Let us pair up the odd terms in α and proceed as in the proof of Proposition 2 to obtain β'. Let $n' = n/2$; this is an integer as n is even. Then β' has sum $3n'$ and at least four 1's, since there are at least four 2's in α. The length $\ell_{\beta'}$ of β' satisfies $\ell_{\beta'} \geq 4 + \frac{\ell-4}{2} = \frac{\ell+4}{2} \geq \frac{n+6}{2} = n'+3$.

We show that each term in β' is at most $n' = n/2$. Suppose not, and let x, y be two odd terms in α such that $x+y > n$. Since n and $x+y$ are both even, $x+y \geq n+2$. Clearly neither x nor y is 1. The sum of α satisfies $3n \geq x+y+x_1+2(\ell-x_1-2)$, which leads to $x_1 \geq 2$. However $x_1 \leq 1$ and a contradiction is reached.

We claim that the trivial algorithm works for β'. Suppose not; then Lemma 3 implies that the critical term of β' is $t' = 2$ (notice that $\ell_{\beta'} \geq n'+3$ excludes the two special cases in Lemma 3 as both of them have length $n'+2$). However this is impossible by Observation (i) and the fact that there are at least four 1's in β'. The contradiction obtained proves that β' and hence α are 3-separable. $\square$

Example (2) with $k=3$ is a 3-inseparable sequence of length $n+1$ for any even $n \geq 4$: $(n/2+1)^3(n/2)1^{n-3}$. We already saw that $3^{n-1}1^3$ is 3-inseparable for $n \equiv 2 \pmod 3$ with length $n+2$. There are other 3-inseparable sequences of length $n+1$, for example, $(n/2+1)^2 2^{n-1}$ for $n \equiv 0 \pmod 4$, $(n/2+2)(n/2)2^{n-1}$for $n \equiv 2 \pmod 4$ (Examples (3) and (4) with $k=3$), and $3^{n-2}2^3$ for $n \equiv 1 \pmod 3$.

Summarizing the results for $k=3$ and n even: we have $\ell_3(n) = n+2$ if $n \geq 4$ is even and $n \equiv 0, 1 \pmod 3$; and $\ell_3(n) = n+3$ if $n \geq 4$ is even and $n \equiv 2 \pmod 3$.

5.3. The Case $k \geq 4$ and n Even

The idea is like before except for additional reasoning when rejecting large values of the critical term t and ensuring that the sum of any two odd terms in α is at most n.

Lemma 4. *Let $n > 8$ and $k \geq 4$ be integers. Suppose the trivial algorithm fails for a positive integer sequence of terms not exceeding n, with sum kn and length $\ell \geq \lceil \frac{n}{2} \rceil (k-1) + 2$, with a critical term t. Then $t = 2$.*

Proof. Consider $g_k(x) = (k-1)x^2 - (\ell + 2k - 1)x + (kn + k)$. The length condition implies $\ell \geq \frac{n}{2}(k-1) + 2$, and we have $g_k(3) = -3\ell + kn + 4k - 6 \leq \frac{(k-3)(8-n)}{2} < 0$, as $k \geq 4$ and $n > 8$. Note that $\frac{\ell+2k-1}{2(k-1)} \geq 3$, which is equivalent to $\ell \geq 4k - 5$, and the latter is guaranteed by the length condition and $n > 8$. By symmetry, $g_k\left(\frac{\ell+2k-1}{k-1} - 3\right) = g_k(3)$, that is, $g_k\left(\frac{\ell-k+2}{k-1}\right) = g_k(3) < 0$. By Observation (iii), $g_k(t) \geq 0$, hence $t < 3$ or $t > \frac{\ell-k+2}{k-1}$. We will reject $t > \frac{\ell-k+2}{k-1}$ and then the remaining case $t < 3$ implies $t = 2$.

Suppose $t > \frac{\ell-k+2}{k-1}$, since $\ell \geq \lceil \frac{n}{2} \rceil (k-1)+2$, we have $t > \lceil \frac{n}{2} \rceil - 1 + \frac{3}{k-1} > \lceil n/2 \rceil - 1$, and as t and $\lceil n/2 \rceil$ are integers, it follows that $t \geq \lceil n/2 \rceil$.

If n is odd, then $t \geq \frac{n+1}{2}$. The estimate of the sum and the length condition $\ell \geq \lceil \frac{n}{2} \rceil (k-1) + 2 = \frac{n+1}{2}(k-1) + 2$ would give $kn \geq (k+1)t + (\ell - k - 1) \geq nk + 1$.

If n is even and supposing $t \geq n/2 + 1$, then the estimate of the sum and $\ell \geq \lceil \frac{n}{2} \rceil (k-1) + 2 = \frac{n}{2}(k-1) + 2$ would give $kn \geq (k+1)t + (\ell - k - 1) \geq nk + 2$.

Suppose n is even and $t = n/2$. Consider the terms before $t = n/2$, each is either at least $n/2 + 1$ or equal to $n/2$. Let x of them be at least $n/2 + 1$ and y of them equal $n/2$. Since $t = n/2$ is a critical term and all the terms before are in the k boxes without exceeding the capacity n, when the trivial algorithm fails with $t = n/2$, each of the k boxes must either contain one term which is at least $n/2 + 1$ or two terms each equals $n/2$ and hence filling up the capacity n. It follows that $x + \frac{y}{2} = k$; in particular, y is even. The sum estimate $kn \geq x(n/2+1) + (y+1)n/2 + (\ell - x - y - 1) \geq kn + \frac{ny}{4} + 1 - y$ gives a contradiction if $\frac{ny}{4} + 1 - y > 0$, which is equivalent to $(n-4)y + 4 > 0$. The last inequality holds as $y \geq 0$ and $n \geq 4$.

This rejects $t > \frac{\ell-k+2}{k-1}$ and concludes the proof. □

Proposition 5. *Let $n > 8$ be an even integer and $k \geq 4$ an integer. Each positive integer sequence of terms not exceeding n, with sum kn and length $\ell \geq \frac{n}{2}(k-1) + 2$ can be separated into k parts each with sum n.*

Proof. Denote the sequence by α and apply the trivial algorithm. The task is complete if the algorithm terminates with all terms placed in the k boxes. If it fails with a critical term t, then Lemma 4 implies that $t = 2$.

We first show that the sum of any two odd terms in α is at most n. Let A be the sum of the terms after the critical term t. The fact that $t = 2$ implies that when the trivial algorithm fails, each of the k boxes must either be full or have a total mass of $n-1$, and there are exactly $A+2$ of the latter kind. Since $n-1$ is odd, there must be at least $A+2$ odd terms before t. Consider the subsequence τ before t. Its sum is $kn-2-A$ and length $\ell_\tau \geq \ell-(1+A)$ as there are at most A terms after t. Let m_o be the number of odd terms in τ, and recall $m_o \geq A+2$. Suppose there exist odd terms x, y in α such that $x+y>n$. Then $x+y \geq n+2$ and $3 \leq x, y \leq n-1$; in particular, both x and y are in τ. Since each odd term in τ is at least 3, its sum satisfies $kn-2-A \geq x+y+3(m_o-2)+2(\ell_\tau-m_o)$. This leads to $kn \geq kn+2$ which is impossible. Therefore, leaving even terms untouched, we may pair up odd terms of α arbitrarily to form β so that each term is at most n.

We need a lower bound on the length of β. To that end it is enough to estimate x_2, the number of 2's in α. Let x_1 denote the number of 1's in α. The sum of α satisfies $kn \geq x_1+2x_2+3(\ell-x_1-x_2)$, which gives $2x_1+x_2 \geq 3\ell-kn$. Apply the length condition, we have $2x_1+x_2 \geq \frac{(k-3)n}{2}+6$. By Observation (i), the sum of the terms after the critical term $t=2$ is at most $(k-1)t-k=k-2$. Hence $x_1 \leq k-2$ and $x_2 \geq \frac{(n-4)(k-3)}{2}+4$. Therefore the length ℓ_β of β satisfies

$$\ell_\beta \geq \frac{\ell-x_2}{2}+x_2=\frac{\ell+x_2}{2} \geq \frac{(n-2)(k-2)}{2}+4.$$

Divide each term of β by 2 to get β' and let $n'=n/2$ (n' is an integer as n is even). The sum of β' is kn' and length $\ell_{\beta'}=\ell_\beta \geq \frac{(n-2)(k-2)}{2}+4=(n'-1)(k-2)+4$. Each term in β' does not exceed n' as each term in β is at most n. Note that $n'=n/2>4$. If $\ell_{\beta'} \geq \lfloor \frac{k(n'+1)}{2} \rfloor$, then we may apply Proposition 1 and conclude that the trivial algorithm succeeds for β', and hence β and α are k-separable. It is sufficient if $(n'-1)(k-2)+4 \geq \frac{k(n'+1)}{2}$, which is equivalent to $(n'-3)(k-4) \geq 0$. The last inequality holds as $k \geq 4$ and $n' \geq 3$. The proof is complete. □

Remark 1. The last part of the proof of Proposition 5 is slightly different from the previous proofs for n even (such as Propositions 2 and 4), in that we did not apply Lemma 4 to β' but were able to use Proposition 1. This is because β' is long enough due to the estimate of the number of 2's in α: $x_2 \geq \frac{(n-4)(k-3)}{2}+4$. The lower bound is a multiple of n if $k \geq 4$, a constant if $k=3$, but useless if $k=2$.

It follows from Proposition 5 and Example (2) that for $k \geq 4$ and even $n \geq 10$, $\ell_k(n)=\frac{n}{2}(k-1)+2$. There remain the cases of small values of even n, $n=4,6,8$. For $n=4,6$, using techniques present one can prove that Lemma 4 holds as well. Then the same argument in the proof of Proposition 5 works with one modification: for $n=4$, in the end $n'=2$ for β'; there is no need to apply Proposition 1, as admissible sequences with capacity 2 are always k-separable. It is a little curious that for $n=8$ there are k-inseparable sequences with length $\ell=\frac{n}{2}(k-1)+2=4k-2$: e.g., $\alpha=3^{2k+1}1^{2k-3}$ for any $k \geq 2$. Clearly the trivial algorithm fails with $t=3$. Moreover, the sequence is k-inseparable because in order to make $n=8$ with 3's and 1's, there are three possibilities: $8 = 3 \cdot 0 + 1 \cdot 8 = 3 \cdot 1 + 1 \cdot 5 = 3 \cdot 2 + 1 \cdot 2$. In each possibility, we need at least two terms equal 1; hence to fill up k boxes of $n = 8$, we need at least $2k$ terms equal 1, but in α there are only $2k - 3$ terms of 1. When $\ell \geq \frac{n}{2}(k-1) + 3 = 4k - 1$, separability for $n = 8$ follows from arguments in the proofs of Lemma 4 and Proposition 5.

In summary, for $k \geq 4$, for any even $n \geq 4$ and $n \neq 8$: $\ell_k(n) = \frac{n}{2}(k-1) + 2$. When $n = 8$, for any $k \geq 2$, $\ell_k(8) = \frac{n}{2}(k-1) + 3 = 4k - 1$ (the bounds obtained before in the cases $k = 2, 3$ obey the same formula).

6. Comments

From a Hungary-Israel contest problem, we solved a certain generalization to its completion with the unexpected observation that the answer for $k \geq 4$ is structurally simpler than the one for $k = 3$, and the case $k = 2$ turns out most intriguing. This suggests that other generalizations might yield more interesting questions. For example, consider sequences with sum kn and no proper subsequence having a sum divisible by n. One can also look at inverse questions such as describing all inseparable sequences of a certain length.

From a different perspective, the problem considered here seems a good exercise for students. It involves some typical features of mathematical investigation and requires minimum knowledge. Such a training may introduce and inspire one to be engaged in more substantial work on additive number theory.

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References

[1] S. Gueron, *Hungary-Israel Mathematics Competition: The First Twelve Years*, Australian Mathematics Trust Publishing, Australia, 2004.

[2] S. Savchev and F. Chen, Long zero-free sequences in finite cyclic groups, *Discrete Math.* **307 (22)** (2007), 2671–2679.