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EVALUATION OF CONVOLUTION SUMS INVOLVING CUBIC DIVISOR FUNCTIONS FOR A CLASS OF LEVELS

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Abstract

For $0 < \alpha, \beta \in \mathbb{N}$ for which $\gcd(\alpha, \beta) = 1$, we tackle an elementary method for the evaluation of the convolution sum $\sum\limits_{\substack{(l,m)\in\mathbb{N}^2 \\ \alpha\, l+\beta\, m=n}} \sigma_3(l)\sigma_3(m)$. Modular forms are used to establish this result. We then apply the convolution sums belonging to this class of levels to determine formulae for the number of representations of a positive integer n by the quadratic form $a\sum\limits_{i=1}^{8} x_i^2 + b\sum\limits_{i=9}^{16} x_i^2$ with $0 < a, b \in \mathbb{N}$ when the level $\alpha\beta \equiv 0 \pmod 4$, and by the quadratic form $c\sum\limits_{i=1}^{4}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2) + d\sum\limits_{i=5}^{8}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2)$ with $0 < c, d \in \mathbb{N}$ when the level $\alpha\beta \equiv 0 \pmod 3$. We illustrate our approach by explicitly evaluating the convolution sums for $\alpha\beta =$ 3, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 25, 27, 32. We then apply some of these convolution sums to determine formulae for the number of representations of a positive integer n by quadratic forms of the above given types.

1. Introduction

The problem of the evaluation of the summation

$$\sigma_r(1)\sigma_s(n-1) + \sigma_r(2)\sigma_s(n-2) + \cdots + \sigma_r(n-2)\sigma_s(2) + \sigma_r(n-1)\sigma_s(1), \qquad (1)$$

wherein $0 < n, r, s, k \in \mathbb{N}$ and $\sigma_k(n)$ denotes the sum of the k^{th} powers of the positive divisors of n, began with the work of Liouville [9]. Hence, research on convolution sums and the number of representation of a natural number by quadratic forms can be traced back to Liouville [19, chap. 12] whose pseudonym was Besge or Besgue. Note that Liouville did state the results on convolution sums and provided the proofs by examples. Results on convolution sums (1) were later obtained by Glaisher [2], Lahiri [6, 7] and Ramanujan [16] for pairs (r, s) such that r and s are odd positive integers and $r + s < 14$.

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, natural numbers without zero, i.e., $\mathbb{N} \setminus \{0\}$, integers, rational numbers, real numbers and complex numbers, respectively.

Suppose that $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Formally, the sum of the k^{th} powers of the positive divisors of n is

$$\sigma_k(n) = \sum_{0<\delta|n} \delta^k. \tag{2}$$

It is obvious from the definition that $\sigma_k(m) = 0$ for all $m \notin \mathbb{N}_0$. We write $d(n)$ and $\sigma(n)$ as a shorthand for $\sigma_0(n)$ and $\sigma_1(n)$, respectively.

Assume that the positive integers α and β with $\alpha \leq \beta$ are given. Then the convolution sum $W^{3,3}_{(\alpha,\beta)}(n)$ is defined by

$$W^{3,3}_{(\alpha,\beta)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ \alpha\, l+\beta\, m=n}} \sigma_3(l)\sigma_3(m). \tag{3}$$

We set $W^{3,3}_{(\alpha,\beta)}(n) = 0$ if for all $(l, m) \in \mathbb{N}^2$ it holds that $\alpha\, l + \beta\, m \neq n$.

In this paper, we evaluate the convolution sum $W^{3,3}_{(\alpha,\beta)}(n)$ for the level $\alpha\beta \in \mathbb{N}_0$. In order to achieve this goal, we let

$$\mathfrak{N} = \{\, 2^{\nu}\mho \,|\, \nu \in \{0, 1, 2, 3\} \text{ and } \mho \text{ is a finite product of distinct odd primes} \,\},$$

then consider the evaluation of the convolution sum $W^{3,3}_{(\alpha,\beta)}(n)$ for the levels $\alpha\beta \in \mathfrak{N}$ and $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$, respectively.

The convolution sums known so far whose evaluation involve cubic divisor functions $\sigma_3(n)$ are displayed in the following table.

Level $\alpha\beta$	**Authors**	**References**
1	Lahiri, Ramanujan	[6, 7, 16]
2,4	Cheng & Williams	[1]

Table 1: Known convolution sums $W^{3,3}_{(\alpha,\beta)}(n)$ of level $\alpha\beta$

The evaluation of convolution sums involving cubic divisor functions for a class of levels is new.

We then apply the result for this class of levels to determine the convolution sum for $\alpha\beta = 3,\ 5,\ 6,\ 8,\ 10,\ 12,\ 15,\ 20 \in \mathfrak{N}$ and $\alpha\beta = 9,\ 16,\ 18,\ 25,\ 27,\ 32 \in \mathbb{N}_0 \setminus \mathfrak{N}$. Again, these explicit convolution sums have not been evaluated as yet.

Certain of these convolution sums are applied to establish explicit formulae for the number of representations of a positive integer n by the quadratic forms

$$a\,(x_1^2 + x_2^2 + \cdots + x_7^2 + x_8^2) + b\,(x_9^2 + x_{10}^2 + \cdots + x_{15}^2 + x_{16}^2) \tag{4}$$

and

$$c\sum_{i=1}^{4}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2) + d\sum_{i=5}^{8}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2), \tag{5}$$

where $(a,b),(c,d) \in \mathbb{N}_0^2$.

Based on the structure of the level $\alpha\beta$, we provide a method to determine all pairs $(a,b),(c,d) \in \mathbb{N}_0^2$ which are neccessary for the determination of the formulae for the number of representations of a positive integer by the quadratic forms (4) and (5), respectively. Then we determine explicit formulae for the number of representations of a positive integer n by the quadratic forms (4) and (5), whenever $\alpha\beta \equiv 0 \pmod 4$ and $\alpha\beta \equiv 0 \pmod 3$, respectively. As an example, we determine formulae for the number of representations of a positive integer n by the quadratic form (4) using the convolution sums for the level $\alpha\beta = 4$, 8, 12, 16, 20 and 32, and that by the quadratic form (5) using the convolution sums for the level $\alpha\beta = 3$, 6, 9, 12, 15, 18 and 27.

The results of this paper are obtained using Software for symbolic scientific computation. This software is composed of the open source software packages *GiNaC*, *Maxima*, *REDUCE*, *SAGE* and the commercial software package *MAPLE*.

2. Essential Knowledge

2.1. Modular Forms

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half-plane and let $\Gamma = G = \mathrm{SL}_2(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a,b,c,d \in \mathbb{R} \text{ and } ad-bc = 1\}$ be the group of 2×2-matrices. Let $N \in \mathbb{N}_0$. Then

$$\Gamma(N) \quad = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N \right\}$$

is a subgroup of G and is called the *principal congruence subgroup of level N*. A subgroup H of G is called a *congruence subgroup of level* N if it contains $\Gamma(N)$.

For our purpose, the following congruence subgroup is relevant:

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod N \right\}.$$

Let $\Gamma' \subseteq \Gamma$ be a congruence subgroup of level N. Let $k \in \mathbb{Z}, \gamma \in \mathrm{SL}_2(\mathbb{Z})$ and f be a meromorphic function on the upper half-plane $\mathbb{H}$. We denote by $f^{[\gamma]_k}$ the function whose value at z is $(cz+d)^{-k}f(\gamma(z))$, i.e., $f^{[\gamma]_k}(z) = (cz+d)^{-k}f(\gamma(z))$. The following definition is according to Koblitz [4, p. 108].

Definition 1. Let $N \in \mathbb{N}_0$, $k \in \mathbb{Z}$, f be a meromorphic function on $\mathbb{H}$ and $\Gamma' \subset \Gamma$ be a congruence subgroup of level N.

(a) f is called a *modular function of weight* k for Γ' if the following two conditions are met.

 (a1) For all $\gamma \in \Gamma'$ it holds that $f^{[\gamma]_k} = f$.

 (a2) For any $\delta \in \Gamma$ it holds that $f^{[\delta]_k}(z)$ has the form $\sum\limits_{n\in\mathbb{Z}} a_n e^{\frac{2\pi i z n}{N}}$ and $a_n \neq 0$ for finitely many $n \in \mathbb{Z} \setminus \mathbb{N}$.

(b) f is called a *modular form of weight* k for Γ' if the following three conditions are met.

 (b1) f is a modular function of weight k for Γ'.

 (b2) f is holomorphic on $\mathbb{H}$.

 (b3) For all $\delta \in \Gamma$ and for all $n \in \mathbb{Z} \setminus \mathbb{N}$ it holds that $a_n = 0$.

(c) f is called a *cusp form of weight k for* Γ' if the following two conditions are met.

 (c1) f is a modular form of weight k for Γ'.

 (c2) For all $\delta \in \Gamma$ it holds that $a_0 = 0$.

Let us denote by $M_k(\Gamma')$ the set of modular forms of weight k for Γ', by $S_k(\Gamma')$ the set of cusp forms of weight k for Γ' and by $E_k(\Gamma')$ the set of Eisenstein series. The sets $M_k(\Gamma')$, $S_k(\Gamma')$ and $E_k(\Gamma')$ are vector spaces over $\mathbb{C}$. Therefore, $M_k(\Gamma_0(N))$ is the space of modular forms of weight k for $\Gamma_0(N)$, $S_k(\Gamma_0(N))$ is the space of cusp forms of weight k for $\Gamma_0(N)$, and $E_k(\Gamma_0(N))$ is the space of Eisenstein series. The decomposition of the space of modular forms as a direct sum of the space of Eisenstein series and the space of cusp forms, i.e., $M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N))$, is well-known; see for example Stein's book (online version) [17, p. 81].

We asume in this paper that $k \in \mathbb{N}_0$ and that χ and ψ are primitive Dirichlet characters with conductors L and R, respectively. Stein [17, p. 86] has noted that

$$E_{2k,\chi,\psi}(q) = C_0 + \sum_{n=1}^{\infty} \Big(\sum_{d|n} \psi(d)\chi(\frac{n}{d})\, d^{2k-1} \Big) q^n, \tag{6}$$

where

$$C_0 = \begin{cases} 0 & \text{if } L > 1 \\ -\frac{B_{2k,\chi}}{4k} & \text{if } L = 1 \end{cases}$$

and $B_{2k,\chi}$ are the generalized Bernoulli numbers. Theorems 5.8 and 5.9 in Section 5.3 of Stein [17, p. 86] are then applicable.

If the primitive Dirichlet characters χ and ψ are trivial then their conductors L and R are one, respectively. In that case (6) can be normalized and then given as follows: $E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, q^n$. This is the case whenever the level $\alpha\beta$ belongs to $\mathfrak{N}$.

Let $m, n \in \mathbb{N}_0$ be such that m is a positive divisor of n, and let $k \in \mathbb{N}_0$. Then we apply Miyake [11, Lemma 2.1.3] to conclude that

$$M_{2k}(\Gamma_0(m)) \subset M_{2k}(\Gamma_0(n)). \tag{7}$$

This implies the same inclusion relation for the bases, the space of Eisenstein forms of weight $2k$ and the spaces of cusp forms of weight $2k$.

2.2. Eta Quotients

On the upper half-plane $\mathbb{H}$, the Dedekind η-function, $\eta(z)$, can be defined as follows: $\eta(z) = e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$. Let us set $q = e^{2\pi i z}$. Then it follows that

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} F(q), \quad \text{where } F(q) = \prod_{n=1}^{\infty} (1 - q^n).$$

Let κ be a finite subset of $\mathbb{N}_0$ and $e_j \in \mathbb{Z}$ with $j \in \kappa$. According to Köhler [5, p. 31] an *η-product* or *η-quotient*, $f(z)$, is a finite product of Dedekind η-functions of the form

$$\prod_{j \in \kappa} \eta(jz)^{e_j}. \tag{8}$$

Note that η-function, η-quotient and η-product are used interchangeably as synonyms.

Based on this definition of an η-quotient, there exists a positive integer N such that $N = \operatorname{lcm}\{j \mid j \in \kappa\}$. We call such an N the *level* of an η-product. Therefore, an η-quotient is simply understood as

$$\prod_{0<j \mid N} \eta(jz)^{e_j}.$$

If $2k = \frac{1}{2} \sum_{0<j \mid N} e_j$ then the η-quotient $f(z)$ behaves like a modular form of weight $2k$ on $\Gamma_0(N)$ with some multiplier system.

Kilford [3, p. 99] and Köhler [5, Corollary 2.3] have formulated the following theorem which is a result of the work of Newman [12, 13] and Ligozat [8]. This theorem is effectively used to exhaustively determine η-quotients, $f(z)$, which belong to $M_{2k}(\Gamma_0(N))$, and in particular those η-quotients which are in $S_{2k}(\Gamma_0(N))$.

Theorem 1 (Newman and Ligozat)**.** *Let $N \in \mathbb{N}_0$, $D(N)$ be the set of all positive divisors of N, $\delta \in D(N)$ and $r_\delta \in \mathbb{Z}$. Furthermore, let $f(z) = \prod_{\delta \in D(N)} \eta^{r_\delta}(\delta z)$ be an η-quotient. If the following four conditions are satisfied:*

(i) $\sum_{\delta \in D(N)} \delta\, r_\delta \equiv 0 \pmod{24}$,

(ii) $\sum_{\delta \in D(N)} \frac{N}{\delta}\, r_\delta \equiv 0 \pmod{24}$,

(iii) $\prod_{\delta \in D(N)} \delta^{r_\delta}$ *is a square in* $\mathbb{Q}$,

(iv) *for all* $d \in D(N)$ *we have* $\sum_{\delta \in D(N)} \frac{\gcd(\delta,d)^2}{\delta}\, r_\delta \geq 0$,

then $f(z) \in M_{2k}(\Gamma_0(N))$*, where* $2k = \frac{1}{2}\sum_{\delta \in D(N)} r_\delta$.

*Moreover, the η-quotient $f(z)$ is an element of $S_{2k}(\Gamma_0(N))$ if **(iv)** is replaced by*

(iv') *for all* $d \in D(N)$ *we have* $\sum_{\delta \in D(N)} \frac{\gcd(\delta,d)^2}{\delta} r_\delta > 0$.

Remark 1. For an η-quotient $f(z) = \prod_{\delta \in D(N)} \eta^{r_\delta}(\delta z)$, it can be shown that if in Theorem 1 either the condition (i) or (ii) is not considered then $f(z)$ belongs to $M_{2k}(\Gamma_0(N))$ under some restrictions. Moreover, $f(z)$ is an element of $S_{2k}(\Gamma_0(N))$ under some restrictions if the condition (iv') is satisfied.

2.3. Convolution Sums $\mathbf{W^{3,3}_{(\alpha,\beta)}(n)}$

Given $\alpha, \beta \in \mathbb{N}_0$ such that $\alpha \leq \beta$, let the convolution sum be defined by (3).

Suppose in addition that $\gcd(\alpha,\beta) = \delta > 1$ for some $\delta \in \mathbb{N}_0$. Then there exist $\alpha_1, \beta_1 \in \mathbb{N}_0$ such that $\gcd(\alpha_1,\beta_1) = 1$, $\alpha = \delta\,\alpha_1$ and $\beta = \delta\,\beta_1$. Hence,

$$W^{3,3}_{(\alpha,\beta)}(n) = \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ \alpha\, k+\beta\, l=n}} \sigma_3(k)\sigma_3(l) = \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ \delta\,\alpha_1\, k+\delta\,\beta_1\, l=n}} \sigma_3(l)\sigma_3(k) = W^{3,3}_{(\alpha_1,\beta_1)}(\frac{n}{\delta}). \qquad (9)$$

Therefore, we may simply assume that $\gcd(\alpha,\beta) = 1$. Moreover, due to the commutativity of the addition and the multiplication, it is obvious that $W^{3,3}_{(\alpha,\beta)}(n) = W^{3,3}_{(\beta,\alpha)}(n)$.

We note that the primitive Dirichlet characters χ and ψ

1. are trivial whenever $\alpha\beta \in \mathfrak{N}$ holds;

2. are such that $\chi = \psi$ and that χ is a Legendre-Jacobi-Kronecker symbol otherwise.

The following Eisenstein series hold:

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n, \tag{10}$$

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{11}$$

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{12}$$

$$E_8(q) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \tag{13}$$

$$\begin{aligned} E_{2k,\chi}(q^{\lambda}) &= E_{2k}(q^{\lambda}) \otimes \chi(\lambda) \\ &= \chi(\lambda) \left(C_0 + \sum_{n=1}^{\infty} \chi(n)\, \sigma_{2k-1}(n)\, q^{\lambda\, n} \right) \\ &= \chi(\lambda)\, C_0 + \sum_{n=1}^{\infty} \chi(\lambda\, n)\, \sigma_{2k-1}(n)\, q^{\lambda\, n}, \\ &= \chi(\lambda)\, C_0 + \sum_{n=1}^{\infty} \chi(n)\, \sigma_{2k-1}(\frac{n}{\lambda})\, q^{n}, \end{aligned} \tag{14}$$

where $\lambda \in \mathbb{N}_0$,

$$C_0 = \begin{cases} 0 & \text{if } L > 1 \\ -\frac{B_{2k,\chi}}{4k} & \text{if } L = 1 \end{cases}$$

and $B_{2k,\chi}$ are the specially generalized Bernoulli numbers.

Note that $E_4(q), E_6(q), E_8(q)$ are special cases of (6) or (14) and hold if $\alpha\beta \in \mathfrak{N}$. We state two relevant results for the sequel of this work.

Lemma 1. *Let $k, \alpha, \beta \in \mathbb{N}_0$. Then*

$$(\alpha\, E_{2k}(q^{\alpha}) - \beta\, E_{2k}(q^{\beta}))^2 \in M_{4k}(\Gamma_0(\alpha\beta)).$$

Proof. If $\alpha = \beta$ then trivially $0 = (\alpha\, E_{2k}(q^{\alpha}) - \alpha\, E_{2k}(q^{\alpha}))^2 \in M_{4k}(\Gamma_0(\alpha))$ and there is nothing to prove. Therefore, we may suppose that $\alpha \neq \beta$ in the sequel. We apply the result proved by Stein [17, Theorems 5.8 and 5.9] and (7) to deduce that

$$E_{2k}(q) - \alpha\, E_{2k}(q^{\alpha}) \in M_{2k}(\Gamma_0(\alpha)) \subseteq M_{2k}(\Gamma_0(\alpha\beta))$$

and

$$E_{2k}(q) - \beta\, E_{2k}(q^{\beta}) \in M_{2k}(\Gamma_0(\beta)) \subseteq M_{2k}(\Gamma_0(\alpha\beta)).$$

Therefore,

$$\alpha\, E_{2k}(q^{\alpha}) - \beta\, E_{2k}(q^{\beta}) = (E_{2k}(q) - \beta\, E_{2k}(q^{\beta})) - (E_{2k}(q) - \alpha\, E_{2k}(q^{\alpha})) \in M_{2k}(\Gamma_0(\alpha\beta))$$

and so $(\alpha\, E_{2k}(q^{\alpha}) - \beta\, E_{2k}(q^{\beta}))^2 \in M_{4k}(\Gamma_0(\alpha\beta))$. □

Theorem 2. *Let $\alpha, \beta \in \mathbb{N}_0$ be such that $\alpha < \beta$, and α and β are relatively prime. Then*

$$\begin{aligned}\Big(\alpha\, E_4(q^{\alpha}) - \beta\, E_4(q^{\beta})\Big)^2 &= (\alpha - \beta)^2 + 480 \sum_{n=1}^{\infty} \Big(\alpha^2\, \sigma_7(\frac{n}{\alpha}) + \beta^2\, \sigma_7(\frac{n}{\beta}) \\ &\quad - \alpha\beta\, \sigma_3(\frac{n}{\alpha}) - \alpha\beta\, \sigma_3(\frac{n}{\beta}) - 240\, \alpha\beta\, W^{3,3}_{(\alpha,\beta)}(n) \Big) q^n. \end{aligned} \quad (15)$$

Proof. We first observe that

$$\Big(\alpha\, E_4(q^{\alpha}) - \beta\, E_4(q^{\beta})\Big)^2 = \alpha^2\, E_4^2(q^{\alpha}) + \beta^2\, E_4^2(q^{\beta}) - 2\,\alpha\beta\, E_4(q^{\alpha}) E_4(q^{\beta}). \quad (16)$$

Lahiri[6] has derived the identity

$$E_4^2(q) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n \quad (17)$$

which we apply to deduce that

$$E_4^2(q^{\alpha}) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(\frac{n}{\alpha}) q^n \quad (18)$$

and

$$E_4^2(q^{\beta}) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(\frac{n}{\beta}) q^n. \quad (19)$$

Since

$$\Big(\sum_{n=1}^{\infty} \sigma_3(\frac{n}{\alpha}) q^n\Big)\Big(\sum_{n=1}^{\infty} \sigma_3(\frac{n}{\beta}) q^n\Big) = \sum_{n=1}^{\infty} \Big(\sum_{\alpha k + \beta l = n} \sigma_3(k)\sigma_3(l) \Big) q^n = \sum_{n=1}^{\infty} W^{3,3}_{(\alpha,\beta)}(n) q^n,$$

we conclude that

$$E_4(q^{\alpha}) E_4(q^{\beta}) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(\frac{n}{\alpha}) q^n + 240 \sum_{n=1}^{\infty} \sigma_3(\frac{n}{\beta}) q^n + 240^2 \sum_{n=1}^{\infty} W^{3,3}_{(\alpha,\beta)}(n) q^n. \quad (20)$$

Therefore,

$$\Big(\alpha\, E_4(q^{\alpha})-\beta\, E_4(q^{\beta})\Big)^2=(\alpha-\beta)^2+480\sum_{n=1}^{\infty}\Big(\ \alpha^2\,\sigma_7(\frac{n}{\alpha})+\beta^2\,\sigma_7(\frac{n}{\beta})$$
$$-\alpha\beta\,\sigma_3(\frac{n}{\alpha})-\alpha\beta\,\sigma_3(\frac{n}{\beta})-240\,\alpha\beta\,W^{3,3}_{(\alpha,\beta)}(n)\ \Big)q^n$$

as asserted. □

If we suppose in the above theorem that $\alpha=\beta$ and combine this fact with Lemma 1 then we deduce the following.

Corollary 1. *Let n be a positive integer. Then for all $\alpha\in\mathbb{N}_0$ it holds that*

$$W^{3,3}_{(\alpha,\alpha)}(n)=W^{3,3}_{(1,1)}(\frac{n}{\alpha})=\frac{1}{120}\sigma_7(\frac{n}{\alpha})-\frac{1}{120}\sigma_3(\frac{n}{\alpha}). \tag{21}$$

In particular, if $\alpha=1$ then we obtain the result proved by Ramanujan [16], namely

$$W^{3,3}_{(1,1)}(n)=\frac{1}{120}\sigma_7(n)-\frac{1}{120}\sigma_3(n). \tag{22}$$

3. Evaluating $\mathbf{W^{3,3}_{(\alpha,\beta)}(n)}$, Where $\boldsymbol{\alpha\beta}\in\mathbb{N}_0$

We carry out an explicit formula for the convolution sum $W^{3,3}_{(\alpha,\beta)}(n)$, where the level $\alpha\beta$ belongs to $\mathbb{N}_0$.

3.1. Bases of $E_8(\Gamma_0(\alpha\beta))$ and $S_8(\Gamma_0(\alpha\beta))$

Let $\mathcal{D}(\alpha\beta)$ denote the set of all positive divisors of $\alpha\beta$.

Pizer [15] has discussed the existence of a basis of the space of cusp forms of weight $2k\in\mathbb{N}_0$ for $\Gamma_0(\alpha\beta)$ when $\alpha\beta$ is not a perfect square. We suppose in the sequel that the weight $2k\in\mathbb{N}_0$. We apply the dimension formulae in Miyake's book [11, Theorem 2.5.2] or Stein's book [17, Proposition 6.1] to conclude the following:

- for the space of Eisenstein series,

$$\dim(E_{2k}(\Gamma_0(\alpha\beta)))=\sum_{d\mid\alpha\beta}\varphi(\gcd(d,\frac{\alpha\beta}{d}))=m_E, \tag{23}$$

  where $m_E\in\mathbb{N}_0$ and φ is the Euler's totient function;

- for the space of cusp forms, $\dim(S_{2k}(\Gamma_0(\alpha\beta)))=m_S$, where $m_S\in\mathbb{N}$.

If $\alpha\beta \in \mathfrak{N}$ then

$$\dim(E_{2k}(\Gamma_0(\alpha\beta))) = \sum_{\delta|\alpha\beta} \varphi(\gcd(\delta, \frac{\alpha\beta}{\delta})) = \sum_{\delta|\alpha\beta} 1 = \sigma_0(\alpha\beta) = d(\alpha\beta). \tag{24}$$

For a fixed $\alpha\beta \in \mathbb{N}_0$, we use Theorem 1 $(i) - (iv')$ to exhaustively determine as many elements of the space of cusp forms, $S_{2k}(\Gamma_0(\alpha\beta))$, as possible. From these elements of the space $S_{2k}(\Gamma_0(\alpha\beta))$, we select relevant ones for the purpose of the determination of a basis of this space.

The so-determined basis of the vector space of cusp forms is in general not unique; however, due to the change of basis which is an automorphism, it is sufficient to only consider this basis for our purpose.

Let $\mathcal{C}$ denote the set of primitive Dirichlet characters $\chi(n) = \left(\frac{m}{n}\right)$ as assumed in (14), where $m, n \in \mathbb{Z}$ and $\left(\frac{m}{n}\right)$ is the Legendre-Jacobi-Kronecker symbol. Let $D_\chi(\alpha\beta) \subseteq \mathcal{D}(\alpha\beta)$ denote the subset of $\mathcal{D}(\alpha\beta)$ associated with the character χ.

The selection of a primitive Dirichlet character as required in (14) is not straightforward. Not all primitive Dirichlet characters are good candidates such that a basis of the space of Eisenstein series for that given level can be found. For example, if the levels are 25 and 32, the primitive Dirichlet characters $\left(\frac{5}{n}\right)$ and $\left(\frac{-4}{n}\right)$ do not permit one to prove the linear independence of a basis of $E_{2k}(\Gamma_0(25))$ and $E_{2k}(\Gamma_0(32))$, respectively.

Let i, κ be natural numbers. The expression of a positive integer in the form $\prod_{i=1}^{\kappa} p_i^{e_i}$ modulo a permutation of the primes p_i, where e_i is in $\mathbb{N}_0$, is standard. In the following, we use this form to express a level $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$.

Definition 2. Let $i, \kappa \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Furthermore, let $C \in \mathbb{Z}$ be fixed. Suppose that the level $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$ is fixed and of the form $\prod_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}_0$. We say that a primitive Dirichlet character $\chi(n) = \left(\frac{C}{n}\right)$ *annihilates* $E_{2k}(\Gamma_0(\alpha\beta))$ or is an *annihilator* of $E_{2k}(\Gamma_0(\alpha\beta))$ if for some $1 \leq j \leq \kappa$ there exists $1 < p_j^{e_j} \in \mathbb{N}_0 \setminus \mathfrak{N}$ positive divisor of $\alpha\beta$ such that $E_{2k,\chi}(q^\delta)$ vanishes for all $1 < \delta$ positive divisor of $p_j^{e_j}$.
A set $\mathcal{C}$ of primitive Dirichlet characters *annihilates* $E_{2k}(\Gamma_0(\alpha\beta))$ or is an *annihilator* of $E_{2k}(\Gamma_0(\alpha\beta))$ if each $\chi(n) \in \mathcal{C}$ is an annihilator of $E_{2k}(\Gamma_0(\alpha\beta))$.

To illustrate the above definition, suppose that $\alpha\beta = 18$ and the primitive Dirichlet character is $\chi(n) = \left(\frac{-3}{n}\right)$. Then $C = -3$ so that $|C|$ is a positive divisor of $9 = 3^2$. Hence, for all $k \in \mathbb{N}$ and for all $1 < \delta \in \mathcal{D}(9)$ it holds that

$$E_{2k,\chi}(q^\delta) = \sum_{n=1}^{\infty} \chi(n)\, \sigma_{2k-1}(\frac{n}{\delta})\, q^n = 0.$$

Therefore, the primitive Dirichlet character χ which is such that $\chi(n) = \left(\frac{-3}{n}\right)$ is an annihilator of $E_{2k}(\Gamma_0(18))$.

The following theorem provides a strong criteria when selecting a primitive Dirichlet character for a given level $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$.

Theorem 3. *Let i, κ be in $\mathbb{N}_0$. Let $C \in \mathbb{Z}$ be fixed. Let χ be a primitive Dirichlet character with conductor $|C| > 1$ and let the level $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$ be fixed and of the form $\prod_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}_0$. Furthermore, suppose that $p_j^{e_j} \in \mathbb{N}_0 \setminus \mathfrak{N}$ is a positive divisor of $\alpha\beta$ for some $1 \leq j \leq \kappa$. If the conductor $|C|$ is a positive divisor of $p_j^{e_j}$ and hence of the level $\alpha\beta$ then $\chi(n) = \left(\frac{C}{n}\right)$, for all $n \in \mathbb{N}$, is an annihilator of $E_{2k}(\Gamma_0(\alpha\beta))$.*

Proof. Suppose that $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$ is fixed and of the form $\prod_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}_0$. As an immediate consequence of the structure of $\alpha\beta$ there exists $1 \leq j \leq \kappa$ such that $p_j^{e_j} \in \mathbb{N}_0 \setminus \mathfrak{N}$ is a positive divisor of $\alpha\beta$. Since the conductor $|C|$ is a positive divisor of the level $p_j^{e_j}$, the existence of $1 < \delta \in \mathcal{D}(p_j^{e_j})$ is given. It is well-known that for each $1 \leq f \leq e_j$ it holds that p_j^f is a positive divisor of $p_j^{e_j}$. On the other hand, It holds that

$$\left(\frac{C}{n}\right) = \begin{cases} 0 & \text{if } \gcd(|C|, n) \neq 1, \\ \text{nonzero} & \text{otherwise.} \end{cases} \tag{25}$$

For each $1 < \delta \in \mathcal{D}(p_j^{e_j})$ it holds that $\gcd(|C|, \delta) \neq 1$. Since the conductor of χ is greater than one, it follows that $C_0 = 0$ in (6); that means

$$E_{2k,\chi}(q^\delta) = \sum_{n=1}^{\infty} \chi(n)\, \sigma_{2k-1}\Big(\frac{n}{\delta}\Big)\, q^n.$$

Since it also holds that

$$\sigma_{2k-1}\Big(\frac{n}{\delta}\Big) = \begin{cases} 0 & \text{if } \frac{n}{\delta} \notin \mathbb{N}_0, \\ \text{nonzero} & \text{otherwise,} \end{cases} \tag{26}$$

we obtain the stated result by simply putting altogether; that is $E_{2k,\chi}(q^\delta) = 0$ for all $1 < \delta \in \mathcal{D}(p_j^{e_j})$. $\square$

If $\alpha\beta \in \mathfrak{N}$ holds then the primitive Dirichlet characters are trivial. Therefore, the set $\mathcal{C}$ is empty. Hence, the case where $\alpha\beta \in \mathfrak{N}$ holds is a special case of the following theorem.

Theorem 4. *Suppose that $\alpha\beta \in \mathbb{N}_0$ is given.*

(a) *Let $\mathcal{C}$ be a set of primitive Dirichlet characters such that for each $\chi \in \mathcal{C}$ it holds that χ is not an annihilator of $E_{2k}(\Gamma_0(\alpha\beta))$. Then the set*

$$\mathcal{B}_E = \{\, E_{2k}(q^t) \mid t \in \mathcal{D}(\alpha\beta) \,\} \cup \bigcup_{\chi \in \mathcal{C}} \{\, E_{2k,\chi}(q^t) \mid t \in D_\chi(\alpha\beta) \,\}$$

is a basis of $E_{2k}(\Gamma_0(\alpha\beta))$.

(b) *Let* $1 \le i \le m_S$ *be positive integers,* $\delta \in D(\alpha\beta)$ *and* $(r(i,\delta))_{i,\delta}$ *be a table of the powers of* $\eta(\delta z)$. *Furthermore, let* $\mathfrak{B}_{\alpha\beta,i}(q) = \prod_{\delta|\alpha\beta} \eta^{r(i,\delta)}(\delta z)$ *be selected elements of* $S_{2k}(\Gamma_0(\alpha\beta))$. *Then the set* $\mathcal{B}_S = \{ \mathfrak{B}_{\alpha\beta,i}(q) \mid 1 \le i \le m_S \}$ *is a basis of* $S_{2k}(\Gamma_0(\alpha\beta))$.

(c) *The set* $\mathcal{B}_M = \mathcal{B}_E \cup \mathcal{B}_S$ *constitutes a basis of* $M_{2k}(\Gamma_0(\alpha\beta))$.

Remark 2. Each $\mathfrak{B}_{\alpha\beta,i}(q)$ can be expressed in the form $\sum_{n=1}^{\infty} \mathfrak{b}_{\alpha\beta,i}(n)q^n$, where $1 \le i \le m_S$ and for each $n \ge 1$ the coefficient $\mathfrak{b}_{\alpha\beta,i}(n)$ is an integer. If we divide the sum, which results from Theorem 1 (iv'), when $d = N$, by 24 then we obtain the smallest positive degree of q in $\mathfrak{B}_{\alpha\beta,i}(q)$.

The existence of a basis of the space of cusp forms for all square-free levels has been proved by Pizer [15], and Theorem 1 provides a method to find as many elements of the space of cusp forms as possible. Hence, the proof of Theorem 4 (b) is essentially restricted to show that the selected elements of the space of cusp forms of the given level are linearly independent.

Proof of Theorem 4. We only consider the case where $\alpha\beta$ is in $\mathbb{N}_0 \setminus \mathfrak{N}$ since the case $\alpha\beta \in \mathfrak{N}$ is proved similarly and even using a relatively straightforward process due to the fact that $\mathcal{C} = \emptyset$.

(a) Stein [17, Theorems 5.8 and 5.9] has shown that for each t positive divisor of $\alpha\beta$ it holds that $E_{2k}(q^t)$ is in $M_{2k}(\Gamma_0(t))$. Since $M_{2k}(\Gamma_0(t))$ is a vector space and the set $\mathcal{C}$ of primitive Dirichlet characters does not annihilate $E_{2k}(\Gamma_0(\alpha\beta))$, it also holds for each Legendre-Jacobi-Kronecker symbol $\chi \in \mathcal{C}$ and $t \in D_\chi(\alpha\beta)$ that $0 \ne E_{2k,\chi}(q^t)$ is in $M_{2k}(\Gamma_0(t))$. Since the dimension of $E_{2k}(\Gamma_0(\alpha\beta))$ is finite, it suffices to show that $\mathcal{B}_E$ is linearly independent. Suppose that for each $\chi \in \mathcal{C}, s \in D_\chi(\alpha\beta)$ we have $z(\chi)_s \in \mathbb{C}$ and that for each $t|\alpha\beta$ we have $x_t \in \mathbb{C}$. Then

$$\sum_{t|\alpha\beta} x_t\, E_{2k}(q^t) + \sum_{\chi\in\mathcal{C}} \Bigg(\sum_{s\in D_\chi(\alpha\beta)} z(\chi)_s E_{2k,\chi}(q^s) \Bigg) = 0.$$

We recall that χ is a Legendre-Jacobi-Kronecker symbol; therefore, for all $0 \ne a \in \mathbb{Z}$ it holds that $\left(\frac{a}{0}\right) = 0$. Since the primitive Dirichlet character χ is not trivial and has a conductor L which we may assume greater than one, we can deduce that $C_0 = 0$ in (14). Then we equate the coefficients of q^n for $n \in D(\alpha\beta) \cup \bigcup_{\chi\in\mathcal{C}} \{s | s \in D_\chi(\alpha\beta)\}$ to obtain the following homogeneous system of linear equations in m_E unknowns:

$$\sum_{u|\alpha\beta}\sigma_{2k-1}(\frac{t}{u})x_u+\sum_{\chi\in\mathcal{C}}\sum_{v\in D_\chi(\alpha\beta)}\chi(t)\sigma_{2k-1}(\frac{t}{v})Z(\chi)_v=0,\qquad t\in D(\alpha\beta).$$

Since $\alpha\beta\in\mathbb{N}_0$ is fixed, one can easily show that the determinant of the matrix of this homogeneous system of linear equations is not zero. Hence, the unique solution is $x_t=z(\chi)_s=0$ for all $t\in D(\alpha\beta)$ and for all $\chi\in\mathcal{C}, s\in D_\chi(\alpha\beta)$. So, the set $\mathcal{B}_E$ is linearly independent and hence is a basis of $E_{2k}(\Gamma_0(\alpha\beta))$.

(b) First, we show that each $\mathfrak{B}_{\alpha\beta,i}(q)$, where $1\leq i\leq m_S$, is in the space $S_{2k}(\Gamma_0(\alpha\beta))$. This is obviously the case since $\mathfrak{B}_{\alpha\beta,i}(q), 1\leq i\leq m_S$, are obtained using an exhaustive search which applies the conditions $(i)-(iv')$ in Theorem 1.

When the level N is a prime number or a composite integer whose dimensions of the composita are all zero, the set $\mathcal{B}_S$ is determined by applying Theorem 1, making appropriate selection of the elements which can build the basis and proceeding as in the fourth paragraph hereafter.

Here is an elegant way to determine $\mathcal{B}_S$ when the level N is not a prime number and a composite integer such that the dimensions of the composita are all zero. In order to achieve this, we apply the following stated and proved by Miyake [11, pp. 153–175]. Let λ be a positive integer and $S_{2k}(\Gamma_0(N),\chi)$ be defined as in Miyake [11, p. 115]. If $f(z)\in S_{2k}(\Gamma_0(N),\chi)$ then $f(\lambda z)\in \mathcal{S}_{2k}(\Gamma_0(\lambda N),\chi)$.

In other words, let the level N be of the form UV, where $0<U,V\in\mathbb{N}$. Furthermore, let $f(z)\in S_{2k}(\Gamma_0(U),\chi)$, $g(z)\in S_{2k}(\Gamma_0(V),\chi)$, and $0<\mu,\nu\in\mathbb{N}$ be such that $\mu U,\nu V\in D(N)$. Then $f(\mu z),g(\nu z)\in\mathcal{S}_{2k}(\Gamma_0(N),\chi)$. In particular, if $f(z)$ and $g(z)$ are basis elements of $S_{2k}(\Gamma_0(U),\chi)$ and $S_{2k}(\Gamma_0(V),\chi)$, respectively, so are $f(\mu z)$ and $g(\nu z)$ basis elements of $S_{2k}(\Gamma_0(N),\chi)$.

We then apply Theorem 1 with the additional condition

$$\sum_{\delta\in D(N)}\frac{\gcd{(\delta,N)}^2}{\delta}\,r_\delta>\max\big(\dim(S_{2k}(\Gamma_0(U))),\dim(S_{2k}(\Gamma_0(V)))\big)$$

to exhaustively search for and determine the rest of the basis elements.

Since the dimension of $S_{2k}(\Gamma_0(\alpha\beta))$ is finite, it suffices to show that the set $\mathcal{B}_S$ is linearly independent. Suppose that $x_i\in\mathbb{C}$ and $\sum\limits_{i=1}^{m_S}x_i\,\mathfrak{B}_{\alpha\beta,i}(q)=0$. Then we apply Remark 2 to obtain $\sum\limits_{i=1}^{m_S}x_i\,\mathfrak{B}_{\alpha\beta,i}(q)=\sum\limits_{n=1}^{\infty}(\sum\limits_{i=1}^{m_S}x_i\,\mathfrak{b}_{\alpha\beta,i}(n)\,)q^n=0$ which gives the following homogeneous system of m_S linear equations in m_S unknowns:

$$\sum_{i=1}^{m_S}\mathfrak{b}_{\alpha\beta,i}(n)\,x_i=0,\qquad 1\leq n\leq m_S.\tag{27}$$

Two cases arise:

The smallest degree of $\mathfrak{B}_{\alpha\beta,i}(q)$ is i for each $1 \le \mathbf{i} \le \mathbf{m_S}$. The square matrix which corresponds to this homogeneous system of m_S linear equations is triangular with 1's on the diagonal. Hence, the determinant of that matrix is 1 and so the unique solution is $x_i = 0$ for all $1 \le i \le m_S$.

The smallest degree of $\mathfrak{B}_{\alpha\beta,i}(q)$ is i for $1 \le \mathbf{i} < \mathbf{m_S}$. Let n' be the largest positive integer such that $1 \le i \le n' < m_S$. Let $\mathcal{B}'_S = \{\, \mathfrak{B}_{\alpha\beta,i}(q) \mid 1 \le i \le n' \,\}$ and $\mathcal{B}''_S = \{\, \mathfrak{B}_{\alpha\beta,i}(q) \mid n' < i \le m_S \,\}$. Then $\mathcal{B}_S = \mathcal{B}'_S \cup \mathcal{B}''_S$ and we may consider $\mathcal{B}_S$ as an ordered set. By the case above, the set $\mathcal{B}'_S$ is linearly independent. Hence, the linear independence of the set $\mathcal{B}_S$ depends on the set $\mathcal{B}''_S$. Let $A = (\mathfrak{b}_{\alpha\beta,i}(n))$ be the $m_S \times m_S$ matrix in (27). If $\det(A) \neq 0$, then $x_i = 0$ for all $1 \le i \le m_S$ and we are done. Suppose that $\det(A) = 0$. Then for some $n' < l \le m_S$ there exists $\mathfrak{B}_{\alpha\beta,l}(q)$ which is causing the system of equations to be inconsistent. We substitute $\mathfrak{B}_{\alpha\beta,l}(q) \in \mathcal{B}''_S$ with, say $\mathfrak{B}'_{\alpha\beta,l}(q)$, which does not occur in $\mathcal{B}''_S$ and compute the determinant of the new matrix A. Since there are finitely many $\mathfrak{B}_{\alpha\beta,l}(q)$ with $n' < l \le m_S$ that may cause the system of linear equations to be inconsistent and finitely many elements of $S_{2k}(\Gamma_0(\alpha\beta)) \setminus \mathcal{B}_S$, the procedure terminates with a consistent system of linear equations. Since Pizer [15] has proved the existence of a basis for the space of cusps, we find a linearly independent set of elements of $S_{2k}(\Gamma_0(\alpha\beta))$.

Therefore, the set $\{\, \mathfrak{B}_{\alpha\beta,i}(q) \mid 1 \le i \le m_S \,\}$ is linearly independent and hence is a basis of $S_{2k}(\Gamma_0(\alpha\beta))$.

(c) Since $M_{2k}(\Gamma_0(\alpha\beta)) = E_{2k}(\Gamma_0(\alpha\beta)) \oplus S_{2k}(\Gamma_0(\alpha\beta))$, the result follows from (a) and (b).

$\square$

If the set $\mathcal{C}$ is empty then the formulation and the proof of Theorem 4 are straightforward.

If the level $\alpha\beta$ belongs to the class $\mathfrak{N}$ then Theorem 4 (a) is provable by induction on the set of positive divisors of $\alpha\beta$; see for example Ntienjem [14]. Note that each positive divisor of $\alpha\beta$ is in $\mathfrak{N}$ whenever the level $\alpha\beta$ belongs to $\mathfrak{N}$. This nice property does not hold in general if the level $\alpha\beta$ belongs to $\mathbb{N}_0 \setminus \mathfrak{N}$. For example 45 is an element of $\mathbb{N}_0 \setminus \mathfrak{N}$; however, 15, which is a positive divisor of 45, does not belong to $\mathbb{N}_0 \setminus \mathfrak{N}$.

The proof of Theorem 4 (b) provides us with an effective and most probably efficient method to determine a basis of the space of cusp forms of weight $2k \in \mathbb{N}$ and of level $\alpha\beta$ whenever $\alpha\beta$ belongs to $\mathbb{N}_0$.

3.2. Evaluating the Convolution Sums $\mathbf{W^{3,3}_{(\alpha,\beta)}(n)}$

We recall that it is sufficient to assume that the primitive Dirichlet character χ is not trivial since the case for which χ is trivial can be deduced as an immediate corollary.

Lemma 2. *Let $\alpha,\beta \in \mathbb{N}_0$ be such that $gcd(\alpha,\beta)=1$. Furthermore, let $\mathcal{B}_M = \mathcal{B}_E \cup \mathcal{B}_S$ be a basis of $M_8(\Gamma_0(\alpha\beta))$. Then there exist $X_\delta, Z(\chi)_s, Y_j \in \mathbb{C}$, where $1 \le j \le m_S, \chi \in \mathcal{C}, s \in D_\chi(\alpha\beta)$ and $\delta \in D(\alpha\beta)$, such that*

$$\begin{aligned}(\alpha\, E_4(q^\alpha) - \beta\, E_4(q^\beta))^2 &= \sum_{\delta|\alpha\beta} X_\delta + \sum_{n=1}^{\infty}\Bigg(480 \sum_{\delta|\alpha\beta}\sigma_7(\frac{n}{\delta})X_\delta \\ &\quad + 480\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha\beta)}\sigma_7(\frac{n}{s})\,Z(\chi)_s + \sum_{j=1}^{m_S}\mathfrak{b}_{\alpha\beta,j}(n)\,Y_j\Bigg)\,q^n. \qquad (28)\end{aligned}$$

Proof. That $(\alpha E_4(q^\alpha) - \beta E_4(q^\beta))^2 \in M_8(\Gamma_0(\alpha\beta))$ follows from Lemma 1. Hence, by Theorem 4 (c), there exist $X_\delta, Z(\chi)_s, Y_j \in \mathbb{C}$, wherein $1 \le j \le m_S, \chi \in \mathcal{C}, s \in D_\chi(\alpha\beta)$ and δ is a divisor of $\alpha\beta$, such that

$$\begin{aligned}(\alpha E_4(q^\alpha) - \beta E_4(q^\beta))^2 &= \sum_{\delta|\alpha\beta} X_\delta\, E_8(q^\delta) + \sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha\beta)} Z(\chi)_s\, E_{8,\chi}(q^s) + \sum_{j=1}^{m_S} Y_j\,\mathfrak{B}_j(q) \\ &= \sum_{\delta|\alpha\beta} X_\delta + \sum_{n=1}^{\infty}\Bigg(480\sum_{\delta|\alpha\beta}\sigma_7(\frac{n}{\delta})\,X_\delta \\ &\quad + 480\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha\beta)}\chi(n)\,\sigma_7(\frac{n}{s})\,Z(\chi)_s + \sum_{j=1}^{m_S}\mathfrak{b}_{\alpha\beta,j}(n)\,Y_j\Bigg)\,q^n.\end{aligned}$$

We equate the right-hand side of (28) with that of (15) to obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}\Bigg(480\sum_{\delta|\alpha\beta}X_\delta\,\sigma_7(\frac{n}{\delta}) + 480\sum_{\chi\in\mathcal{C}}\Bigg(\sum_{s\in D_\chi(\alpha\beta)}\chi(n)\,\sigma_7(\frac{n}{s})\,Z(\chi)_s\Bigg) + \sum_{j=1}^{m_S}Y_j\,\mathfrak{b}_{\alpha\beta,j}(n)\Bigg)\,q^n \\ &= \sum_{n=1}^{\infty}\Bigg(480\,\alpha^2\,\sigma_7(\frac{n}{\alpha}) + 480\,\beta^2\,\sigma_7(\frac{n}{\beta}) - 480\,\alpha\beta\,\sigma_3(\frac{n}{\alpha}) \\ &\quad - 480\,\alpha\beta\,\sigma_3(\frac{n}{\beta}) - 2\times 240^2\,\alpha\beta\,W^{3,3}_{(\alpha,\beta)}(n)\Bigg)q^n.\end{aligned}$$

We then take the coefficients of q^n such that n is in $D(\alpha\beta)$ and $1 \le n \le m_S$, but as many as the unknowns, X_δ for all $0 < \delta|\alpha\beta$, $Z(\chi)_s$ for all $\chi \in \mathcal{C}$ and $s \in D_\chi(\alpha\beta)$, and $Y_1, \ldots, Y_{m_S}$, to obtain a system of $m_E + m_S$ linear equations whose unique solution determines the values of the unknowns. Hence, we obtain the result. $\square$

For the following theorem and for the sake of simplicity, let $X_\delta, Z(\chi)_s$ and Y_j stand for their values obtained in the previous lemma.

Theorem 5. *Let n be a positive integer. Then*

$$\begin{aligned} W_{(\alpha,\beta)}^{3,3}(n) = &- \frac{1}{240\,\alpha\beta} \sum_{\substack{\delta|\alpha\beta\\ \delta\neq\alpha,\,\beta}} X_\delta\,\sigma_7(\frac{n}{\delta}) - \sum_{j=1}^{m_S} \frac{1}{2\times 240^2\,\alpha\beta}\,Y_j\,\mathfrak{b}_{\alpha\beta,j}(n) \\ &+ \frac{1}{240\,\alpha\,\beta}\,(\alpha^2 - X_\alpha)\,\sigma_7(\frac{n}{\alpha}) + \frac{1}{240\,\alpha\,\beta}\,(\beta^2 - X_\beta)\,\sigma_7(\frac{n}{\beta}) \\ &- \frac{1}{2\times 240^2\,\alpha\beta} \sum_{\chi\in\mathcal{C}} \sum_{s\in D_\chi(\alpha\beta)} Z(\chi)_s\,\sigma_7(\frac{n}{s}) \\ &- \frac{1}{240}\,\sigma_3(\frac{n}{\alpha}) - \frac{1}{240}\,\sigma_3(\frac{n}{\beta}). \end{aligned}$$

Proof. We equate the right-hand side of (28) with that of (15) to yield

$$\begin{aligned} 2\times 240^2\,\alpha\beta\,W_{(\alpha,\beta)}^{3,3}(n) = &- 480 \sum_{\delta|\alpha\beta} X_\delta\,\sigma_7(\frac{n}{\delta}) - 480 \sum_{\chi\in\mathcal{C}} \sum_{s\in D_\chi(\alpha\beta)} Z(\chi)_s\,\sigma_7(\frac{n}{s}) \\ &- \sum_{j=1}^{m_S} Y_j\,\mathfrak{b}_{\alpha\beta,j}(n) + 480\,\alpha^2\,\sigma_7(\frac{n}{\alpha}) + 480\,\beta^2\,\sigma_7(\frac{n}{\beta}) \\ &+ 480\,\alpha\beta\,\sigma_3(\frac{n}{\alpha}) + 480\,\alpha\beta\,\sigma_3(\frac{n}{\beta}). \end{aligned}$$

We then solve for $W_{(\alpha,\beta)}^{3,3}(n)$ to obtain the stated result. □

Remark 3. We observe that

$$\frac{1}{240}\,\sigma_3(\frac{n}{\alpha}) + \frac{1}{240}\,\sigma_3(\frac{n}{\beta}),$$

which is part of Theorem 5, depends only on n, α and β; it does not rely on the basis of the modular space $M_8(\Gamma_0(\alpha\beta))$. For all $\chi\in\mathcal{C}$ and for all $s\in D_\chi(\alpha\beta)$ the value of $Z(\chi)_s$ appears to be zero in all explicit examples evaluated as yet. Will the value of $Z(\chi)_s$ always vanish for all $\alpha\beta$ belonging to $\mathbb{N}_0\setminus\mathfrak{N}$?

Now, we have the prerequisite to determine a formula for the number of representations of a positive integer n by a quadratic form.

4. Number of Representations of a Positive Integer for this Class of Levels

We discuss the determination of formulae for the number of representations of a positive integer by the quadratic forms (4) and (5).

4.1. Representations of a Positive Integer by the Quadratic Form (4)

We determine formulae for the number of representations of a positive integer by the quadratic form (4).

4.1.1. Formulae for the Number of Representations by (4)

Let $n \in \mathbb{N}$ and let the number of representations of n by the quadratic form $x_1^2 + x_2^2 + \cdots + x_7^2 + x_8^2$ be

$$r_8(n) = \operatorname{card}(\{(x_1, x_2, \cdots, x_7, x_8) \in \mathbb{Z}^8 \mid n = x_1^2 + x_2^2 + \cdots + x_7^2 + x_8^2\}).$$

It follows from the definition that $r_8(0) = 1$. For each $n \in \mathbb{N}_0$, the identity $r_8(n)$ is

$$r_8(n) = 16\,\sigma_3(n) - 32\,\sigma_3(\frac{n}{2}) + 256\,\sigma_3(\frac{n}{4}) \tag{29}$$

and its arithmetic proof is given by Williams [18].

Now, let the number of representations of n by the quadratic form (4) be

$$\begin{aligned} N^{8,8}_{(a,b)}(n) = \operatorname{card}(\{&(x_1, x_2, \cdots, x_{15}, x_{16}) \in \mathbb{Z}^{16} \mid \\ &n = a\,(x_1^2 + x_2^2 + \cdots + x_7^2 + x_8^2) + b\,(x_9^2 + x_{10}^2 + \cdots + x_{15}^2 + x_{16}^2)\}), \end{aligned}$$

where $a, b \in \mathbb{N}_0$.

It is clear from the definition that for all $a, b \in \mathbb{N}_0$ it holds that $N^{8,8}_{(a,b)}(0) = 1$. It immediately follows from the definition of $N^{8,8}_{(a,b)}(n)$ that $N^{8,8}_{(a,b)}(n) = N^{8,8}_{(b,a)}(n)$. If $a, b \in \mathbb{N}_0$ are such that $\gcd(a, b) = d > 1$ for some $d \in \mathbb{N}_0$ then $N^{8,8}_{(a,b)}(n) = N^{8,8}_{(\frac{a}{d},\frac{b}{d})}(\frac{n}{d})$. Therefore, one may simply assume that $a, b \in \mathbb{N}_0$ are relatively prime.

We then derive the following result:

Theorem 6. *Let $n \in \mathbb{N}_0$ and let $a, b \in \mathbb{N}_0$ be relatively prime. Then*

$$\begin{aligned} N^{8,8}_{(a,b)}(n) = {} & 16\,\sigma_3(\frac{n}{a}) + 16\,\sigma_3(\frac{n}{b}) - 32\,\sigma_3(\frac{n}{2a}) - 32\,\sigma_3(\frac{n}{2b}) + 256\sigma_3(\frac{n}{4a}) + 256\sigma_3(\frac{n}{4b}) \\ & + 256\,W^{3,3}_{(a,b)}(n) - 512\,W^{3,3}_{(a,2b)}(n) + 4096\,W^{3,3}_{(a,4b)}(n) - 512\,W^{3,3}_{(2a,b)}(n) \\ & + 1024\,W^{3,3}_{(a,b)}(\frac{n}{2}) - 8192\,W^{3,3}_{(a,2b)}(\frac{n}{2}) + 4096\,W^{3,3}_{(4a,b)}(n) \\ & - 8192\,W^{3,3}_{(2a,b)}(\frac{n}{2}) + 65536\,W^{3,3}_{(a,b)}(\frac{n}{4}). \end{aligned}$$

Proof. From the definition of $N^{8,8}_{(a,b)}(n)$ it follows that

$$N^{8,8}_{(a,b)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ a\,l+b\,m=n}} r_8(l)r_8(m) = r_8(\frac{n}{a})r_8(0) + r_8(0)r_8(\frac{n}{b}) + \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ a\,l+b\,m=n}} r_8(l)r_8(m).$$

We make use of (29) to obtain

$$N^{8,8}_{(a,b)}(n) = 16\,\sigma_3(\frac{n}{a})+16\,\sigma_3(\frac{n}{b})-32\,\sigma_3(\frac{n}{2a})-32\,\sigma_3(\frac{n}{2b})+256\sigma_3(\frac{n}{4a})+256\sigma_3(\frac{n}{4b})$$
$$+\sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ al+bm=n}}\Big(16\,\sigma_3(l)-32\,\sigma_3(\frac{l}{2})+256\,\sigma_3(\frac{l}{4})\Big)\Big(16\,\sigma_3(m)-32\,\sigma_3(\frac{m}{2})+256\,\sigma_3(\frac{m}{4})\Big).$$

We know that

$$\Big(16\,\sigma_3(l)-32\,\sigma_3(\frac{l}{2})+256\,\sigma_3(\frac{l}{4})\Big)\Big(16\,\sigma_3(m)-32\,\sigma_3(\frac{m}{2})+256\,\sigma_3(\frac{m}{4})\Big)=$$
$$256\sigma_3(l)\sigma_3(m)-512\sigma_3(l)\sigma_3(\frac{m}{2})+4096\sigma_3(l)\sigma_3(\frac{m}{4})$$
$$-512\sigma_3(\frac{l}{2})\sigma_3(m)+1024\sigma_3(\frac{l}{2})\sigma_3(\frac{m}{2})-8192\sigma_3(\frac{l}{2})\sigma_3(\frac{m}{4})$$
$$+4096\sigma_3(\frac{l}{4})\sigma_3(m)-8192\sigma_3(\frac{l}{4})\sigma_3(\frac{m}{2})+65536\,\sigma_3(\frac{l}{4})\sigma_3(\frac{m}{4}).$$

In the sequel of this proof, we assume that the evaluation of

$$W^{3,3}_{(a,b)}(n)=\sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ al+bm=n}}\sigma_3(l)\sigma_3(m),$$

$W^{3,3}_{(2a,b)}(n)$, $W^{3,3}_{(a,2b)}(n)$, $W^{3,3}_{(4a,b)}(n)$ and $W^{3,3}_{(a,4b)}(n)$ are known.

Let $u,v\in\mathbb{N}_0$ and $f,g:\mathbb{N}\mapsto\mathbb{N}$ be injective functions such that $f(n)=u\cdot n$ and $g(n)=v\cdot n$ for each $n\in\mathbb{N}$.

When we simultaneously apply the functions f and g with l and m as argument, respectively, we derive

$$\sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ al+bm=n}}\sigma_3(\frac{l}{u})\sigma_3(\frac{m}{v})=\sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ ua\,l+vb\,m=n}}\sigma_3(l)\sigma_3(m)=W^{3,3}_{(ua,vb)}(n).$$

We set $(u,v)=(1,1),(1,2),(2,1),(2,2),(1,4),(2,4),(4,1),(4,2),(4,4)$, respectively, and put all these evaluations together to obtain the stated result for $N^{8,8}_{(a,b)}(n)$. □

From this proof, one immediately observes that a formula for the number of representations of a positive integer n by the quadratic form (4) depends on the evaluated convolution sums for some given levels ab and $4ab$ with $a,b\in\mathbb{N}_0$.

Based on this observation, we only take into consideration those levels $\alpha\beta$ which are multiple of 4; that is $\alpha\beta\equiv 0 \pmod 4$.

4.1.2. Determination of All Relevant $(\mathbf{a},\mathbf{b}) \in \mathbb{N}_0^2$ for $\mathbf{N}^{8,8}_{(\mathbf{a},\mathbf{b})}(\mathbf{n})$ for a Level $\boldsymbol{\alpha\beta} \in \mathbb{N}_0$

For a given level $\alpha\beta \in \mathbb{N}_0$ such that $\alpha\beta \equiv 0 \pmod 4$ holds, we carry out a method to determine all pairs $(a,b) \in \mathbb{N}_0^2$ which are necessary for the determination of $N^{8,8}_{(a,b)}(n)$.

Let $\Lambda = \frac{\alpha\beta}{4} = 2^{\nu-2}\mho$, $P_4 = \{p_0 = 2^{\nu-2}\} \cup \bigcup_{j>1} \{p_j \mid p_j \text{ is a prime divisor of } \mho\}$ and $\mathcal{P}(P_4)$ be the power set of P_4. Then for each $Q \in \mathcal{P}(P_4)$ we define $\mu(Q) = \prod_{p\in Q} p$. We set $\mu(Q) = 1$ if Q is an empty set. Now let

$$\Omega_4 = \{(\mu(Q_1), \mu(Q_2)) \mid \text{ there exist } Q_1, Q_2 \in \mathcal{P}(P_4) \text{ such that } \gcd(\mu(Q_1), \mu(Q_2)) = 1 \text{ and } \mu(Q_1)\,\mu(Q_2) = \Lambda\}.$$

Observe that $\Omega_4 \neq \emptyset$ since $(1, \Lambda) \in \Omega_4$.

To illustrate our method, suppose that $\alpha\beta = 2^3 \cdot 3 \cdot 5$. Then $\Lambda = 2 \cdot 3 \cdot 5$, $P_4 = \{2, 3, 5\}$ and $\Omega_4 = \{(1, 30), (2, 15), (3, 10), (5, 6)\}$.

Proposition 1. *Suppose that the level $\alpha\beta \in \mathbb{N}_0$ and $\alpha\beta \equiv 0 \pmod 4$. Furthermore, suppose that Ω_4 is defined as above. Then for all $n \in \mathbb{N}_0$ the set Ω_4 contains all pairs $(a,b) \in \mathbb{N}_0^2$ such that $N^{8,8}_{(a,b)}(n)$ can be obtained by applying $W^{3,3}_{(\alpha,\beta)}(n)$ and some other evaluated convolution sums.*

Proof. We prove this by induction on the structure of the level $\alpha\,\beta$.

Suppose that $\alpha\beta = 2^\nu p_2$, where $\nu \geq 2$ and p_2 is an odd prime. Then by the above definitions we have

$$\Lambda = 2^{\nu-2}p_2,$$

$$P_4 = \{\, 2^{\nu-2}, p_2 \,\},$$

$$\mathcal{P}(P_4) = \left\{ \emptyset, \{2^{\nu-2}\}, \{p_2\}, \{\, 2^{\nu-2}, p_2\} \right\} \text{ and}$$

$$\Omega_4 = \{\, (1, 2^{\nu-2}p_2), (2^{\nu-2}, p_2) \,\}.$$

Following the observation made at the end of the proof of Theorem 6, we note that $\alpha\beta = 4ab = 2^\nu p_2$. Hence, $ab = 2^{\nu-2}p_2$ which leads immediately to $N^{8,8}_{(a,b)}(n)$.

We show that Ω_4 is the largest such set. Assume now that there exists another set, say Ω_4', which results from the above definitions. Then there are two cases.

Case $\boldsymbol{\Omega_4'} \subseteq \boldsymbol{\Omega_4}$ There is nothing to show. So, we are done.

Case $\boldsymbol{\Omega_4} \subset \boldsymbol{\Omega_4'}$ Let $(e,f) \in \Omega_4' \setminus \Omega_4$. Since $ef = 2^{\nu-2}p_2$ and $\gcd(e,f) = 1$, we must have either $(e,f) = (1, 2^{\nu-2}p_2)$ or $(e,f) = (2^{\nu-2}, p_2)$. So, $(e,f) \in \Omega_4$. Hence, $\Omega_4 = \Omega_4'$.

Suppose now that $\alpha\beta = 2^\nu p_2 p_3$, where $\nu \geq 2$ and p_2, p_3 are distinct odd primes. Then by the induction hypothesis and by the above definitions we have essentially

$$\Omega_4 = \{\,(1, 2^{\nu-2}p_2p_3), (2^{\nu-2}, p_2p_3), (2^{\nu-2}p_2, p_3), (2^{\nu-2}p_3, p_2)\,\}.$$

One notes that $\alpha\beta = 4ab = 2^\nu p_2p_3$. Hence, $ab = 2^{\nu-2}p_2p_3$ which immediately gives $N^{8,8}_{(a,b)}(n)$.

Again, we show that Ω_4 is the largest such set. Suppose that there exists another set, say Ω_4', which results from the above definitions. Two cases arise.

Case $\mathbf{\Omega_4' \subseteq \Omega_4}$ There is nothing to prove. So, we are done.

Case $\mathbf{\Omega_4 \subset \Omega_4'}$ Let $(e, f) \in \Omega_4' \setminus \Omega_4$. Since $ef = 2^{\nu-2}p_2p_3$ and $\gcd(e, f) = 1$, we must have $(e, f) = (1, 2^{\nu-2}p_2p_3)$ or $(e, f) = (2^{\nu-2}, p_2p_3)$ or $(e, f) = (2^{\nu-2}p_2, p_3)$ or $(e, f) = (2^{\nu-2}p_3, p_2)$. So, $(e, f) \in \Omega_4$. Hence, $\Omega_4 = \Omega_4'$.

□

We then deduce the following.

Corollary 2. *Let $n, \alpha\beta \in \mathbb{N}_0$ with $\alpha\beta \equiv 0 \pmod 4$ and Ω_4 be determined as above. Then for each $(a, b) \in \Omega_4$ it holds that*

$$\begin{aligned} N^{8,8}_{(a,b)}(n) = {}& 16\,\sigma_3(\frac{n}{a}) + 16\,\sigma_3(\frac{n}{b}) - 32\,\sigma_3(\frac{n}{2a}) - 32\,\sigma_3(\frac{n}{2b}) + 256\sigma_3(\frac{n}{4a}) + 256\sigma_3(\frac{n}{4b}) \\ & + 256\,W^{3,3}_{(a,b)}(n) - 512\,W^{3,3}_{(a,2b)}(n) + 4096\,W^{3,3}_{(a,4b)}(n) - 512\,W^{3,3}_{(2a,b)}(n) \\ & + 1024\,W^{3,3}_{(a,b)}(\frac{n}{2}) - 8192\,W^{3,3}_{(a,2b)}(\frac{n}{2}) + 4096\,W^{3,3}_{(4a,b)}(n) \\ & - 8192\,W^{3,3}_{(2a,b)}(\frac{n}{2}) + 65536\,W^{3,3}_{(a,b)}(\frac{n}{4}). \end{aligned}$$

4.2. Representations of a Positive Integer by the Quadratic Form (5)

We now determine formulae for the number of representations of a positive integer by the quadratic form (5).

4.2.1. Formulae for the Number of Representations by (5)

Let $n \in \mathbb{N}$ and let $s_8(n)$ denote the number of representations of n by the quadratic form $\sum_{i=1}^{4}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2)$, that is,

$$s_8(n) = \operatorname{card}(\{(x_1, x_2, \ldots, x_7, x_8) \in \mathbb{Z}^8 \mid n = \sum_{i=1}^{4}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2)\}).$$

It is obvious that $s_8(0)=1$. It is well-known that

$$s_8(n) = 24\,\sigma_3(n) + 216\,\sigma_3(\frac{n}{3}). \tag{30}$$

A proof of this identity is given by Lomadze [10].

Now, let the number of representations of n by the quadratic form (5) be

$$R^{8,8}_{(c,d)}(n) = \mathrm{card}(\{(x_1,x_2,\ldots,x_{15},x_{16}) \in \mathbb{Z}^{16} \mid \\ n = c\sum_{i=1}^{4}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2) + d\sum_{i=5}^{8}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2)\}),$$

where $c,d \in \mathbb{N}_0$.

It is obvious from the definition that for all $c,d \in \mathbb{N}_0$ it holds that $R^{8,8}_{(c,d)}(0)=1$. From this definition of $R^{8,8}_{(c,d)}(n)$, suppose that $c,d \in \mathbb{N}_0$ are such that $\gcd(c,d) = e > 1$ for some $e \in \mathbb{N}_0$. Then $R^{8,8}_{(c,d)}(n) = R^{8,8}_{(\frac{c}{e},\frac{d}{e})}(\frac{n}{e})$ and $R^{8,8}_{(c,d)}(n) = R^{8,8}_{(d,c)}(n)$. Hence, one can simply assume that $c,d \in \mathbb{N}_0$ are relatively prime.

We infer the following.

Theorem 7. *Let $n \in \mathbb{N}_0$ and $c,d \in \mathbb{N}_0$ be relatively prime. Then*

$$R^{8,8}_{(c,d)}(n) = 24\sigma_3(\frac{n}{c}) + 216\sigma_3(\frac{n}{3c}) + 24\sigma_3(\frac{n}{d}) + 216\sigma_3(\frac{n}{3d}) \\ + 576\left(W^{3,3}_{(c,d)}(n) + 9\,W^{3,3}_{(c,3d)}(n) + 9\,W^{3,3}_{(3c,d)}(n) + 81\,W^{3,3}_{(c,d)}(\frac{n}{3})\right).$$

Proof. From the definition of $R^{8,8}_{(c,d)}(n)$ it holds that

$$R^{8,8}_{(c,d)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ cl+dm=n}} s_8(l)s_8(m) = s_8(\frac{n}{c})s_8(0) + s_8(0)s_8(\frac{n}{d}) + \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ cl+dm=n}} s_8(l)s_8(m).$$

We apply (30) to derive

$$R^{8,8}_{(c,d)}(n) = 24\sigma_3(\frac{n}{c}) + 216\sigma_3(\frac{n}{3c}) + 24\sigma_3(\frac{n}{d}) + 216\sigma_3(\frac{n}{3d}) \\ + \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ cl+dm=n}} \left(24\sigma_3(l) + 216\sigma_3(\frac{l}{3})\right)\left(24\sigma_3(m) + 216\sigma_3(\frac{m}{3})\right).$$

We know that

$$\left(24\sigma_3(l) + 216\sigma_3(\frac{l}{3})\right)\left(24\sigma_3(m) + 216\sigma_3(\frac{m}{3})\right) = 576\,\sigma_3(l)\sigma_3(m) + 5184\,\sigma_3(\frac{l}{3})\sigma_3(m)$$

$$+\,5184\,\sigma_3(l)\sigma_3(\frac{m}{3}) + 46656\,\sigma_3(\frac{l}{3})\sigma_3(\frac{m}{3}).$$

We assume that the evaluation of

$$W^{3,3}_{(c,d)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ cl+dm=n}} \sigma_3(l)\sigma_3(m),$$

$W^{3,3}_{(c,3d)}(n)$ and $W^{3,3}_{(3c,d)}(n)$ are known. We set $\lambda = 3$ in the sequel. We apply the function τ to m to derive

$$\sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ cl+dm=n}} \sigma_3(l)\sigma_3(\frac{m}{3}) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ cl+3d\,m=n}} \sigma_3(l)\sigma_3(m) = W^{3,3}_{(c,3d)}(n).$$

Let λ and τ be defined as in Subsection 4.1.1. We make use of the function τ with l as argument to conclude

$$\sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ cl+dm=n}} \sigma_3(m)\sigma_3(\frac{l}{3}) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ 3c\,l+dm=n}} \sigma_3(l)\sigma_3(m) = W^{3,3}_{(3c,d)}(n).$$

We simultaneously apply the function τ to l and to m as arguments, respectively, to infer

$$\sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ cl+dm=n}} \sigma_3(\frac{m}{3})\sigma_3(\frac{l}{3}) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ cl+dm=\frac{n}{3}}} \sigma_3(l)\sigma_3(m) = W^{3,3}_{(c,d)}(\frac{n}{3}).$$

Finally, we bring all these evaluations together to obtain the stated result for $R^{8,8}_{(c,d)}(n)$. $\square$

From this proof, we note that a formula for the number of representations of a positive integer n by the quadratic form (5) depends on the evaluated convolution sums for some given levels cd and $3cd$ with $c, d \in \mathbb{N}_0$.

As a consequence, we do consider only the levels $\alpha\beta$ which are divisible by 3, that is, $\alpha\beta \equiv 0 \pmod 3$.

4.2.2. Determination of All Relevant $(\mathbf{c},\mathbf{d}) \in \mathbb{N}_0^2$ for $\mathbf{R}^{8,8}_{(\mathbf{c},\mathbf{d})}(\mathbf{n})$ for a Level $\boldsymbol{\alpha\beta} \in \mathbb{N}_0$

The following method determines all pairs $(c, d) \in \mathbb{N}_0^2$ necessary for the determination of $R^{8,8}_{(c,d)}(n)$ for a given $\alpha\beta \in \mathbb{N}_0$. This method is quasi-similar to the one in Subsection 4.1.2.

Let $\Delta = \frac{\alpha\beta}{3} = \frac{2^\nu\mho}{3}$. Let $P_3 = \{p_0 = 2^\nu\} \cup \bigcup_{j>2}\{p_j \mid p_j \text{ is a prime divisor of } \mho\}$. Let $\mathcal{P}(P_3)$ be the power set of P_3. Then for each $Q \in \mathcal{P}(P_3)$ we define $\mu(Q) = \prod_{p\in Q} p$.

We set $\mu(Q)=1$ if Q is an empty set. Now let Ω_3 be defined in a similar way as Ω_4 in Subsection 4.1.2; however, with Δ instead of Λ, i.e.,

$$\Omega_3 = \{(\mu(Q_1),\mu(Q_2)) \mid \text{ there exist } Q_1,Q_2\in\mathcal{P}(P_3) \text{ such that } \gcd(\mu(Q_1),\mu(Q_2))=1 \text{ and } \mu(Q_1)\,\mu(Q_2)=\Delta\}.$$

Note that $\Omega_3\neq\emptyset$ since $(1,\Delta)\in\Omega_3$.

As an example, suppose again that $\alpha\beta=2^3\cdot 3\cdot 5$. Then $\Delta=2^3\cdot 5$, $P_3=\{2^3,5\}$ and $\Omega_3=\{(1,40),(5,8)\}$.

The proof of the next proposition is omitted, as it is very similar to that of Proposition 1.

Proposition 2. *Suppose that the level $\alpha\beta\in\mathbb{N}_0$ and $\alpha\beta\equiv 0 \pmod 3$. Suppose in addition that Ω_3 is defined as above. Then for all $n\in\mathbb{N}_0$ the set Ω_3 contains all pairs $(c,d)\in\mathbb{N}_0^2$ such that $R^{8,8}_{(c,d)}(n)$ can be obtained by applying $W_{(\alpha,\beta)}(n)$ and some other evaluated convolution sums.*

We then infer the following.

Corollary 3. *Let $n\in\mathbb{N}_0$, $\alpha\beta\in\mathbb{N}_0$ with $\alpha\beta\equiv 0 \pmod 3$ and Ω_3 be determined as above. Then for each $(c,d)\in\Omega_3$ we obtain*

$$\begin{aligned} R^{8,8}_{(c,d)}(n) =&24\sigma_3(\frac{n}{c})+216\sigma_3(\frac{n}{3c})+24\sigma_3(\frac{n}{d})+216\sigma_3(\frac{n}{3d})\\ &+576\left(W^{3,3}_{(c,d)}(n)+9\,W^{3,3}_{(c,3d)}(n)+9\,W^{3,3}_{(3c,d)}(n)+81\,W^{3,3}_{(c,d)}(\frac{n}{3})\right). \end{aligned}$$

5. Evaluation of the Convolution Sums When $\alpha\beta = 3,5,6,8,10,12,15,20$

In this section, we give explicit formulae for the convolution sum $W^{3,3}_{(\alpha,\beta)}(n)$ when $\alpha\beta=3$, 5, 6, 8, 10, 12, 15 and $\alpha\beta=20$. These levels belong to $\mathfrak{N}$. Hence, the primitive Dirichlet characters are trivial.

The following graphical illustration shows the relationship induced by (7).

In the directed graph illustrating the inclusion relation, we interpret an edge (k,l) with $k,l\in\mathbb{N}_0$ as k is a positive divisor of l; for example $(4,20)$. Only levels whose basis of the space of cusp forms is non-empty are taken into consideration in the representation as a directed graph.

5.1. Bases of $E_8(\Gamma_0(\alpha\beta))$ and $S_8(\Gamma_0(\alpha\beta))$ for $\alpha\beta = 12, 15, 20$

When we apply (7), it suffices to only consider the basis for the levels 12, 15 and 20. Observe, using again (7), that $M_8(\Gamma_0(8))\subset M_8(\Gamma_0(16))\subset M_8(\Gamma_0(32))$. This implies that the bases of $M_8(\Gamma_0(2))$, $M_8(\Gamma_0(4))$ and $M_8(\Gamma_0(8))$ are contained in the basis of $M_8(\Gamma_0(32))$; see the following section.

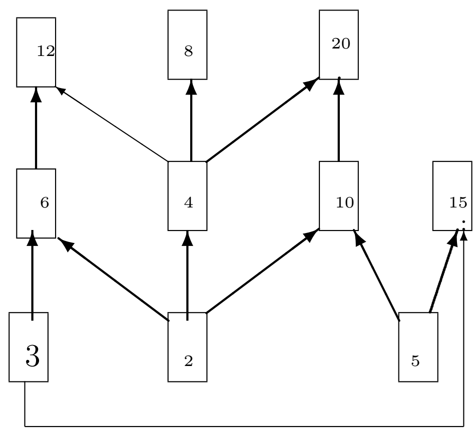

Figure 1: Inclusion relation of the modular space of weight 8 for considered levels

We apply the dimension formulae in Miyake's book [11, Theorem 2.5.2] or in Stein's book [17, Proposition 6.1] to deduce that

$$\dim(S_8(\Gamma_0(12))) = 11,\ \dim(S_8(\Gamma_0(15))) = 12 \ \text{ and } \ \dim(S_8(\Gamma_0(20))) = 18.$$

We use (24) to infer that

$$\dim(E_8(\Gamma_0(12))) = \dim(E_8(\Gamma_0(20))) = 6 \ \text{ and } \ \dim(E_8(\Gamma_0(15))) = 4.$$

We apply Theorem 1 as mentioned in the third paragraph of Subsection 3.1 to determine as many elements of $S_8(\Gamma_0(12))$, $S_8(\Gamma_0(15))$ and $S_8(\Gamma_0(20))$ as possible. Then we apply Remark 2 and (7) when selecting basis elements of a given space of cusp forms as stated in the proof of Theorem 4 (b). Tables containing the powers of the η-quotients are displayed in Appendices.

Corollary 4. ***(a)*** *The sets $\mathcal{B}_{E,12} = \{\, E_8(q^t) \mid t|12 \,\}$, $\mathcal{B}_{E,15} = \{\, E_8(q^t) \mid t|15 \,\}$ and $\mathcal{B}_{E,20} = \{\, E_8(q^t) \mid t|20 \,\}$ are bases of $E_8(\Gamma_0(12))$, $E_8(\Gamma_0(15))$ and $E_8(\Gamma_0(20))$, respectively.*

(b) *Let $j, k, l \in \mathbb{N}_0$ satisfy $1 \le j \le 11$, $1 \le k \le 12$ and $1 \le l \le 18$.*
Let $\delta_2 \in D(12)$ and $(r(j,\delta_2))_{j,\delta_2}$ be Table 2 of the powers of $\eta(\delta_2 z)$.
Let $\delta_3 \in D(15)$ and $(r(k,\delta_3))_{k,\delta_3}$ be Table 3 of the powers of $\eta(\delta_3 z)$.
Let $\delta_4 \in D(20)$ and $(r(l,\delta_4))_{l,\delta_4}$ be Table 5 of the powers of $\eta(\delta_4 z)$.
Furthermore, let $\mathfrak{B}_{12,j}(q) = \prod\limits_{\delta_2|12} \eta^{r(j,\delta_2)}(\delta_2 z)$, $\quad \mathfrak{B}_{15,k}(q) = \prod\limits_{\delta_3|15} \eta^{r(k,\delta_3)}(\delta_3 z)$, and $\mathfrak{B}_{20,l}(q) = \prod\limits_{\delta_4|20} \eta^{r(l,\delta_4)}(\delta_4 z)$ be selected elements of $S_8(\Gamma_0(12))$, $S_8(\Gamma_0(15))$ and $S_8(\Gamma_0(20))$, respectively.

Then the sets $\mathcal{B}_{S,12} = \{\, \mathfrak{B}_{12,j}(q) \mid 1 \le j \le 11 \,\}$, $\mathcal{B}_{S,15} = \{\, \mathfrak{B}_{15,k}(q) \mid 1 \le k \le 12 \,\}$,
and $\mathcal{B}_{S,20} = \{\, \mathfrak{B}_{20,l}(q) \mid 1 \le l \le 18 \,\}$ *are bases of* $S_8(\Gamma_0(12))$, $S_8(\Gamma_0(15))$ *and* $S_8(\Gamma_0(20))$*, repectively.*

(c) *The sets* $\mathcal{B}_{M,12} = \mathcal{B}_{E,12} \cup \mathcal{B}_{S,12}$, $\mathcal{B}_{M,15} = \mathcal{B}_{E,15} \cup \mathcal{B}_{S,15}$ *and* $\mathcal{B}_{M,20} = \mathcal{B}_{E,20} \cup \mathcal{B}_{S,20}$ *constitute bases of* $M_8(\Gamma_0(12))$, $M_8(\Gamma_0(15))$ *and* $M_8(\Gamma_0(20))$*, respectively.*

By Remark 2, the basis elements $\mathfrak{B}_{12,j}(q)$, $\mathfrak{B}_{15,k}(q)$ and $\mathfrak{B}_{20,l}(q)$ can be expressed in the form $\sum_{n=1}^{\infty} \mathfrak{b}_{12,j}(n)q^n$, $\sum_{n=1}^{\infty} \mathfrak{b}_{15,k}(n)q^n$ and $\sum_{n=1}^{\infty} \mathfrak{b}_{20,l}(n)q^n$, respectively. In these expressions, $\mathfrak{b}_{12,j}(n)$, $\mathfrak{b}_{15,k}(n)$ and $\mathfrak{b}_{20,l}(n)$ belong to the set of integers.

Proof of Corollary 4. It follows immediately from Theorem 4. $\square$

5.2. Evaluation of $\mathbf{W}^{3,3}_{(\alpha,\beta)}(\mathbf{n})$ When $\alpha\beta = 3, 5, 6, 8, 10, 12, 15, 20$

A basis of the space $S_8(\Gamma_0(3))$ is $\mathcal{B}_{S,3} = \{\, \mathfrak{B}_{18,1}(q) = \eta^{12}(z)\,\eta^4(3\,z) \,\}$. We are able to determine it by omitting the condition (ii) in Theorem 1. Therefore, this basis element does belong to $M_8(\Gamma_0(3))$, and so to $S_8(\Gamma_0(3))$ as per Remark 1, provided $n \equiv 0 \pmod 3$ for all $n \in \mathbb{N}_0$.

Corollary 5. *The following equations are derived.*

$$(E_4(q) - 2\,E_4(q^2))^2 = 1 + \sum_{n=1}^{\infty}\Big(\frac{13}{17}\,\sigma_7(n) + \frac{4}{17}\,\sigma_7(\frac{n}{2}) - \frac{14400}{17}\,\mathfrak{b}_{32,1}(n) \Big) q^n, \quad (31)$$

$$(E_4(q) - 3\,E_4(q^3))^2 = 4 + \sum_{n=1}^{\infty}\Big(\frac{38}{41}\,\sigma_7(n) + \frac{126}{41}\,\sigma_7(\frac{n}{3}) + \frac{28800}{41}\,\mathfrak{b}_{18,1}(n) \Big) q^n, \quad (32)$$

$$(E_4(q) - 4\,E_4(q^4))^2 = 9 + \sum_{n=1}^{\infty}\Big(\frac{33}{34}\,\sigma_7(n) - \frac{15}{34}\,\sigma_7(\frac{n}{2}) + \frac{144}{17}\,\sigma_7(\frac{n}{4}) - \frac{32400}{17}\,\mathfrak{b}_{32,1}(n) - \frac{518400}{17}\,\mathfrak{b}_{32,2}(n) \Big) q^n, \quad (33)$$

$$(E_4(q) - 5\,E_4(q^5))^2 = 16 + \sum_{n=1}^{\infty}\Big(\frac{308}{313}\,\sigma_7(n) + \frac{4700}{313}\,\sigma_7(\frac{n}{5}) - \frac{748800}{313}\,\mathfrak{b}_{15,1}(n) - \frac{16934400}{313}\,\mathfrak{b}_{15,2}(n) - \frac{93600000}{313}\,\mathfrak{b}_{15,3}(n) \Big) q^n, \quad (34)$$

$$
\begin{aligned}
(E_4(q) - 6\,E_4(q^6))^2 = 25 + \sum_{n=1}^{\infty}\Big(&\frac{691}{697}\,\sigma_7(n) - \frac{96}{697}\,\sigma_7(\frac{n}{2}) - \frac{486}{697}\,\sigma_7(\frac{n}{3}) \\
&+ \frac{17316}{697}\,\sigma_7(\frac{n}{6}) - \frac{2004480}{697}\,\mathfrak{b}_{12,1}(n) - \frac{1981440}{41}\,\mathfrak{b}_{12,2}(n) - \frac{160355520}{697}\,\mathfrak{b}_{12,3}(n) \\
&- 46080\,\mathfrak{b}_{12,4}(n) + 184320\,\mathfrak{b}_{12,5}(n)\Big)\,q^n, \quad (35)
\end{aligned}
$$

$$
\begin{aligned}
(2\,E_4(q^2) - 3\,E_4(q^3))^2 = 1 + \sum_{n=1}^{\infty}\Big(&-\frac{6}{697}\,\sigma_7(n) + \frac{2692}{697}\,\sigma_7(\frac{n}{2}) \\
&+ \frac{5787}{697}\,\sigma_7(\frac{n}{3}) - \frac{7776}{697}\,\sigma_7(\frac{n}{6}) + \frac{2880}{697}\,\mathfrak{b}_{12,1}(n) - \frac{92160}{41}\,\mathfrak{b}_{12,2}(n) \\
&- \frac{1774080}{697}\,\mathfrak{b}_{12,3}(n) + 46080\,\mathfrak{b}_{12,4}(n) - 184320\,\mathfrak{b}_{12,5}(n)\Big)\,q^n, \quad (36)
\end{aligned}
$$

$$
\begin{aligned}
(E_4(q) - 8\,E_4(q^8))^2 = 49 + \sum_{n=1}^{\infty}\Big(&\frac{271}{272}\,\sigma_7(n) - \frac{15}{272}\,\sigma_7(\frac{n}{2}) - \frac{15}{17}\,\sigma_7(\frac{n}{4}) + \frac{832}{17}\,\sigma_7(\frac{n}{8}) \\
&- \frac{65250}{17}\,\mathfrak{b}_{32,1}(n) - \frac{1105200}{17}\,\mathfrak{b}_{32,2}(n) - 57600\,\mathfrak{b}_{32,3}(n) \\
&- \frac{16704000}{17}\,\mathfrak{b}_{32,4}(n) - \frac{66816000}{17}\,\mathfrak{b}_{32,5}(n)\Big)\,q^n, \quad (37)
\end{aligned}
$$

$$
\begin{aligned}
(E_4(q) - 10\,E_4(q^{10}))^2 = 81 + \sum_{n=1}^{\infty}\Big(&\frac{59454261}{65613251}\,\sigma_7(n) + \frac{1543161120}{65613251}\,\sigma_7(\frac{n}{2}) \\
&- \frac{71033070}{65613251}\,\sigma_7(\frac{n}{5}) + \frac{3783091020}{65613251}\,\sigma_7(\frac{n}{10}) - \frac{311987289600}{65613251}\,\mathfrak{b}_{20,1}(n) \\
&- \frac{334690790400}{3859603}\,\mathfrak{b}_{20,2}(n) - \frac{2595535488000}{3859603}\,\mathfrak{b}_{20,3}(n) - \frac{1424231654400}{3859603}\,\mathfrak{b}_{20,4}(n) \\
&+ \frac{81674848872000}{65613251}\,\mathfrak{b}_{20,5}(n) - \frac{82312327296000}{3859603}\,\mathfrak{b}_{20,6}(n) + \frac{564939648000}{12331}\,\mathfrak{b}_{20,7}(n) \\
&+ \frac{31190957568000}{3859603}\,\mathfrak{b}_{20,8}(n) - \frac{27189941760000}{3859603}\,\mathfrak{b}_{20,9}(n)\Big)\,q^n, \quad (38)
\end{aligned}
$$

$$
\begin{aligned}
(\,2\,E_4(q^2) - 5\,E_4(q^5))^2 = 9 + \sum_{n=1}^{\infty}\Big(& -\frac{49230}{65613251}\,\sigma_7(n) + \frac{241515564}{65613251}\,\sigma_7(\frac{n}{2}) \\
& + \frac{1563188445}{65613251}\,\sigma_7(\frac{n}{5}) - \frac{1214135520}{65613251}\,\sigma_7(\frac{n}{10}) + \frac{23630400}{65613251}\,\mathfrak{b}_{20,1}(n) \\
& - \frac{17744486400}{3859603}\,\mathfrak{b}_{20,2}(n) - \frac{138931200000}{3859603}\,\mathfrak{b}_{20,3}(n) + \frac{9418521600}{3859603}\,\mathfrak{b}_{20,4}(n) \\
& - \frac{788634432000}{65613251}\,\mathfrak{b}_{20,5}(n) + \frac{278326656000}{3859603}\,\mathfrak{b}_{20,6}(n) - \frac{6933888000}{12331}\,\mathfrak{b}_{20,7}(n) \\
& + \frac{5667945984000}{3859603}\,\mathfrak{b}_{20,8}(n) - \frac{10401523200000}{3859603}\,\mathfrak{b}_{20,9}(n)\Big)\,q^n, \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
(\,E_4(q) - 12\,E_4(q^{12}))^2 = 121 + \sum_{n=1}^{\infty}\Big(& \frac{2785}{2788}\,\sigma_7(n) - \frac{45}{2788}\,\sigma_7(\frac{n}{2}) - \frac{243}{2788}\,\sigma_7(\frac{n}{3}) \\
& - \frac{192}{697}\,\sigma_7(\frac{n}{4}) - \frac{3645}{2788}\,\sigma_7(\frac{n}{6}) + \frac{84816}{697}\,\sigma_7(\frac{n}{12}) - \frac{4014360}{697}\,\mathfrak{b}_{12,1}(n) \\
& - \frac{68195520}{697}\,\mathfrak{b}_{12,2}(n) - \frac{336205080}{697}\,\mathfrak{b}_{12,3}(n) - \frac{165813120}{697}\,\mathfrak{b}_{12,4}(n) \\
& - \frac{454273920}{697}\,\mathfrak{b}_{12,5}(n) - \frac{5856330240}{697}\,\mathfrak{b}_{12,6}(n) + \frac{6855632640}{697}\,\mathfrak{b}_{12,7}(n) \\
& - \frac{14998348800}{697}\,\mathfrak{b}_{12,8}(n) - \frac{6908613120}{697}\,\mathfrak{b}_{12,9}(n) - \frac{1926696960}{697}\,\mathfrak{b}_{12,10}(n) \\
& - \frac{1874085120}{697}\,\mathfrak{b}_{12,11}(n)\Big)\,q^n, \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
(\,3\,E_4(q^3) - 4\,E_4(q^4))^2 = 1 + \sum_{n=1}^{\infty}\Big(& -\frac{3}{2788}\,\sigma_7(n) - \frac{45}{2788}\,\sigma_7(\frac{n}{2}) \\
& + \frac{24849}{2788}\,\sigma_7(\frac{n}{3}) + \frac{10960}{697}\,\sigma_7(\frac{n}{4}) - \frac{3645}{2788}\,\sigma_7(\frac{n}{6}) - \frac{15552}{697}\,\sigma_7(\frac{n}{12}) + \frac{360}{697}\,\mathfrak{b}_{12,1}(n) \\
& + \frac{54720}{697}\,\mathfrak{b}_{12,2}(n) - \frac{2983320}{697}\,\mathfrak{b}_{12,3}(n) + \frac{2805120}{697}\,\mathfrak{b}_{12,4}(n) + \frac{35521920}{697}\,\mathfrak{b}_{12,5}(n) \\
& + \frac{262103040}{697}\,\mathfrak{b}_{12,6}(n) - \frac{1189866240}{697}\,\mathfrak{b}_{12,7}(n) + \frac{1381708800}{697}\,\mathfrak{b}_{12,8}(n) \\
& + \frac{2052241920}{697}\,\mathfrak{b}_{12,9}(n) - \frac{2183639040}{697}\,\mathfrak{b}_{12,10}(n) + \frac{1064689920}{697}\,\mathfrak{b}_{12,11}(n)\Big)\,q^n, \qquad (41)
\end{aligned}
$$

$$\begin{aligned}
(\,E_4(q)-15\,E_4(q^{15}))^2 = 196+\sum_{n=1}^{\infty}\Big(&\frac{1017763622}{1044336707}\,\sigma_7(n)-\frac{75000}{3336539}\,\sigma_7(\frac{n}{3})\\
&+\frac{10141314825000}{1044336707}\,\sigma_7(\frac{n}{5})-\frac{31749581850}{3336539}\,\sigma_7(\frac{n}{15})-\frac{7506469209600}{1044336707}\,\mathfrak{b}_{15,1}(n)\\
&-\frac{171118182124800}{1044336707}\,\mathfrak{b}_{15,2}(n)-\frac{973566223200000}{1044336707}\,\mathfrak{b}_{15,3}(n)\\
&-\frac{732348000000}{3336539}\,\mathfrak{b}_{15,4}(n)-\frac{18505802419200}{3336539}\,\mathfrak{b}_{15,5}(n)\\
&-\frac{41495373446400}{3336539}\,\mathfrak{b}_{15,6}(n)+\frac{2622520108800}{3336539}\,\mathfrak{b}_{15,7}(n)\\
&+\frac{18375785625600}{3336539}\,\mathfrak{b}_{15,8}(n)+\frac{187616606745600}{3336539}\,\mathfrak{b}_{15,9}(n)\\
&-\frac{1518886974240000}{3336539}\,\mathfrak{b}_{15,10}(n)-\frac{3849572436710400}{3336539}\,\mathfrak{b}_{15,11}(n)\\
&-\frac{46658163801600}{81379}\,\mathfrak{b}_{15,12}(n)\Big)\,q^n, \qquad (42)
\end{aligned}$$

$$\begin{aligned}
(\,3\,E_4(q^3)-5\,E_4(q^5))^2 = 4+\sum_{n=1}^{\infty}\Big(&\frac{23283120}{1044336707}\,\sigma_7(n)+\frac{29794566}{3336539}\,\sigma_7(\frac{n}{3})\\
&-\frac{9307656835450}{1044336707}\,\sigma_7(\frac{n}{5})+\frac{29720400000}{3336539}\,\sigma_7(\frac{n}{15})-\frac{11175897600}{1044336707}\,\mathfrak{b}_{15,1}(n)\\
&-\frac{1598153356800}{1044336707}\,\mathfrak{b}_{15,2}(n)-\frac{43861572000000}{1044336707}\,\mathfrak{b}_{15,3}(n)\\
&-\frac{774254592000}{3336539}\,\mathfrak{b}_{15,4}(n)+\frac{11360058624000}{3336539}\,\mathfrak{b}_{15,5}(n)\\
&+\frac{12396170880000}{3336539}\,\mathfrak{b}_{15,6}(n)-\frac{19221873696000}{3336539}\,\mathfrak{b}_{15,7}(n)\\
&-\frac{94553354592000}{3336539}\,\mathfrak{b}_{15,8}(n)-\frac{415246609728000}{3336539}\,\mathfrak{b}_{15,9}(n)\\
&+\frac{1324243124064000}{3336539}\,\mathfrak{b}_{15,10}(n)+\frac{3711541061952000}{3336539}\,\mathfrak{b}_{15,11}(n)\\
&+\frac{42939419328000}{81379}\,\mathfrak{b}_{15,12}(n)\Big)\,q^n, \qquad (43)
\end{aligned}$$

$$
\begin{aligned}
(\,E_4(q) - 20\,E_4(q^{20}))^2 = 361 + \sum_{n=1}^{\infty}\Big(& \frac{349063}{351888}\,\sigma_7(n) + \frac{220208065}{110140944}\,\sigma_7(\frac{n}{2}) \\
& - \frac{45200}{21993}\,\sigma_7(\frac{n}{4}) - \frac{831695}{5982096}\,\sigma_7(\frac{n}{5}) - \frac{7873822505}{1872396048}\,\sigma_7(\frac{n}{10}) \\
& + \frac{136245280}{373881}\,\sigma_7(\frac{n}{20}) - \frac{70349350}{7331}\,\mathfrak{b}_{20,1}(n) - \frac{375469902000}{2294603}\,\mathfrak{b}_{20,2}(n) \\
& - \frac{3336969500800}{2294603}\,\mathfrak{b}_{20,3}(n) + \frac{500846721600}{2294603}\,\mathfrak{b}_{20,4}(n) \\
& - \frac{170775562301850}{39008251}\,\mathfrak{b}_{20,5}(n) - \frac{25589550927200}{2294603}\,\mathfrak{b}_{20,6}(n) \\
& + \frac{46439225094400}{2294603}\,\mathfrak{b}_{20,7}(n) + \frac{366667176043200}{2294603}\,\mathfrak{b}_{20,8}(n) \\
& - \frac{1163337163523200}{2294603}\,\mathfrak{b}_{20,9}(n) - \frac{15436341936000}{124627}\,\mathfrak{b}_{20,10}(n) \\
& + \frac{1066465872000}{7331}\,\mathfrak{b}_{20,11}(n) - \frac{2089807776000}{7331}\,\mathfrak{b}_{20,12}(n) \\
& + \frac{7405984992000}{7331}\,\mathfrak{b}_{20,13}(n) - \frac{6563369184000}{7331}\,\mathfrak{b}_{20,14}(n) \\
& + \frac{29137235712000}{7331}\,\mathfrak{b}_{20,15}(n) - \frac{82106975872000}{7331}\,\mathfrak{b}_{20,16}(n) \\
& + \frac{419141760000}{7331}\,\mathfrak{b}_{20,17}(n) + \frac{70301805696000}{7331}\,\mathfrak{b}_{20,18}(n)\Big)\,q^n, \quad (44)
\end{aligned}
$$

$$
\begin{aligned}
(\,4\,E_4(q^4) - 5\,E_4(q^5))^2 = 1 + \sum_{n=1}^{\infty}\Big(& -\frac{145}{351888}\,\sigma_7(n) + \frac{4626185}{110140944}\,\sigma_7(\frac{n}{2}) \\
& + \frac{349568}{21993}\,\sigma_7(\frac{n}{4}) + \frac{148675145}{5982096}\,\sigma_7(\frac{n}{5}) - \frac{4208930545}{1872396048}\,\sigma_7(\frac{n}{10}) \\
& - \frac{14036080}{373881}\,\sigma_7(\frac{n}{20}) + \frac{1450}{7331}\,\mathfrak{b}_{20,1}(n) + \frac{15915600}{2294603}\,\mathfrak{b}_{20,2}(n) \\
& + \frac{1114902400}{2294603}\,\mathfrak{b}_{20,3}(n) - \frac{18502910400}{2294603}\,\mathfrak{b}_{20,4}(n) + \frac{251617430550}{39008251}\,\mathfrak{b}_{20,5}(n) \\
& - \frac{122216284000}{2294603}\,\mathfrak{b}_{20,6}(n) + \frac{371298156800}{2294603}\,\mathfrak{b}_{20,7}(n) + \frac{3862407710400}{2294603}\,\mathfrak{b}_{20,8}(n) \\
& - \frac{10638944566400}{2294603}\,\mathfrak{b}_{20,9}(n) - \frac{244946966400}{124627}\,\mathfrak{b}_{20,10}(n) + \frac{51017040000}{7331}\,\mathfrak{b}_{20,11}(n) \\
& - \frac{38416492800}{7331}\,\mathfrak{b}_{20,12}(n) + \frac{150328032000}{7331}\,\mathfrak{b}_{20,13}(n) - \frac{136435526400}{7331}\,\mathfrak{b}_{20,14}(n) \\
& + \frac{662308608000}{7331}\,\mathfrak{b}_{20,15}(n) - \frac{1724891571200}{7331}\,\mathfrak{b}_{20,16}(n) + \frac{6566016000}{7331}\,\mathfrak{b}_{20,17}(n) \\
& + \frac{1240614835200}{7331}\,\mathfrak{b}_{20,18}(n)\Big)\,q^n. \quad (45)
\end{aligned}
$$

Proof. These equations follow immediately when one sets $(\alpha,\beta) = (1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,3)$, $(1,8)$,$(1,10)$, $(2,5)$, $(1,12)$, $(3,4)$, $(1,15)$, $(3,5)$, $(1,20)$, and $(4,5)$ in Lemma 2. As an example let $\alpha\beta = 12$. Then take all n in $\{1,2,\dots,16,24\}$ to obtain a system of 17 linear equations with unknowns X_δ and Y_j in each case $(1,12)$ and $(3,4)$, where $\delta \in D(12)$ and $1 \le j \le 11$. These systems of linear equations each have a unique solution displayed in (40) and (41), respectively. □

Now, we are in a better posture to state and prove our main result of this section.

Corollary 6. *Let n be a positive integer. Then*

$$W^{3,3}_{(1,2)}(n) = \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\,\sigma_7(\frac{n}{2}) - \frac{1}{240}\,\sigma_3(n) - \frac{1}{240}\,\sigma_3(\frac{n}{2}) + \frac{1}{272}\,\mathfrak{b}_{32,1}(n), \tag{46}$$

$$\begin{aligned} W^{3,3}_{(1,3)}(n) = {} & \frac{1}{9840}\sigma_7(n) + \frac{27}{3280}\,\sigma_7(\frac{n}{3}) - \frac{1}{240}\,\sigma_3(n) - \frac{1}{240}\,\sigma_3(\frac{n}{3}) \\ & - \frac{1}{492}\,\mathfrak{b}_{18,1}(n) \qquad \textit{if } n \equiv 0 \pmod 3, \end{aligned} \tag{47}$$

$$\begin{aligned} W^{3,3}_{(1,4)}(n) = {} & \frac{1}{32640}\,\sigma_7(n) + \frac{1}{2176}\,\sigma_7(\frac{n}{2}) + \frac{2}{255}\,\sigma_7(\frac{n}{4}) - \frac{1}{240}\,\sigma_3(n) \\ & - \frac{1}{240}\,\sigma_3(\frac{n}{4}) + \frac{9}{2176}\,\mathfrak{b}_{32,1}(n) + \frac{9}{136}\,\mathfrak{b}_{32,2}(n), \end{aligned} \tag{48}$$

$$\begin{aligned} W^{3,3}_{(1,5)}(n) = {} & \frac{1}{75120}\,\sigma_7(n) + \frac{125}{15024}\,\sigma_7(\frac{n}{5}) - \frac{1}{240}\,\sigma_3(n) - \frac{1}{240}\,\sigma_3(\frac{n}{5}) \\ & + \frac{13}{3130}\,\mathfrak{b}_{15,1}(n) + \frac{147}{1565}\,\mathfrak{b}_{15,2}(n) + \frac{325}{626}\,\mathfrak{b}_{15,3}(n), \end{aligned} \tag{49}$$

$$\begin{aligned} W^{3,3}_{(1,6)}(n) = {} & \frac{1}{167280}\,\sigma_7(n) + \frac{1}{10455}\,\sigma_7(\frac{n}{2}) + \frac{27}{55760}\,\sigma_7(\frac{n}{3}) + \frac{27}{3485}\,\sigma_7(\frac{n}{6}) \\ & - \frac{1}{240}\,\sigma_3(n) - \frac{1}{240}\,\sigma_3(\frac{n}{6}) + \frac{29}{6970}\,\mathfrak{b}_{12,1}(n) + \frac{43}{615}\,\mathfrak{b}_{12,2}(n) \\ & + \frac{55679}{167280}\,\mathfrak{b}_{12,3}(n) + \frac{1}{15}\,\mathfrak{b}_{12,4}(n) - \frac{4}{15}\,\mathfrak{b}_{12,5}(n), \end{aligned} \tag{50}$$

$$\begin{aligned} W^{3,3}_{(2,3)}(n) = {} & \frac{1}{167280}\,\sigma_7(n) + \frac{1}{10455}\,\sigma_7(\frac{n}{2}) + \frac{27}{55760}\,\sigma_7(\frac{n}{3}) + \frac{27}{3485}\,\sigma_3(\frac{n}{6}) \\ & - \frac{1}{240}\,\sigma_3(\frac{n}{2}) - \frac{1}{240}\,\sigma_3(\frac{n}{3}) - \frac{1}{167280}\,\mathfrak{b}_{12,1}(n) + \frac{2}{615}\,\mathfrak{b}_{12,2}(n) \\ & + \frac{77}{20910}\,\mathfrak{b}_{12,3}(n) - \frac{1}{15}\,\mathfrak{b}_{12,4}(n) + \frac{4}{15}\,\mathfrak{b}_{12,5}(n), \end{aligned} \tag{51}$$

$$\begin{aligned}W^{3,3}_{(1,8)}(n) =&\frac{1}{522240}\,\sigma_7(n)+\frac{1}{34816}\,\sigma_7(\frac{n}{2})+\frac{1}{2176}\,\sigma_7(\frac{n}{4})+\frac{2}{255}\,\sigma_7(\frac{n}{8})\\&-\frac{1}{240}\,\sigma_3(n)-\frac{1}{240}\,\sigma_3(\frac{n}{8})+\frac{145}{34816}\,\mathfrak{b}_{32,1}(n)+\frac{307}{4352}\,\mathfrak{b}_{32,2}(n)\\&+\frac{1}{16}\,\mathfrak{b}_{32,3}(n)+\frac{145}{136}\,\mathfrak{b}_{32,4}(n)+\frac{145}{34}\,\mathfrak{b}_{32,5}(n),\end{aligned}\tag{52}$$

$$\begin{aligned}W^{3,3}_{(1,10)}(n) =&\frac{615899}{15747180240}\,\sigma_7(n)-\frac{3214919}{328066255}\,\sigma_7(\frac{n}{2})+\frac{2367769}{5249060080}\,\sigma_7(\frac{n}{5})\\&+\frac{17363963}{984198765}\,\sigma_7(\frac{n}{10})-\frac{1}{240}\,\sigma_3(n)-\frac{1}{240}\,\sigma_3(\frac{n}{10})\\&+\frac{2708223}{656132510}\,\mathfrak{b}_{20,1}(n)+\frac{1452651}{19298015}\,\mathfrak{b}_{20,2}(n)+\frac{2253069}{3859603}\,\mathfrak{b}_{20,3}(n)\\&+\frac{6181561}{19298015}\,\mathfrak{b}_{20,4}(n)-\frac{1134372901}{1049812016}\,\mathfrak{b}_{20,5}(n)+\frac{71451673}{3859603}\,\mathfrak{b}_{20,6}(n)\\&-\frac{490399}{12331}\,\mathfrak{b}_{20,7}(n)-\frac{27075484}{3859603}\,\mathfrak{b}_{20,8}(n)+\frac{23602380}{3859603}\,\mathfrak{b}_{20,9}(n),\end{aligned}\tag{53}$$

$$\begin{aligned}W^{3,3}_{(2,5)}(n) =&\frac{1641}{5249060080}\,\sigma_7(n)+\frac{130859}{984198765}\,\sigma_7(\frac{n}{2})+\frac{7714283}{15747180240}\,\sigma_7(\frac{n}{5})\\&+\frac{2529449}{328066255}\,\sigma_7(\frac{n}{10})-\frac{1}{240}\,\sigma_3(\frac{n}{2})-\frac{1}{240}\,\sigma_3(\frac{n}{5})\\&-\frac{1641}{5249060080}\,\mathfrak{b}_{20,1}(n)+\frac{77016}{19298015}\,\mathfrak{b}_{20,2}(n)+\frac{120600}{3859603}\,\mathfrak{b}_{20,3}(n)\\&-\frac{40879}{19298015}\,\mathfrak{b}_{20,4}(n)+\frac{1369157}{131226502}\,\mathfrak{b}_{20,5}(n)-\frac{241603}{3859603}\,\mathfrak{b}_{20,6}(n)\\&+\frac{6019}{12331}\,\mathfrak{b}_{20,7}(n)-\frac{4920092}{3859603}\,\mathfrak{b}_{20,8}(n)+\frac{9029100}{3859603}\,\mathfrak{b}_{20,9}(n),\end{aligned}\tag{54}$$

$$\begin{aligned}W^{3,3}_{(1,12)}(n) =&\frac{1}{2676480}\,\sigma_7(n)+\frac{1}{178432}\,\sigma_7(\frac{n}{2})+\frac{27}{892160}\,\sigma_7(\frac{n}{3})+\frac{1}{10455}\,\sigma_7(\frac{n}{4})\\&+\frac{81}{178432}\,\sigma_7(\frac{n}{6})+\frac{27}{3485}\,\sigma_7(\frac{n}{12})-\frac{1}{240}\,\sigma_3(n)-\frac{1}{240}\,\sigma_3(\frac{n}{12})\\&+\frac{3717}{892160}\,\mathfrak{b}_{12,1}(n)+\frac{7893}{111520}\,\mathfrak{b}_{12,2}(n)+\frac{311301}{892160}\,\mathfrak{b}_{12,3}(n)+\frac{28787}{167280}\,\mathfrak{b}_{12,4}(n)\\&+\frac{26289}{55760}\,\mathfrak{b}_{12,5}(n)+\frac{84727}{13940}\,\mathfrak{b}_{12,6}(n)-\frac{198369}{27880}\,\mathfrak{b}_{12,7}(n)+\frac{21699}{1394}\,\mathfrak{b}_{12,8}(n)\\&+\frac{99951}{13940}\,\mathfrak{b}_{12,9}(n)+\frac{20906}{10455}\,\mathfrak{b}_{12,10}(n)+\frac{54227}{27880}\,\mathfrak{b}_{12,11}(n),\end{aligned}\tag{55}$$

$$\begin{aligned}W^{3,3}_{(3,4)}(n) =&\frac{1}{2676480}\,\sigma_7(n)+\frac{1}{178432}\,\sigma_7(\frac{n}{2})+\frac{27}{892160}\,\sigma_7(\frac{n}{3})+\frac{1}{10455}\,\sigma_7(\frac{n}{4})\\&+\frac{81}{178432}\,\sigma_7(\frac{n}{6})+\frac{27}{3485}\,\sigma_7(\frac{n}{12})-\frac{1}{240}\,\sigma_3(\frac{n}{3})-\frac{1}{240}\,\sigma_3(\frac{n}{4})\\&-\frac{1}{2676480}\,\mathfrak{b}_{12,1}(n)-\frac{19}{334560}\,\mathfrak{b}_{12,2}(n)+\frac{8287}{2676480}\,\mathfrak{b}_{12,3}(n)-\frac{487}{167280}\,\mathfrak{b}_{12,4}(n)\\&-\frac{6167}{167280}\,\mathfrak{b}_{12,5}(n)-\frac{948}{3485}\,\mathfrak{b}_{12,6}(n)+\frac{34429}{27880}\,\mathfrak{b}_{12,7}(n)-\frac{1999}{1394}\,\mathfrak{b}_{12,8}(n)\\&-\frac{29691}{13940}\,\mathfrak{b}_{12,9}(n)+\frac{7898}{3485}\,\mathfrak{b}_{12,10}(n)-\frac{30807}{27880}\,\mathfrak{b}_{12,11}(n),\end{aligned}\tag{56}$$

$$\begin{aligned}W^{3,3}_{(1,15)}(n) =&\frac{590513}{83546936560}\,\sigma_7(n)+\frac{125}{20019234}\,\sigma_7(\frac{n}{3})-\frac{16902191375}{6266020242}\,\sigma_7(\frac{n}{5})\\&+\frac{433337375}{160153872}\,\sigma_7(\frac{n}{15})-\frac{1}{240}\,\sigma_3(n)-\frac{1}{240}\,\sigma_3(\frac{n}{12})\\&+\frac{65160323}{15665050605}\,\mathfrak{b}_{15,1}(n)+\frac{2970801773}{31330101210}\,\mathfrak{b}_{15,2}(n)+\frac{3380438275}{6266020242}\,\mathfrak{b}_{15,3}(n)\\&+\frac{847625}{6673078}\,\mathfrak{b}_{15,4}(n)+\frac{53546882}{16682695}\,\mathfrak{b}_{15,5}(n)+\frac{240135263}{33365390}\,\mathfrak{b}_{15,6}(n)\\&-\frac{15176621}{33365390}\,\mathfrak{b}_{15,7}(n)-\frac{53170676}{16682695}\,\mathfrak{b}_{15,8}(n)-\frac{542872126}{16682695}\,\mathfrak{b}_{15,9}(n)\\&+\frac{1757971035}{6673078}\,\mathfrak{b}_{15,10}(n)+\frac{11138809134}{16682695}\,\mathfrak{b}_{15,11}(n)\\&+\frac{135006261}{406895}\,\mathfrak{b}_{15,12}(n),\end{aligned}\tag{57}$$

$$\begin{aligned}W^{3,3}_{(3,5)}(n) =&-\frac{97013}{15665050605}\,\sigma_7(n)+\frac{15619}{800769360}\,\sigma_7(\frac{n}{3})+\frac{41483401125}{16709387312}\,\sigma_7(\frac{n}{5})\\&-\frac{24767000}{10009617}\,\sigma_7(\frac{n}{15})-\frac{1}{240}\,\sigma_3(\frac{n}{3})-\frac{1}{240}\,\sigma_3(\frac{n}{5})\\&+\frac{97013}{15665050605}\,\mathfrak{b}_{15,1}(n)+\frac{13872859}{15665050605}\,\mathfrak{b}_{15,2}(n)+\frac{152297125}{6266020242}\,\mathfrak{b}_{15,3}(n)\\&+\frac{448064}{3336539}\,\mathfrak{b}_{15,4}(n)-\frac{6574108}{3336539}\,\mathfrak{b}_{15,5}(n)-\frac{7173710}{3336539}\,\mathfrak{b}_{15,6}(n)\\&+\frac{22247539}{6673078}\,\mathfrak{b}_{15,7}(n)+\frac{109436753}{6673078}\,\mathfrak{b}_{15,8}(n)+\frac{240304751}{3336539}\,\mathfrak{b}_{15,9}(n)\\&-\frac{1532688801}{6673078}\,\mathfrak{b}_{15,10}(n)-\frac{2147882559}{3336539}\,\mathfrak{b}_{15,11}(n)\\&-\frac{24849201}{81379}\,\mathfrak{b}_{15,12}(n),\end{aligned}\tag{58}$$

$$\begin{aligned}
W^{3,3}_{(1,20)}(n) =& \frac{113}{67562496}\,\sigma_7(n) - \frac{44041613}{105735306240}\,\sigma_7(\frac{n}{2}) + \frac{113}{263916}\,\sigma_7(\frac{n}{4}) \\
&+ \frac{166339}{5742812160}\,\sigma_7(\frac{n}{5}) + \frac{1574764501}{1797500206080}\,\sigma_7(\frac{n}{10}) + \frac{166339}{22432860}\,\sigma_7(\frac{n}{20}) \\
&- \frac{1}{240}\,\sigma_3(n) - \frac{1}{240}\,\sigma_3(\frac{n}{20}) + \frac{1406987}{337812480}\,\mathfrak{b}_{20,1}(n) \\
&+ \frac{20859439}{293709184}\,\mathfrak{b}_{20,2}(n) + \frac{1042802969}{1652114160}\,\mathfrak{b}_{20,3}(n) - \frac{104343067}{1101409440}\,\mathfrak{b}_{20,4}(n) \\
&+ \frac{1138503748679}{599166735360}\,\mathfrak{b}_{20,5}(n) + \frac{31986938659}{6608456640}\,\mathfrak{b}_{20,6}(n) - \frac{7256128921}{826057080}\,\mathfrak{b}_{20,7}(n) \\
&- \frac{76388995009}{1101409440}\,\mathfrak{b}_{20,8}(n) + \frac{363542863601}{1652114160}\,\mathfrak{b}_{20,9}(n) + \frac{107196819}{1994032}\,\mathfrak{b}_{20,10}(n) \\
&- \frac{7406013}{117296}\,\mathfrak{b}_{20,11}(n) + \frac{7256277}{58648}\,\mathfrak{b}_{20,12}(n) - \frac{77145677}{175944}\,\mathfrak{b}_{20,13}(n) \\
&+ \frac{68368429}{175944}\,\mathfrak{b}_{20,14}(n) - \frac{37939109}{21993}\,\mathfrak{b}_{20,15}(n) + \frac{641460749}{131958}\,\mathfrak{b}_{20,16}(n) \\
&- \frac{1091515}{43986}\,\mathfrak{b}_{20,17}(n) - \frac{61025873}{14662}\,\mathfrak{b}_{20,18}(n),
\end{aligned} \tag{59}$$

$$\begin{aligned}
W^{3,3}_{(4,5)}(n) =& \frac{29}{337812480}\,\sigma_7(n) - \frac{925237}{105735306240}\,\sigma_7(\frac{n}{2}) + \frac{29}{1319580}\,\sigma_7(\frac{n}{4}) \\
&+ \frac{175451}{5742812160}\,\sigma_7(\frac{n}{5}) + \frac{841786109}{1797500206080}\,\sigma_7(\frac{n}{10}) + \frac{175451}{22432860}\,\sigma_7(\frac{n}{20}) \\
&- \frac{1}{240}\,\sigma_3(\frac{n}{4}) - \frac{1}{240}\,\sigma_3(\frac{n}{5}) - \frac{29}{337812480}\,\mathfrak{b}_{20,1}(n) \\
&- \frac{4421}{1468545920}\,\mathfrak{b}_{20,2}(n) - \frac{348407}{1652114160}\,\mathfrak{b}_{20,3}(n) + \frac{3854773}{1101409440}\,\mathfrak{b}_{20,4}(n) \\
&- \frac{1677449537}{599166735360}\,\mathfrak{b}_{20,5}(n) + \frac{30554071}{1321691328}\,\mathfrak{b}_{20,6}(n) - \frac{58015337}{826057080}\,\mathfrak{b}_{20,7}(n) \\
&- \frac{804668273}{1101409440}\,\mathfrak{b}_{20,8}(n) + \frac{3324670177}{1652114160}\,\mathfrak{b}_{20,9}(n) + \frac{8505103}{9970160}\,\mathfrak{b}_{20,10}(n) \\
&- \frac{354285}{117296}\,\mathfrak{b}_{20,11}(n) + \frac{666953}{293240}\,\mathfrak{b}_{20,12}(n) - \frac{1565917}{175944}\,\mathfrak{b}_{20,13}(n) \\
&+ \frac{7106017}{879720}\,\mathfrak{b}_{20,14}(n) - \frac{862381}{21993}\,\mathfrak{b}_{20,15}(n) + \frac{67378577}{659790}\,\mathfrak{b}_{20,16}(n) \\
&- \frac{17099}{43986}\,\mathfrak{b}_{20,17}(n) - \frac{5384613}{73310}\,\mathfrak{b}_{20,18}(n).
\end{aligned} \tag{60}$$

Proof. These identities follow from Theorem 5 when we set $(\alpha, \beta) = (1, 2)$, $(1, 4)$, $(1, 5)$, $(1, 6)$, $(2, 3)$, $(1, 8)$, $(1, 10)$, $(2, 5)$, $(1, 12)$, $(3, 4)$, $(1, 15)$, $(3, 5)$, $(1, 20)$, and $(4, 5)$. $\square$

6. Evaluation of the Convolution Sums $\mathbf{W}^{3,3}_{(\alpha,\beta)}(\mathbf{n})$ When $\alpha\beta = 9, 16, 18, 25, 27, 32$

In this Section, we give explicit formulae for the convolution sums $W_{(1,9)}(n)$, $W_{(1,16)}(n)$, $W_{(1,18)}(n)$, $W_{(2,9)}(n)$, $W_{(1,25)}(n)$, $W_{(1,27)}(n)$ and $W_{(1,32)}(n)$. These levels belong to $\mathbb{N}_0 \setminus \mathfrak{N}$. Hence, the primitive Dirichlet characters are non-trivial.

The convolution sum $W_{(1,25)}(n)$ is interesting due to the fact that the positive divisors of 25 which are associated with the Dirichlet character for the formation of a basis of the space of Eisenstein series constitute the entire set of positive divisors of 25.

6.1. Bases for $E_8(\Gamma_0(\alpha\beta))$ and $S_8(\Gamma_0(\alpha\beta))$ When $\alpha\beta = 9, 16, 18, 25, 27, 32$

When we apply (7), it is then sufficient to only consider the basis for the levels 18, 25, 27 and 32. By still using (7) we obtain

$$M_8(\Gamma_0(3)) \subset M_8(\Gamma_0(6)) \subset M_8(\Gamma_0(18)), \tag{61}$$

$$M_8(\Gamma_0(3)) \subset M_8(\Gamma_0(9)) \subset M_8(\Gamma_0(18)), \tag{62}$$

$$M_8(\Gamma_0(5)) \subset M_8(\Gamma_0(25)), \tag{63}$$

$$M_8(\Gamma_0(3)) \subset M_8(\Gamma_0(9)) \subset M_8(\Gamma_0(27)), \tag{64}$$

$$M_8(\Gamma_0(2)) \subset M_8(\Gamma_0(4)) \subset M_8(\Gamma_0(8)) \subset M_8(\Gamma_0(16)) \subset M_8(\Gamma_0(32)). \tag{65}$$

The dimension formulae for the space of cusp forms as given in Miyake's book [11, Theorem 2.5.2] and Stein's book [17, Proposition 6.1] and (23) are applied to compute

$$\begin{aligned}
\dim(E_8(\Gamma_0(18))) &= 8, \quad \dim(S_8(\Gamma_0(18))) = 17,\\
\dim(E_8(\Gamma_0(25))) &= 6, \quad \dim(S_8(\Gamma_0(25))) = 15,\\
\dim(E_8(\Gamma_0(27))) &= 6, \quad \dim(S_8(\Gamma_0(27))) = 18,\\
\dim(E_8(\Gamma_0(32))) &= 8, \quad \dim(S_8(\Gamma_0(32))) = 24.
\end{aligned}$$

We use Theorem 1 to determine many η-quotients which are elements of the vector spaces $S_8(\Gamma_0(18))$, $S_8(\Gamma_0(27))$ and $S_8(\Gamma_0(32))$, respectively.

Let $D(18)$, $D(25)$, $D(27)$ and $D(32)$ denote the sets of positive divisors of 18, 25, 27 and 32, respectively.

Corollary 7. *Let n be a positive integer.*

(a) *Let $\chi(n) = \left(\frac{-4}{n}\right)$ and $\psi(n) = \left(\frac{-3}{n}\right)$ be primitive Dirichlet characters such that χ is not an annihilator of $E_8(\Gamma_0(9))$, $E_8(\Gamma_0(18))$, $E_8(\Gamma_0(25))$ and $E_8(\Gamma_0(27))$, and ψ is not an annihilator of $E_8(\Gamma_0(16))$ and $E_8(\Gamma_0(32))$. Then the sets*

$$\mathcal{B}_{E,18} = \{\, E_8(q^t) \mid t|18 \,\} \cup \{\, E_{8,\left(\frac{-4}{n}\right)}(q^s) \mid s = 1, 3 \,\},$$

$$\mathcal{B}_{E,25} = \{\, E_8(q^t) \mid t|25 \,\} \cup \{\, E_{8,(\frac{-3}{n})}(q^s) \mid s \in D(25) \,\},$$
$$\mathcal{B}_{E,27} = \{\, E_8(q^t) \mid t|27 \,\} \cup \{\, E_{8,(\frac{-4}{n})}(q^s) \mid s = 1,3 \,\} \text{ and}$$
$$\mathcal{B}_{E,32} = \{E_8(q^t) \mid t|32 \,\} \cup \{\, E_{8,(\frac{-3}{n})}(q^s) \mid s = 1,2 \,\}$$

constitute bases of $E_8(\Gamma_0(18))$, $E_8(\Gamma_0(25))$, $E_8(\Gamma_0(27))$ *and* $E_8(\Gamma_0(32))$, *respectively.*

(b) *Let* $1 \le i \le 17$, $1 \le j \le 15$, $1 \le k \le 18$ *and* $1 \le l \le 24$ *be positive integers.*
Let $\delta_1 \in D(18)$ *and* $(r(i,\delta_1))_{i,\delta_1}$ *be Table 4 of the powers of* $\eta(\delta_1 z)$.
Let $\delta_2 \in D(25)$ *and* $(r(j,\delta_2))_{j,\delta_2}$ *be Table 6 of the powers of* $\eta(\delta_2 z)$.
Let $\delta_3 \in D(27)$ *and* $(r(k,\delta_3))_{k,\delta_3}$ *be Table 7 of the powers of* $\eta(\delta_3 z)$.
Let $\delta_4 \in D(32)$ *and* $(r(l,\delta_4))_{l,\delta_4}$ *be Table 8 of the powers of* $\eta(\delta_4 z)$.
Furthermore, let

$$\mathfrak{B}_{18,i}(q) = \prod_{\delta_1|18} \eta^{r(i,\delta_1)}(\delta_1 z), \quad \mathfrak{B}_{25,j}(q) = \prod_{\delta_2|25} \eta^{r(j,\delta_2)}(\delta_2 z),$$
$$\mathfrak{B}_{27,k}(q) = \prod_{\delta_3|27} \eta^{r(k,\delta_3)}(\delta_3 z) \text{ and } \quad \mathfrak{B}_{32,l}(q) = \prod_{\delta_4|32} \eta^{r(l,\delta_4)}(\delta_4 z)$$

be selected elements of $S_8(\Gamma_0(18))$, $S_8(\Gamma_0(25))$, $S_8(\Gamma_0(27))$ *and* $S_8(\Gamma_0(32))$, *respectively. The sets*

$$\mathcal{B}_{S,18} = \{\mathfrak{B}_{18,i}(q) \mid \ 1 \le i \le 17\}, \quad \mathcal{B}_{S,25} = \{\mathfrak{B}_{25,j}(q) \mid \ 1 \le j \le 15\},$$
$$\mathcal{B}_{S,27} = \{\mathfrak{B}_{27,k}(q) \mid \ 1 \le k \le 18\} \quad \text{ and } \quad \mathcal{B}_{S,32} = \{\mathfrak{B}_{32,l}(q) \mid \ 1 \le l \le 24\}$$

are bases of $S_8(\Gamma_0(18))$, $S_8(\Gamma_0(25))$, $S_8(\Gamma_0(27))$ *and* $S_8(\Gamma_0(32))$, *respectively.*

(c) *The sets*

$$\mathcal{B}_{M,18} = \mathcal{B}_{E,18} \cup \mathcal{B}_{S,18}, \quad \mathcal{B}_{M,25} = \mathcal{B}_{E,25} \cup \mathcal{B}_{S,25},$$
$$\mathcal{B}_{M,27} = \mathcal{B}_{E,27} \cup \mathcal{B}_{S,27} \quad \text{ and } \quad \mathcal{B}_{M,32} = \mathcal{B}_{E,32} \cup \mathcal{B}_{S,32}$$

constitute bases of $M_8(\Gamma_0(18))$, $M_8(\Gamma_0(25))$, $M_8(\Gamma_0(27))$ *and* $M_8(\Gamma_0(32))$, *respectively.*

By Remark 2, each $\mathfrak{B}_{\alpha\beta,i}(q)$ is expressible in the form $\sum\limits_{n=1}^{\infty} \mathfrak{b}_{\alpha\beta,i}(n)q^n$, where $\alpha\beta = 18, 25, 27, 32$ and for each $n \in \mathbb{N}_0$ it holds that $\mathfrak{b}_{\alpha\beta,i}(n) \in \mathbb{Z}$.

The basis element $\mathfrak{B}_{32,24}(q) = \eta^2(4\,z)\,\eta^2(8\,z)\,\eta^{-4}(16\,z)\,\eta^8(32\,z)$ of the vector space $S_8(\Gamma_0(32))$ belongs in fact to the space $S_4(\Gamma_0(32))$ which is contained in $S_8(\Gamma_0(32))$.

Proof of Corollary 7. It holds that $18 = 3^2 \times 2$ and $27 = 3^3$. Since $\gcd(4,3) = 1$, it is clear that the primitive Dirichlet character $\chi(n) = \left(\frac{-4}{n}\right)$ is not an annihilator of $E_8(\Gamma_0(3^2))$ and $E_8(\Gamma_0(3^3))$. Hence, $\chi(n) = \left(\frac{-4}{n}\right)$ is not an annihilator of $E_8(\Gamma_0(18))$. Since $\gcd(3,5) = 1$, the primitive Dirichlet character $\psi(n) = \left(\frac{-3}{n}\right)$ is not an annihilator of the space $E_8(\Gamma_0(5^2))$.

One also observes that the primitive Dirichlet character $\psi(n) = \left(\frac{-3}{n}\right)$ is not an annihilator of the spaces $E_8(\Gamma_0(2^4))$ and $E_8(\Gamma_0(2^5))$.

We only give the proof for $\mathcal{B}_{M,25} = \mathcal{B}_{E,25} \cup \mathcal{B}_{S,25}$ since the other cases are proved similarly.

(a) Suppose that $x_\delta, z_\delta \in \mathbb{C}$ with $0 < \delta | 25$. Let

$$\sum_{\delta|25} \left(x_\delta\, E_8(q^\delta) + z_\delta\, E_{8,\left(\frac{-3}{n}\right)}(q) \right) = 0.$$

We observe that

$$\left(\frac{-3}{n}\right) = \begin{cases} -1 & \text{if } n \equiv 2 \pmod 3, \\ 0 & \text{if } \gcd(3,n) \neq 1, \\ 1 & \text{if } n \equiv 1 \pmod 3, \end{cases} \tag{66}$$

and we recall that for all $0 \neq a \in \mathbb{Z}$ it holds that $\left(\frac{a}{0}\right) = 0$. Since the conductor of the Dirichlet character $\left(\frac{-3}{n}\right)$ is 3, we infer from (6) that $C_0 = 0$. We then deduce

$$\sum_{\delta|25} x_\delta + \sum_{n=1}^{\infty} \left(480 \sum_{\delta|25} \sigma_7(\frac{n}{\delta}) x_\delta + \left(\frac{-3}{n}\right) \sum_{\delta|25} \sigma_7(\frac{n}{\delta}) z_\delta \right) q^n = 0.$$

Then we equate the coefficients of q^n for $n \in D(25)$ to obtain a system of 6 linear equations whose unique solution is $x_\delta = z_\delta = 0$ with $\delta \in D(25)$. So, the set $\mathcal{B}_E$ is linearly independent. Hence, the set $\mathcal{B}_E$ is a basis of $E_8(\Gamma_0(25))$.

(b) Suppose that $x_i \in \mathbb{C}$ with $1 \leq i \leq 15$. Let $\sum\limits_{i=1}^{15} x_i\, \mathfrak{B}_{25,i}(q) = 0$. Then

$$\sum_{i=1}^{15} x_i\, \mathfrak{B}_{25,i}(q) = \sum_{i=1}^{15} x_i \sum_{n=1}^{\infty} \mathfrak{b}_{25,i}(n) q^n = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{15} \mathfrak{b}_{25,i}(n)\, x_i \right) q^n = 0.$$

So, we equate the coefficients of q^n for $1 \leq n \leq 15$ to obtain a system of 15 linear equations whose unique solution is $x_i = 0$ for all $1 \leq i \leq 15$. It follows that the set $\mathcal{B}_S$ is linearly independent. Hence, the set $\mathcal{B}_S$ is a basis of $S_8(\Gamma_0(25))$.

(c) Since $M_8(\Gamma_0(25)) = E_8(\Gamma_0(25)) \oplus S_8(\Gamma_0(25))$, the result follows from (a) and (b).

$\square$

6.2. Evaluation of $\mathbf{W}^{3,3}_{(\alpha,\beta)}(\mathbf{n})$ for $\alpha\beta = 9, 16, 18, 25, 27, 32$

In this section, the evaluation of the convolution sum $W_{(\alpha,\beta)}(n)$ is discussed for $(\alpha,\beta) = (1,9)$, $(1,16)$, $(1,18)$, $(2,9)$, $(1,25)$, $(1,27)$ and $(1,32)$.

Corollary 8. *The following equations are obtained.*

$$\begin{aligned}(E_4(q) - 9\, E_4(q^9))^2 = 64 + \sum_{n=1}^{\infty}\Big(& \frac{368}{369}\,\sigma_7(n) - \frac{80}{369}\,\sigma_7(\frac{n}{3}) + \frac{2592}{41}\,\sigma_7(\frac{n}{9}) \\ - \frac{531200}{123}\,\mathfrak{b}_{27,1}(n) - \frac{3712000}{41}\,\mathfrak{b}_{27,2}(n) & + 115200\,\mathfrak{b}_{27,3}(n) - \frac{129081600}{41}\,\mathfrak{b}_{27,5}(n)\Big)\, q^n,\end{aligned} \tag{67}$$

$$\begin{aligned}(E_4(q) - 16\, E_4(q^{16}))^2 = 225 + \sum_{n=1}^{\infty}\Big(& \frac{2175}{2176}\,\sigma_7(n) - \frac{15}{2176}\,\sigma_7(\frac{n}{2}) - \frac{15}{136}\,\sigma_7(\frac{n}{4}) \\ & - \frac{30}{17}\,\sigma_7(\frac{n}{8}) + \frac{3840}{17}\,\sigma_7(\frac{n}{16}) - \frac{522225}{68}\,\mathfrak{b}_{32,1}(n) - \frac{2218950}{17}\,\mathfrak{b}_{32,2}(n) \\ & - 122400\,\mathfrak{b}_{32,3}(n) - \frac{35503200}{17}\,\mathfrak{b}_{32,4}(n) - 115200\,\mathfrak{b}_{32,5}(n) - 1036800\,\mathfrak{b}_{32,6}(n) \\ & - 3686400\,\mathfrak{b}_{32,7}(n) - \frac{284083200}{17}\,\mathfrak{b}_{32,8}(n) - 27648000\,\mathfrak{b}_{32,9}(n) \\ & - 117964800\,\mathfrak{b}_{32,11}(n)\Big)\, q^n,\end{aligned} \tag{68}$$

$$\begin{aligned}(E_4(q) - 18\, E_4(q^{18}))^2 = 289 + \sum_{n=1}^{\infty}\Big(& \frac{6271}{6273}\,\sigma_7(n) - \frac{32}{6273}\,\sigma_7(\frac{n}{2}) \\ - \frac{160}{6273}\,\sigma_7(\frac{n}{3}) - \frac{2560}{6273}\,\sigma_7(\frac{n}{6}) & - \frac{1458}{697}\,\sigma_7(\frac{n}{9}) + \frac{202500}{697}\,\sigma_7(\frac{n}{18}) - \frac{18065920}{2091}\,\mathfrak{b}_{18,1}(n) \\ & - \frac{379340800}{2091}\,\mathfrak{b}_{18,2}(n) + \frac{3824640}{17}\,\mathfrak{b}_{18,3}(n) - \frac{50790400}{697}\,\mathfrak{b}_{18,4}(n) \\ & - \frac{13933150720}{2091}\,\mathfrak{b}_{18,5}(n) - \frac{472227840}{697}\,\mathfrak{b}_{18,6}(n) - \frac{1377505280}{697}\,\mathfrak{b}_{18,7}(n) \\ & - \frac{4103741440}{697}\,\mathfrak{b}_{18,8}(n) - \frac{14596514880}{697}\,\mathfrak{b}_{18,9}(n) - \frac{28631792640}{697}\,\mathfrak{b}_{18,10}(n) \\ & - \frac{20046658560}{697}\,\mathfrak{b}_{18,11}(n) - \frac{24250060800}{697}\,\mathfrak{b}_{18,12}(n) + \frac{6335267840}{697}\,\mathfrak{b}_{18,13}(n) \\ & - \frac{523814824960}{2091}\,\mathfrak{b}_{18,14}(n) + \frac{15482880}{697}\,\mathfrak{b}_{18,15}(n) - \frac{11564154880}{697}\,\mathfrak{b}_{18,16}(n) \\ & + \frac{14231700480}{697}\,\mathfrak{b}_{18,17}(n)\Big)\, q^n,\end{aligned} \tag{69}$$

$$
\begin{aligned}
(\,2\,E_4(q^2)-9\,E_4(q^9))^2 = 49 + \sum_{n=1}^{\infty}\Big(& -\frac{2}{6273}\,\sigma_7(n)+\frac{25060}{6273}\,\sigma_7(\frac{n}{2}) \\
& -\frac{160}{6273}\,\sigma_7(\frac{n}{3})-\frac{2560}{6273}\,\sigma_7(\frac{n}{6})+\frac{54999}{697}\,\sigma_7(\frac{n}{9})-\frac{23328}{697}\,\sigma_7(\frac{n}{18}) \\
& +\frac{320}{2091}\,\mathfrak{b}_{18,1}(n)-\frac{18016000}{2091}\,\mathfrak{b}_{18,2}(n)+\frac{5760}{17}\,\mathfrak{b}_{18,3}(n) \\
& -\frac{50790400}{697}\,\mathfrak{b}_{18,4}(n)-\frac{726729280}{2091}\,\mathfrak{b}_{18,5}(n)-\frac{134991360}{697}\,\mathfrak{b}_{18,6}(n) \\
& +\frac{140058880}{697}\,\mathfrak{b}_{18,7}(n)+\frac{87626240}{697}\,\mathfrak{b}_{18,8}(n)+\frac{410508480}{697}\,\mathfrak{b}_{18,9}(n) \\
& -\frac{592988160}{697}\,\mathfrak{b}_{18,10}(n)-\frac{1113239040}{697}\,\mathfrak{b}_{18,11}(n)+\frac{2728857600}{697}\,\mathfrak{b}_{18,12}(n) \\
& -\frac{5082595840}{697}\,\mathfrak{b}_{18,13}(n)+\frac{5309212160}{2091}\,\mathfrak{b}_{18,14}(n)-\frac{10776084480}{697}\,\mathfrak{b}_{18,15}(n) \\
& +\frac{6935674880}{697}\,\mathfrak{b}_{18,16}(n)-\frac{12217274880}{697}\,\mathfrak{b}_{18,17}(n)\Big)\,q^n, \qquad (70)
\end{aligned}
$$

$$
\begin{aligned}
(\,E_4(q)-25\,E_4(q^{25}))^2 = 576 + \sum_{n=1}^{\infty}\Big(& \frac{7824}{7825}\,\sigma_7(n)-\frac{624}{7825}\,\sigma_7(\frac{n}{5}) \\
& +\frac{180000}{313}\,\sigma_7(\frac{n}{25})-\frac{18779904}{1565}\,\mathfrak{b}_{25,1}(n)-\frac{431926272}{1565}\,\mathfrak{b}_{25,2}(n) \\
& -\frac{506997504}{313}\,\mathfrak{b}_{25,3}(n)-778752\,\mathfrak{b}_{25,4}(n)-\frac{1222490880}{313}\,\mathfrak{b}_{25,5}(n) \\
& -8801280\,\mathfrak{b}_{25,6}(n)-40435200\,\mathfrak{b}_{25,7}(n)-307008000\,\mathfrak{b}_{25,8}(n) \\
& +284544000\,\mathfrak{b}_{25,9}(n)-\frac{42272064000}{313}\,\mathfrak{b}_{25,10}(n)-2021760000\,\mathfrak{b}_{25,11}(n) \\
& -5241600000\,\mathfrak{b}_{25,12}(n)-10670400000\,\mathfrak{b}_{25,13}(n)-936000000\,\mathfrak{b}_{25,14}(n) \\
& -\frac{293436000000}{313}\,\mathfrak{b}_{25,15}(n)\Big)\,q^n, \qquad (71)
\end{aligned}
$$

$$
\begin{aligned}
(\,E_4(q) - 27\,E_4(q^{27}))^2 = 676 + \sum_{n=1}^{\infty}\Big(& \frac{9962}{9963}\,\sigma_7(n) - \frac{58400}{9963}\,\sigma_7(\frac{n}{3}) \\
& + \frac{640}{123}\,\sigma_7(\frac{n}{9}) + \frac{27702}{41}\,\sigma_7(\frac{n}{27}) - \frac{43040000}{3321}\,\mathfrak{b}_{27,1}(n) \\
& - \frac{301273600}{1107}\,\mathfrak{b}_{27,2}(n) + \frac{41804800}{123}\,\mathfrak{b}_{27,3}(n) - \frac{4032000}{41}\,\mathfrak{b}_{27,4}(n) \\
& - \frac{409939200}{41}\,\mathfrak{b}_{27,5}(n) - 729600\,\mathfrak{b}_{27,6}(n) - 3110400\,\mathfrak{b}_{27,7}(n) \\
& - 9331200\,\mathfrak{b}_{27,8}(n) - \frac{538963200}{41}\,\mathfrak{b}_{27,9}(n) - 52876800\,\mathfrak{b}_{27,10}(n) \\
& - 55987200\,\mathfrak{b}_{27,11}(n) - \frac{3701376000}{41}\,\mathfrak{b}_{27,12}(n) - 102643200\,\mathfrak{b}_{27,13}(n) \\
& - 419904000\,\mathfrak{b}_{27,14}(n) + \frac{6499180800}{41}\,\mathfrak{b}_{27,15}(n) - 839808000\,\mathfrak{b}_{27,16}(n) \\
& - 755827200\,\mathfrak{b}_{27,17}(n) + 755827200\,\mathfrak{b}_{27,18}(n)\Big)\,q^n, \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
(\,E_4(q) - 32\,E_4(q^{32}))^2 = 961 + \sum_{n=1}^{\infty}\Big(& \frac{17407}{17408}\,\sigma_7(n) - \frac{15}{17408}\,\sigma_7(\frac{n}{2}) \\
& - \frac{15}{1088}\,\sigma_7(\frac{n}{4}) - \frac{15}{68}\,\sigma_7(\frac{n}{8}) - \frac{60}{17}\,\sigma_7(\frac{n}{16}) \\
& + \frac{16384}{17}\,\sigma_7(\frac{n}{32}) - \frac{8355825}{544}\,\mathfrak{b}_{32,1}(n) - \frac{17755875}{68}\,\mathfrak{b}_{32,2}(n) \\
& - 245700\,\mathfrak{b}_{32,3}(n) - \frac{71268300}{17}\,\mathfrak{b}_{32,4}(n) - 244800\,\mathfrak{b}_{32,5}(n) \\
& - 2203200\,\mathfrak{b}_{32,6}(n) - 7833600\,\mathfrak{b}_{32,7}(n) - \frac{603691200}{17}\,\mathfrak{b}_{32,8}(n) \\
& - 2300313600\,\mathfrak{b}_{32,9}(n) - 2073600\,\mathfrak{b}_{32,10}(n) - 9223372800\,\mathfrak{b}_{32,11}(n) \\
& - 16819200\,\mathfrak{b}_{32,12}(n) - 29491200\,\mathfrak{b}_{32,13}(n) - 66355200\,\mathfrak{b}_{32,14}(n) \\
& - 117964800\,\mathfrak{b}_{32,15}(n) - \frac{4578969600}{17}\,\mathfrak{b}_{32,16}(n) - 704102400\,\mathfrak{b}_{32,17}(n) \\
& - 497664000\,\mathfrak{b}_{32,18}(n) - 2713190400\,\mathfrak{b}_{32,19}(n) - 8980070400\,\mathfrak{b}_{32,21}(n) \\
& - 2123366400\,\mathfrak{b}_{32,22}(n) - 25480396800\,\mathfrak{b}_{32,23}(n) + 2241331200\,\mathfrak{b}_{32,24}(n)\Big)\,q^n. \qquad (73)
\end{aligned}
$$

Proof. We only provide the proof for the case $\alpha = 2$ and $\beta = 9$ since the other cases can be done similarly.

The proof follows in principle immediately when we set $\alpha = 2$ and $\beta = 9$ in Lemma 2. However, we briefly show it for $(\,2\,E_4(q^2) - 9\,E_4(q^9)\,)^2$ as an example.

Let $\chi = \left(\frac{-4}{n}\right)$ be the primitive Dirichlet character. One obtains

$$\begin{aligned}
(\,2\,E_4(q^2) - 9\,E_4(q^9)\,)^2 &= \sum_{0<\delta|18} x_\delta E_8(q^\delta) + z_1\,E_{8,\chi}(q) + z_3\,E_{8,\chi}(q^3) \\
&\quad + \sum_{j=1}^{17} y_j\,\mathfrak{B}_{18,j}(q) \\
&= \sum_{0<\delta|18} x_\delta + \sum_{n=1}^{\infty}\Bigg(\sum_{0<\delta|18} 480\,\sigma_7(\frac{n}{\delta})\,x_\delta + \left(\frac{-4}{n}\right)\sigma_7(n)\,z_1 \\
&\quad + \left(\frac{-4}{n}\right)\sigma_7(\frac{n}{3})\,z_3 \quad + \sum_{j=1}^{17} \mathfrak{b}_{18,j}(n)\,y_j \Bigg)\,q^n. \qquad (74)
\end{aligned}$$

Since the conductor of the Dirichlet character $\left(\frac{-4}{n}\right)$ is 4, from (6) we have $C_0 = 0$. Now, when we equate the right-hand side of (74) with that of (15) and when we take the coefficients of q^n for which $1 \leq n \leq 17$ and $n = 18, 19, 20, 21, 36, 54$ for example, we obtain a system of linear equations with a unique solution. Hence, we obtain the stated result. □

We can state and prove our main result of this subsection.

Corollary 9. *Let n be a positive integer. Then*

$$\begin{aligned}
W_{(1,9)}^{3,3}(n) =& \frac{1}{797040}\,\sigma_7(n) + \frac{1}{9963}\,\sigma_7(\frac{n}{3}) + \frac{27}{3280}\,\sigma_7(\frac{n}{9}) - \frac{1}{240}\,\sigma_3(n) \\
& - \frac{1}{240}\,\sigma_3(\frac{n}{9}) + \frac{83}{19926}\,\mathfrak{b}_{27,1}(n) + \frac{290}{3321}\,\mathfrak{b}_{27,2}(n) - \frac{1}{9}\,\mathfrak{b}_{27,3}(n) \\
& + \frac{249}{82}\,\mathfrak{b}_{27,5}(n), \qquad (75)
\end{aligned}$$

$$\begin{aligned}
W_{(1,16)}^{3,3}(n) =& \frac{1}{8355840}\,\sigma_7(n) + \frac{1}{557056}\,\sigma_7(\frac{n}{2}) + \frac{1}{34816}\,\sigma_7(\frac{n}{4}) + \frac{1}{2176}\,\sigma_7(\frac{n}{8}) \\
& + \frac{2}{255}\,\sigma_7(\frac{n}{16}) - \frac{1}{240}\,\sigma_3(n) - \frac{1}{240}\,\sigma_3(\frac{n}{16}) + \frac{2321}{557056}\,\mathfrak{b}_{32,1}(n) \\
& + \frac{4931}{69632}\,\mathfrak{b}_{32,2}(n) + \frac{17}{256}\,\mathfrak{b}_{32,3}(n) + \frac{4931}{4352}\,\mathfrak{b}_{32,4}(n) + \frac{1}{16}\,\mathfrak{b}_{32,5}(n) \\
& + \frac{9}{16}\,\mathfrak{b}_{32,6}(n) + 2\,\mathfrak{b}_{32,7}(n) + \frac{1233}{136}\,\mathfrak{b}_{32,8}(n) + 15\,\mathfrak{b}_{32,9}(n) \\
& + 64\,\mathfrak{b}_{32,11}(n), \qquad (76)
\end{aligned}$$

$$\begin{aligned}
W^{3,3}_{(1,18)}(n) =&\frac{1}{13549680}\,\sigma_7(n) + \frac{1}{846855}\,\sigma_7(\frac{n}{2}) + \frac{1}{169371}\,\sigma_7(\frac{n}{3}) + \frac{16}{169371}\,\sigma_7(\frac{n}{6})\\
&+ \frac{27}{55760}\,\sigma_7(\frac{n}{9}) + \frac{27}{3485}\,\sigma_7(\frac{n}{18}) - \frac{1}{240}\,\sigma_3(n) - \frac{1}{240}\,\sigma_3(\frac{n}{18})\\
&+ \frac{7057}{1693710}\,\mathfrak{b}_{18,1}(n) + \frac{14818}{169371}\,\mathfrak{b}_{18,2}(n) - \frac{83}{765}\,\mathfrak{b}_{18,3}(n) + \frac{1984}{56457}\,\mathfrak{b}_{18,4}(n)\\
&+ \frac{5442637}{1693710}\,\mathfrak{b}_{18,5}(n) + \frac{3416}{10455}\,\mathfrak{b}_{18,6}(n) + \frac{269044}{282285}\,\mathfrak{b}_{18,7}(n) + \frac{801512}{282285}\,\mathfrak{b}_{18,8}(n)\\
&+ \frac{15204703}{1505520}\,\mathfrak{b}_{18,9}(n) + \frac{1864049}{94095}\,\mathfrak{b}_{18,10}(n) + \frac{1305121}{94095}\,\mathfrak{b}_{18,11}(n)\\
&+ \frac{35084}{2091}\,\mathfrak{b}_{18,12}(n) - \frac{1237357}{282285}\,\mathfrak{b}_{18,13}(n) + \frac{102307583}{846855}\,\mathfrak{b}_{18,14}(n)\\
&- \frac{112}{10455}\,\mathfrak{b}_{18,15}(n) + \frac{2258624}{282285}\,\mathfrak{b}_{18,16}(n) - \frac{926543}{94095}\,\mathfrak{b}_{18,17}(n), \qquad (77)
\end{aligned}$$

$$\begin{aligned}
W^{3,3}_{(2,9)}(n) =&\frac{1}{13549680}\,\sigma_7(n) + \frac{1}{846855}\,\sigma_7(\frac{n}{2}) + \frac{1}{169371}\,\sigma_7(\frac{n}{3})\\
&+ \frac{16}{169371}\,\sigma_7(\frac{n}{6}) + \frac{27}{55760}\,\sigma_7(\frac{n}{9}) + \frac{27}{3485}\,\sigma_7(\frac{n}{18}) - \frac{1}{240}\,\sigma_3(\frac{n}{2})\\
&- \frac{1}{240}\,\sigma_3(\frac{n}{9}) - \frac{1}{13549680}\,\mathfrak{b}_{18,1}(n) + \frac{2815}{677484}\,\mathfrak{b}_{18,2}(n)\\
&- \frac{1}{6120}\,\mathfrak{b}_{18,3}(n) + \frac{1984}{56457}\,\mathfrak{b}_{18,4}(n) + \frac{2271029}{13549680}\,\mathfrak{b}_{18,5}(n)\\
&+ \frac{651}{6970}\,\mathfrak{b}_{18,6}(n) - \frac{109421}{1129140}\,\mathfrak{b}_{18,7}(n) - \frac{34229}{564570}\,\mathfrak{b}_{18,8}(n)\\
&- \frac{427613}{1505520}\,\mathfrak{b}_{18,9}(n) + \frac{38606}{94095}\,\mathfrak{b}_{18,10}(n) + \frac{144953}{188190}\,\mathfrak{b}_{18,11}(n)\\
&- \frac{1316}{697}\,\mathfrak{b}_{18,12}(n) + \frac{1985389}{564570}\,\mathfrak{b}_{18,13}(n) - \frac{2073911}{1693710}\,\mathfrak{b}_{18,14}(n)\\
&+ \frac{25984}{3485}\,\mathfrak{b}_{18,15}(n) - \frac{1354624}{282285}\,\mathfrak{b}_{18,16}(n) + \frac{1590791}{188190}\,\mathfrak{b}_{18,17}(n), \qquad (78)
\end{aligned}$$

$$\begin{aligned}
W^{3,3}_{(1,25)}(n) =&\frac{1}{46950000}\,\sigma_7(n) + \frac{13}{978125}\,\sigma_7(\frac{n}{5}) + \frac{125}{15024}\,\sigma_7(\frac{n}{25})\\
&- \frac{1}{240}\,\sigma_3(n) - \frac{1}{240}\,\sigma_3(\frac{n}{25}) + \frac{8151}{1956250}\,\mathfrak{b}_{25,1}(n) + \frac{93734}{978125}\,\mathfrak{b}_{25,2}(n)\\
&+ \frac{220051}{391250}\,\mathfrak{b}_{25,3}(n) + \frac{169}{625}\,\mathfrak{b}_{25,4}(n) + \frac{106119}{78250}\,\mathfrak{b}_{25,5}(n) + \frac{382}{125}\,\mathfrak{b}_{25,6}(n)\\
&+ \frac{351}{25}\,\mathfrak{b}_{25,7}(n) + \frac{533}{5}\,\mathfrak{b}_{25,8}(n) - \frac{494}{5}\,\mathfrak{b}_{25,9}(n) + \frac{73389}{1565}\,\mathfrak{b}_{25,10}(n)\\
&+ 702\,\mathfrak{b}_{25,11}(n) + 1820\,\mathfrak{b}_{25,12}(n) + 3705\,\mathfrak{b}_{25,13}(n) + 325\,\mathfrak{b}_{25,14}(n)\\
&+ \frac{203775}{626}\,\mathfrak{b}_{25,15}(n), \qquad (79)
\end{aligned}$$

$$\begin{aligned}
W_{(1,27)}^{3,3}(n) =& \frac{1}{64560240}\,\sigma_7(n) + \frac{730}{807003}\,\sigma_7(\frac{n}{3}) - \frac{8}{9963}\,\sigma_7(\frac{n}{9}) \\
&+ \frac{27}{3280}\,\sigma_7(\frac{n}{27}) - \frac{1}{240}\,\sigma_3(n) - \frac{1}{240}\,\sigma_3(\frac{n}{27}) + \frac{6725}{1614006}\,\mathfrak{b}_{27,1}(n) \\
&+ \frac{23537}{269001}\,\mathfrak{b}_{27,2}(n) - \frac{3266}{29889}\,\mathfrak{b}_{27,3}(n) + \frac{35}{1107}\,\mathfrak{b}_{27,4}(n) + \frac{7117}{2214}\,\mathfrak{b}_{27,5}(n) \\
&+ \frac{19}{81}\,\mathfrak{b}_{27,6}(n) + 1\,\mathfrak{b}_{27,7}(n) + 3\,\mathfrak{b}_{27,8}(n) + \frac{3119}{738}\,\mathfrak{b}_{27,9}(n) \\
&+ 17\,\mathfrak{b}_{27,10}(n) + 18\,\mathfrak{b}_{27,11}(n) + \frac{1190}{41}\,\mathfrak{b}_{27,12}(n) + 33\,\mathfrak{b}_{27,13}(n) \\
&+ 135\,\mathfrak{b}_{27,14}(n) - \frac{4179}{82}\,\mathfrak{b}_{27,15}(n) + 270\,\mathfrak{b}_{27,16}(n) + 243\,\mathfrak{b}_{27,17}(n) \\
&- 243\,\mathfrak{b}_{27,18}(n), \qquad (80)
\end{aligned}$$

$$\begin{aligned}
W_{(1,32)}^{3,3}(n) =& \frac{1}{133693440}\,\sigma_7(n) + \frac{1}{8912896}\,\sigma_7(\frac{n}{2}) + \frac{1}{557056}\,\sigma_7(\frac{n}{4}) \\
&+ \frac{1}{34816}\,\sigma_7(\frac{n}{8}) + \frac{1}{2176}\,\sigma_7(\frac{n}{16}) + \frac{2}{255}\,\sigma_7(\frac{n}{32}) - \frac{1}{240}\,\sigma_3(n) \\
&- \frac{1}{240}\,\sigma_3(\frac{n}{32}) + \frac{37137}{8912896}\,\mathfrak{b}_{32,1}(n) + \frac{78915}{1114112}\,\mathfrak{b}_{32,2}(n) + \frac{273}{4096}\,\mathfrak{b}_{32,3}(n) \\
&+ \frac{79187}{69632}\,\mathfrak{b}_{32,4}(n) + \frac{17}{256}\,\mathfrak{b}_{32,5}(n) + \frac{153}{256}\,\mathfrak{b}_{32,6}(n) + \frac{17}{8}\,\mathfrak{b}_{32,7}(n) \\
&+ \frac{41923}{4352}\,\mathfrak{b}_{32,8}(n) + 624\,\mathfrak{b}_{32,9}(n) + \frac{9}{16}\,\mathfrak{b}_{32,10}(n) + 2502\,\mathfrak{b}_{32,11}(n) \\
&+ \frac{73}{16}\,\mathfrak{b}_{32,12}(n) + 8\,\mathfrak{b}_{32,13}(n) + 18\,\mathfrak{b}_{32,14}(n) + 32\,\mathfrak{b}_{32,15}(n) \\
&+ \frac{9937}{136}\,\mathfrak{b}_{32,16}(n) + 191\,\mathfrak{b}_{32,17}(n) + 135\,\mathfrak{b}_{32,18}(n) + 736\,\mathfrak{b}_{32,19}(n) \\
&+ 2436\,\mathfrak{b}_{32,21}(n) + 576\,\mathfrak{b}_{32,22}(n) + 6912\,\mathfrak{b}_{32,23}(n) - 608\,\mathfrak{b}_{32,24}(n). \qquad (81)
\end{aligned}$$

Proof. It follows immediately when we set $(\alpha, \beta) = (1, 9)$, $(1, 16)$, $(1, 18)$, $(2, 9)$, $(1, 25)$, $(1, 27)$, and $(1, 32)$ in Theorem 5. □

7. Formulae for the Number of Representations of a Positive Integer

We make use of the convolution sums evaluated in Sections 5 and 6 among others to determine explicit formulae for the number of representations of a positive integer n by the quadratic forms (4) and (5), respectively.

7.1. Representations by the Quadratic Form (5)

We determine formulae for the number of representations of a positive integer n by the quadratic form (5). We mainly apply the evaluation of the convolution sums of levels $3, 6, 9, 12, 15, 18, 27$ and other well-known convolution sums to determine these formulae. In order to do that, we recall that the levels $3, 6, 9, 12, 15, 18, 27$ are each congruent to zero modulo 3. These levels are therefore of the restricted form in Section 4.2. Hence, from Proposition 2 we derive that $\Omega_3 = \{(1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (2,3),\ (1,9)\,\}$. We then deduce the following result.

Corollary 10. *Let $n \in \mathbb{N}_0$. Then*

$$\begin{aligned}
R^{8,8}_{(1,1)}(n) &= 48\sigma_3(n) + 432\sigma_3(\frac{n}{3}) \\
&\quad + 576\left(W^{3,3}_{(1,1)}(n) + 18\,W^{3,3}_{(1,3)}(n) + 81\,W^{3,3}_{(1,1)}(\frac{n}{3})\right) \\
&= \frac{240}{41}\,\sigma_7(n) + \frac{19440}{41}\,\sigma_7(\frac{n}{3}) - \frac{864}{41}\,\mathfrak{b}_{18,1}(n) \\
&= s_{16}(n) \qquad \textit{if } n \equiv 0 \pmod 3, \\
&\quad \textit{wherein } \ \mathfrak{B}_{18,1}(q) = \eta^{12}(z)\,\eta^{4}(3\,z) \ = \sum_{n=1}^{\infty} \mathfrak{b}_{18,1}(n)\,q^n.
\end{aligned}$$

$$\begin{aligned}
R^{8,8}_{(1,2)}(n) =&24\sigma_3(n) + 216\sigma_3(\frac{n}{3}) + 24\sigma_3(\frac{n}{2}) + 216\sigma_3(\frac{n}{6}) \\
&+ 576\left(W^{3,3}_{(1,2)}(n) + 9\,W^{3,3}_{(1,6)}(n) + 9\,W^{3,3}_{(2,3)}(n) + 81\,W^{3,3}_{(1,2)}(\frac{n}{3})\right),
\end{aligned}$$

$$\begin{aligned}
R^{8,8}_{(1,3)}(n) =&24\sigma_3(n) + 216\sigma_3(\frac{n}{3}) + 24\sigma_3(\frac{n}{3}) + 216\sigma_3(\frac{n}{9}) \\
&+ 576\left(W^{3,3}_{(1,3)}(n) + 9\,W^{3,3}_{(1,9)}(n) + 9\,W^{3,3}_{(1,1)}(\frac{n}{3}) + 81\,W^{3,3}_{(1,3)}(\frac{n}{3})\right),
\end{aligned}$$

$$\begin{aligned}
R^{8,8}_{(1,4)}(n) =&24\sigma_3(n) + 216\sigma_3(\frac{n}{3}) + 24\sigma_3(\frac{n}{4}) + 216\sigma_3(\frac{n}{12}) \\
&+ 576\left(W^{3,3}_{(1,4)}(n) + 9\,W^{3,3}_{(1,12)}(n) + 9\,W^{3,3}_{(3,4)}(n) + 81\,W^{3,3}_{(1,4)}(\frac{n}{3})\right),
\end{aligned}$$

$$\begin{aligned}
R^{8,8}_{(1,5)}(n) =&24\sigma_3(n) + 216\sigma_3(\frac{n}{3}) + 24\sigma_3(\frac{n}{5}) + 216\sigma_3(\frac{n}{15}) \\
&+ 576\left(W^{3,3}_{(1,5)}(n) + 9\,W^{3,3}_{(1,15)}(n) + 9\,W^{3,3}_{(3,5)}(n) + 81\,W^{3,3}_{(1,5)}(\frac{n}{3})\right),
\end{aligned}$$

$$R^{8,8}_{(1,6)}(n) = 24\sigma_3(n) + 216\sigma_3(\frac{n}{3}) + 24\sigma_3(\frac{n}{6}) + 216\sigma_3(\frac{n}{18}) + 576\left(W^{3,3}_{(1,6)}(n) + 9\,W^{3,3}_{(1,18)}(n) + 9\,W^{3,3}_{(1,2)}(\frac{n}{3}) + 81\,W^{3,3}_{(1,6)}(\frac{n}{3})\right),$$

$$R^{8,8}_{(2,3)}(n) = 24\sigma_3(\frac{n}{2}) + 216\sigma_3(\frac{n}{9}) + 24\sigma_3(\frac{n}{3}) + 216\sigma_3(\frac{n}{9}) + 576\left(W^{3,3}_{(2,3)}(n) + 9\,W^{3,3}_{(2,9)}(n) + 9\,W^{3,3}_{(1,2)}(\frac{n}{3}) + 81\,W^{3,3}_{(2,3)}(\frac{n}{3})\right),$$

$$R^{8,8}_{(1,9)}(n) = 24\sigma_3(n) + 216\sigma_3(\frac{n}{3}) + 24\sigma_3(\frac{n}{9}) + 216\sigma_3(\frac{n}{27}) + 576\left(W^{3,3}_{(1,9)}(n) + 9\,W^{3,3}_{(1,27)}(n) + 9\,W^{3,3}_{(1,3)}(\frac{n}{3}) + 81\,W^{3,3}_{(1,9)}(\frac{n}{3})\right).$$

Proof. We only consider the case $(c, d) = (1, 2)$ since the other cases can be proved in a similar way.

It follows immediately from Theorem 7 with $(c, d) = (1, 2)$. One can then make use of (46), (50) and (51) to simplify $R^{8,8}_{(1,2)}(n)$. □

7.2. Representations by Quadratic Form (4)

We give formulae for the number of representations of a positive integer n by the quadratic form (4). We apply among others the evaluated convolution sums $W^{3,3}_{(1,4)}(n)$, $W^{3,3}_{(1,8)}(n)$, $W^{3,3}_{(1,12)}(n)$, $W^{3,3}_{(3,4)}(n)$, $W^{3,3}_{(1,20)}(n)$, $W^{3,3}_{(4,5)}(n)$, $W^{3,3}_{(1,16)}(n)$ and $W^{3,3}_{(1,32)}(n)$. To achieve this, we recall that the levels $4, 8, 12, 16, 20, 32$ are all divisible by 4. Hence, they are of the restricted form in Section 4.1. Therefore, we apply Proposition 1 to arrive at the conclusion that $\Omega_4 = \{\,(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,8)\,\}$.

Corollary 11. *Let $n \in \mathbb{N}_0$. Then*

$$\begin{aligned}
N^{8,8}_{(1,1)}(n) = {} & 32\,\sigma_3(n) - 64\,\sigma_3(\frac{n}{2}) + 512\sigma_3(\frac{n}{4}) + 256\,W^{3,3}_{(1,1)}(n) - 1024\,W^{3,3}_{(1,2)}(n) \\
& + 1024\,W^{3,3}_{(1,1)}(\frac{n}{2}) - 16384\,W^{3,3}_{(1,2)}(\frac{n}{2}) + 8192\,W^{3,3}_{(1,4)}(n) + 65536\,W^{3,3}_{(1,1)}(\frac{n}{4}) \\
= {} & \frac{32}{17}\,\sigma_7(n) - \frac{64}{17}\,\sigma_7(\frac{n}{2}) + \frac{8192}{17}\,\sigma_7(\frac{n}{4}) + \frac{512}{17}\,\mathfrak{b}_{32,1}(n) + \frac{8192}{17}\,\mathfrak{b}_{32,1}(\frac{n}{2}) \\
= {} & r_{16}(n),
\end{aligned}$$

$$\textit{wherein}\ \ \mathfrak{B}_{32,1}(q) = \eta^8(z)\,\eta^8(2\,z) = \sum_{n=1}^{\infty} \mathfrak{b}_{32,1}(n)\,q^n,$$

$$
\begin{aligned}
N^{8,8}_{(1,2)}(n) = {} & 16\,\sigma_3(n) + 16\,\sigma_3(\frac{n}{2}) - 32\,\sigma_3(\frac{n}{2}) - 32\,\sigma_3(\frac{n}{4}) + 256\sigma_3(\frac{n}{4}) + 256\sigma_3(\frac{n}{8}) \\
& + 256\,W^{3,3}_{(1,2)}(n) - 512\,W^{3,3}_{(1,4)}(n) + 4096\,W^{3,3}_{(1,8)}(n) - 512\,W^{3,3}_{(1,1)}(\frac{n}{2}) \\
& + 5120\,W^{3,3}_{(1,2)}(\frac{n}{2}) - 8192\,W^{3,3}_{(1,4)}(\frac{n}{2}) - 8192\,W^{3,3}_{(1,1)}(\frac{n}{4}) \\
& + 65536\,W^{3,3}_{(1,2)}(\frac{n}{4}),
\end{aligned}
$$

$$
\begin{aligned}
N^{8,8}_{(1,3)}(n) = {} & 16\,\sigma_3(n) + 16\,\sigma_3(\frac{n}{3}) - 32\,\sigma_3(\frac{n}{2}) - 32\,\sigma_3(\frac{n}{6}) + 256\sigma_3(\frac{n}{4}) + 256\sigma_3(\frac{n}{12}) \\
& + 256\,W^{3,3}_{(1,3)}(n) - 512\,W^{3,3}_{(1,6)}(n) + 4096\,W^{3,3}_{(1,12)}(n) - 512\,W^{3,3}_{(2,3)}(n) \\
& + 1024\,W^{3,3}_{(1,3)}(\frac{n}{2}) - 8192\,W^{3,3}_{(1,6)}(\frac{n}{2}) + 4096\,W^{3,3}_{(3,4)}(n) \\
& - 8192\,W^{3,3}_{(2,3)}(\frac{n}{2}) + 65536\,W^{3,3}_{(1,3)}(\frac{n}{4}),
\end{aligned}
$$

$$
\begin{aligned}
N^{8,8}_{(1,4)}(n) = {} & 16\,\sigma_3(n) + 16\,\sigma_3(\frac{n}{4}) - 32\,\sigma_3(\frac{n}{2}) - 32\,\sigma_3(\frac{n}{8}) + 256\sigma_3(\frac{n}{4}) + 256\sigma_3(\frac{n}{16}) \\
& + 256\,W^{3,3}_{(1,4)}(n) - 512\,W^{3,3}_{(1,8)}(n) + 4096\,W^{3,3}_{(1,16)}(n) - 512\,W^{3,3}_{(1,2)}(\frac{n}{2}) \\
& + 1024\,W^{3,3}_{(1,4)}(\frac{n}{2}) - 8192\,W^{3,3}_{(1,8)}(\frac{n}{2}) + 4096\,W^{3,3}_{(1,1)}(\frac{n}{4}) \\
& - 8192\,W^{3,3}_{(1,2)}(\frac{n}{4}) + 65536\,W^{3,3}_{(1,4)}(\frac{n}{4}),
\end{aligned}
$$

$$
\begin{aligned}
N^{8,8}_{(1,5)}(n) = {} & 16\,\sigma_3(n) + 16\,\sigma_3(\frac{n}{5}) - 32\,\sigma_3(\frac{n}{2}) - 32\,\sigma_3(\frac{n}{10}) + 256\sigma_3(\frac{n}{4}) + 256\sigma_3(\frac{n}{20}) \\
& + 256\,W^{3,3}_{(1,5)}(n) - 512\,W^{3,3}_{(1,10)}(n) + 4096\,W^{3,3}_{(1,20)}(n) - 512\,W^{3,3}_{(2,5)}(n) \\
& + 1024\,W^{3,3}_{(1,5)}(\frac{n}{2}) - 8192\,W^{3,3}_{(1,10)}(\frac{n}{2}) + 4096\,W^{3,3}_{(4,5)}(n) \\
& - 8192\,W^{3,3}_{(2,5)}(\frac{n}{2}) + 65536\,W^{3,3}_{(1,5)}(\frac{n}{4}),
\end{aligned}
$$

$$
\begin{aligned}
N^{8,8}_{(1,8)}(n) = {} & 16\,\sigma_3(n) + 16\,\sigma_3(\frac{n}{8}) - 32\,\sigma_3(\frac{n}{2}) - 32\,\sigma_3(\frac{n}{16}) + 256\sigma_3(\frac{n}{4}) + 256\sigma_3(\frac{n}{32}) \\
& + 256\,W^{3,3}_{(1,8)}(n) - 512\,W^{3,3}_{(1,16)}(n) + 4096\,W^{3,3}_{(1,32)}(n) - 512\,W^{3,3}_{(1,4)}(\frac{n}{2}) \\
& + 1024\,W^{3,3}_{(1,8)}(\frac{n}{2}) - 8192\,W^{3,3}_{(1,16)}(\frac{n}{2}) + 4096\,W^{3,3}_{(1,2)}(\frac{n}{4}) \\
& - 8192\,W^{3,3}_{(1,4)}(\frac{n}{4}) + 65536\,W^{3,3}_{(1,8)}(\frac{n}{4}).
\end{aligned}
$$

Proof. These formulae follow immediately from Theorem 6 when we set $(a, b) = (1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$ and $(1, 8)$, respectively. One can then use the result of (21), (46), 48 and (52) to simplify $N_{(1,2)}$. $\square$

8. Concluding Remark

The evaluation of the convolution sums $\sum_{\substack{(l,m)\in\mathbb{N}^2\\ \alpha\, l+\beta\, m=n}} \sigma_1(l)\sigma_5(m)$ and $\sum_{\substack{(l,m)\in\mathbb{N}^2\\ \alpha\, l+\beta\, m=n}} \sigma_5(l)\sigma_1(m)$ for those $0 < \alpha, \beta \in \mathbb{N}$ for which $\gcd(\alpha,\beta) = 1$ is complete and will be submitted for publication very soon.

As suggested by an anonymous referee, a future work will focus on the properties and arithmetic of the η-quotients resulting from a path in a graph of cusp spaces. Such a work may therefore explain some properties and arithmetic of their Fourier coefficients.

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Appendix

	1	**2**	**3**	**4**	**6**	**12**
1	8	8	0	0	0	0
2	4	4	4	0	4	0
3	0	0	8	0	8	0
4	3	-3	-1	0	17	0
5	6	-6	-10	0	26	0
6	0	0	0	0	8	8
7	0	0	0	6	-4	14
8	0	3	0	-3	-1	17
9	0	4	0	-2	-8	22
10	0	6	0	-6	-10	26
11	0	8	0	-10	-12	30

Table 2: Power of η-quotients being basis elements of $S_8(\Gamma_0(12))$

	1	**3**	**5**	**15**
1	14	0	2	0
2	8	0	8	0
3	2	0	14	0
4	0	2	12	2
5	2	6	2	6
6	0	8	0	8
7	3	5	-3	11
8	2	0	2	12
9	0	2	0	14
10	3	-1	-3	17
11	1	1	-5	19
12	4	-2	-8	22

Table 3: Power of η-quotients being basis elements of $S_8(\Gamma_0(15))$

	1	**2**	**3**	**6**	**9**	**18**
1	12	0	4	0	0	0
2	0	0	16	0	0	0
3	6	0	4	0	6	0
4	3	0	4	0	9	0
5	0	0	4	0	12	0
6	0	0	4	4	4	4
7	0	3	0	0	8	5
8	0	3	0	4	0	9
9	0	0	0	0	8	8
10	0	0	0	4	0	12
11	0	0	9	-1	-11	19
12	0	0	3	-3	-1	17
13	0	0	4	-4	-4	20
14	0	0	4	0	-12	24
15	0	0	6	-6	-10	26
16	0	0	7	-7	-13	29
17	0	0	8	-8	-16	32

Table 4: Power of η-quotients being basis elements for $S_8(\Gamma_0(18))$

	1	**2**	**4**	**5**	**10**	**20**
1	8	8	0	0	0	0
2	8	0	0	8	0	0
3	2	0	0	14	0	0
4	0	8	0	0	8	0
5	0	0	0	8	8	0
6	0	2	0	0	14	0
7	10	-6	0	-10	22	0
8	2	0	0	-10	24	0
9	4	-4	0	-12	28	0
10	0	0	0	0	8	8
11	0	3	-3	-8	17	7
12	0	0	2	0	0	14
13	0	4	-4	-8	12	12
14	0	10	-6	0	-10	22
15	0	1	-3	8	-13	23
16	0	2	0	0	-10	24
17	2	3	-5	-2	-7	25
18	0	4	-4	0	-12	28

Table 5: Power of η-quotients being basis elements of $S_8(\Gamma_0(20))$

	1	**5**	**25**
1	14	2	0
2	8	8	0
3	2	14	0
4	1	14	1
5	0	14	2
6	4	8	4
7	8	2	6
8	2	8	6
9	1	8	7
10	0	8	8
11	4	2	10
12	3	2	11
13	2	2	12
14	1	2	13
15	0	2	14

Table 6: Power of η-quotients being basis elements for $S_8(\Gamma_0(25))$

	1	**3**	**9**	**27**
1	12	4	0	0
2	0	16	0	0
3	6	4	6	0
4	3	4	9	0
5	0	4	12	0
6	0	0	16	0
7	0	5	8	3
8	0	1	12	3
9	0	6	4	6
10	0	2	8	6
11	0	7	0	9
12	0	3	4	9
13	0	8	-4	12
14	0	4	0	12
15	0	0	4	12
16	0	5	-4	15
17	0	1	0	15
18	3	2	-7	18

Table 7: Power of η-functions being basis elements of $S_8(\Gamma_0(27))$

	1	**2**	**4**	**8**	**16**	**32**
1	8	8	0	0	0	0
2	0	8	8	0	0	0
3	8	0	0	8	0	0
4	0	0	8	8	0	0
5	0	4	-4	16	0	0
6	0	8	0	0	8	0
7	0	4	0	4	8	0
8	0	0	0	8	8	0
9	0	4	4	-8	16	0
10	0	0	4	-4	16	0
11	0	0	6	-10	20	0
12	0	0	8	0	0	8
13	0	0	2	6	0	8
14	0	0	4	0	4	8
15	0	4	0	-4	8	8
16	0	0	0	0	8	8
17	0	4	0	-2	2	12
18	0	0	4	4	-8	16
19	0	4	0	0	-4	16
20	0	0	0	4	-4	16
21	0	4	0	2	-10	20
22	0	0	0	6	-10	20
23	0	0	2	0	-6	20
24	0	0	2	2	-4	8

Table 8: Power of η-functions being basis elements of $S_8(\Gamma_0(32))$