ON A GENERALIZED FAMILY OF EULER-GENOCCHI POLYNOMIALS

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Abstract

As an extension to Euler-Genocchi numbers and polynomials introduced recently by Belbachir et al., we define and study the generalized Euler-Genocchi numbers and polynomials. These polynomials are also a generalization of the polynomials already studied by Frontczak and Tomovski. In this work we give the explicit formulae of the generalized Euler-Genocchi polynomials and other interesting identities. The obtained results are improvements of several properties and identities established by Belbachir et al.

1. Introduction

The Euler polynomials $E_n(x)$ and Genocchi polynomials $G_n(x)$ (see [8]) admit the following generating functions, respectively;

$$\frac{2}{e^z+1}e^{xz} = \sum_{n\geq 0} E_n(x)\frac{z^n}{n!} \quad (|z|<\pi) \tag{1}$$

and

$$\frac{2z}{e^z+1}e^{xz} = \sum_{n\geq 0} G_n(x)\frac{z^n}{n!} \quad (|z|<\pi). \tag{2}$$

It is obvious that

$$\frac{2z}{e^z+1}e^{xz} = \sum_{n\geq 0} E_n(x)\frac{z^{n+1}}{n!} = \sum_{n\geq 1} nE_{n-1}(x)\frac{z^n}{n!}.$$

Then $G_0(x) = 0$ and $G_n(x) = nE_{n-1}(x)$ for $n \geq 1$. In [1] and [2] Belbachir et al. introduced and studied the Euler-Genocchi polynomials defined by means of the generating function

$$\frac{2z^r}{e^z+1}e^{xz} = \sum_{n\geq 0} A_n^{(r)}(x)\frac{z^n}{n!}. \tag{3}$$

The numbers $A_n^{(r)} = A_n^{(r)}(0)$ are the so-called Euler-Genocchi numbers. It is easy to see that $A_n^{(0)}(x) = E_n(x)$ and $A_n^{(1)}(x) = G_n(x)$. In [2] the authors showed that

$$A_n^{(r)}(x+1) = \sum_{k=0}^{n} \binom{n}{k} A_k^{(r)}(x), \tag{4}$$

$$x^n = \frac{1}{2(n+r)_r}\left[A_{n+r}^{(r)}(x) + \sum_{k=0}^{n}\binom{n+r}{k+r}A_{k+r}^{(r)}(x)\right] \tag{5}$$

and

$$\sum_{j=0}^{m-1}(-1)^j A_n^{(r)}\left(\frac{x+j}{m}\right) = \frac{1}{m^{n-r}}A_n^{(r)}(x) \quad m \text{ odd}, \tag{6}$$

where

$$(x)_r = \prod_{k=0}^{r-1}(x-k) = \begin{cases} x(x-1)\cdots(x-r+1)\ , & r\geq 1 \\ 1\ , & r=0. \end{cases}$$

denotes the falling factorial of $x \in \mathbb{C}$. By convention, we set $(n)_r = 0$ if $n \in \mathbb{N}$ and $n < r$. Frontczak and Tomovski [3] extended the definition of Euler-Genocchi polynomials to generalized Euler-Genocchi polynomials $A_n^{(r,m)}(x)$ of order $m \in \mathbb{N}$, by means of the generating function

$$z^{mr}\left(\frac{2}{e^z+1}\right)^m e^{xz} = \sum_{n\geq 0} A_n^{(r,m)}(x)\frac{z^n}{n!} \quad (|z| < \pi). \tag{7}$$

The authors derived many interesting properties of polynomials $A_n^{(r,m)}(x)$ such as the linear recurrence

$$A_n^{(r,m)}(x) = \sum_{k=0}^{n}\binom{n}{k}A_k^{r,m}x^{n-k} \tag{8}$$

and the derivative formula

$$\frac{d}{dx}A_n^{(r,m)}(x) = n\frac{d}{dx}A_{n-1}^{(r,m)}(x). \tag{9}$$

We have $A_n^{(r,0)}(x) = x^n$ and $A_n^{(r,m)}(x) = 0$ for $n < rm$. The explicit formula of $A_n^{(r,m)}(x)$ remains an open problem. In this work we introduce a generalized class of Euler-Genocchi polynomials, where we compute the explicit formula and give a corresponding result for Euler-Genocchi polynomials.

2. Generalized Euler-Genocchi Polynomials

We begin by describing the generalized class of the Euler-Genocchi polynomials, which includes the polynomials $A_n^{(r,m)}(x)$.

Definition 1. Let $\alpha \neq 0$ be a complex number and r a positive integer. The *generalized Euler-Genocchi polynomials* $A_n^{r,\alpha}(x)$ are defined by the generating function

$$z^r \left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n\geq 0} A_n^{r,\alpha}(x)\frac{z^n}{n!} \quad (|z| < \pi). \tag{10}$$

The corresponding numbers $A_n^{r,\alpha} = A_n^{r,\alpha}(0)$ are given by the generating function

$$z^r \left(\frac{2}{e^z+1}\right)^{\alpha} = \sum_{n\geq 0} A_n^{r,\alpha}\frac{z^n}{n!}. \tag{11}$$

The natural connection of the polynomial $A_n^{r,\alpha}(x)$ with the numbers $A_n^{r,\alpha}$ is given by the following formula:

$$A_n^{r,\alpha}(x) = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}^{r,\alpha} x^k. \tag{12}$$

A key step in obtaining Equation (12) is the Cauchy product of the generating functions e^{xz} and $z^r\left(\frac{2}{e^z+1}\right)^{\alpha}$. If $\alpha = -m$ is a negative integer we have

$$\frac{1}{2^m} z^r \left(e^z+1\right)^m e^{xz} = \frac{1}{2^m} z^r \sum_{k=0}^{m}\binom{m}{k} e^{(k+x)z} = \frac{1}{2^m}\sum_{n\geq 0}\sum_{k=0}^{m}\binom{m}{k}(k+x)^n \frac{z^{n+r}}{n!}.$$

Then

$$\frac{1}{2^m} z^r \left(e^z+1\right)^m e^{xz} = \frac{1}{2^m}\sum_{n\geq r}\sum_{k=0}^{m}\binom{m}{k}(n)_r (k+x)^{n-r} \frac{z^{n}}{n!}.$$

So we deduce that

$$A_n^{r,-m}(x) = \frac{(n)_r}{2^m}\sum_{k=0}^{m}\binom{m}{k}(k+x)^{n-r}. \tag{13}$$

The expression of the numbers $A_n^{r,-m}$ is immediate and we have

$$A_n^{r,-m} = \frac{(n)_r}{2^m}\sum_{k=0}^{m}\binom{m}{k}k^{n-r},$$

which implies that

$$A_n^{r,-m}(x) = \frac{1}{2^m}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}(n-k)_r j^{n-r-k}x^k. \tag{14}$$

We recall that the generalized Euler polynomials $E_n^{(\alpha)}(x)$ are defined by means of the generating function

$$\left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n\geq 0} E_n^{(\alpha)}(x)\frac{z^n}{n!},\ |z|<\pi. \tag{15}$$

The corresponding generalized Euler numbers are $E_n^{(\alpha)} = E_n^{(\alpha)}(0)$. Polynomials $A_n^{r,\alpha}(x)$ are related to polynomials $E_n^{(\alpha)}(x)$ by the following identity:

$$A_n^{r,\alpha}(x) = (n)_r E_{n-r}^{(\alpha)}(x),\ n\geq r. \tag{16}$$

Therefore we have

$$A_n^{r,\alpha} = (n)_r E_{n-r}^{(\alpha)},\ n\geq r. \tag{17}$$

It is obvious that $A_n^{r,\alpha}(x) = A_n^{r,\alpha} = 0$ for $1\leq n<r$ and $A_r^{r,\alpha}(x) = A_r^{r,\alpha} = r!$. The following theorem gives a closed formula of the polynomial $A_n^{r,\alpha}(x)$.

Theorem 1. *We have, for $n\geq 1$,*

$$A_n^{r,\alpha}(x) = (n)_r x^{n-r} + (n)_r \sum_{k=0}^{n-r-1}\binom{n-r}{k} E_{n-k-r}^{(\alpha)} x^k. \tag{18}$$

Proof. The combination of identities (12) and (17) leads to the following expression

$$A_n^{r,\alpha}(x) = \sum_{k=0}^{n}\binom{n}{k}(n-k)_r E_{n-r-k}^{(\alpha)} x^k = r!\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{r} E_{n-r-k}^{(\alpha)} x^k.$$

But we have

$$\binom{n}{k}\binom{n-k}{r} = \binom{n}{r}\binom{n-r}{k},$$

and then

$$A_n^{r,\alpha}(x) = (n)_r \sum_{k=0}^{n-r}\binom{n-r}{k} E_{n-r-k}^{(\alpha)} x^k.$$

Finally we have

$$A_n^{r,\alpha}(x) = (n)_r x^{n-r} + (n)_r \sum_{k=0}^{n-r-1}\binom{n-r}{k} E_{n-r-k}^{(\alpha)} x^k.$$

□

2.1. Closed Formula of Generalized Euler Numbers

Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a formal generating function with $(a_n)_{n\in\mathbb{N}}$ a sequence of numbers or polynomials, such that $a_0 \neq 0$. For $\alpha \in \mathbb{C}$; $f^\alpha(z)$ is also a generating function. In [4] we showed that

$$f^m(z) = \sum_{n\geq 0} \sum_{a_{i_1}+...+a_{i_n}=m} a_{i_1}...a_{i_n} z^n, \ m \in \mathbb{N}. \tag{19}$$

Our proof makes substantial use of classical umbral calculus. In [5] we studied the general case where we obtained the following explicit formula:

$$f^\alpha(z) = a_0^\alpha + \sum_{n\geq 1} \sum_{k=1}^{n} \sum_{s_n(k)} \binom{\alpha}{k} \binom{k}{k_1, ..., k_n} a_0^{\alpha-k} a_1^{k_1} ... a_n^{k_n} z^n, \tag{20}$$

where $s_n(k)$ is the set of all $(k_1, \cdots, k_n) \in \mathbb{N}^n$ satisfying the conditions $k_1+...+k_n = k$ and $k_1 + 2k_2 + ... + nk_n = n$. It is obvious that $k_j = 0$ for $j > n-k+1$, and $s_n(k)$ reduces to the $(n-k+1)$-tuplet $(k_1, ..., k_{n-k+1})$. The exponential partial Bell polynomials are the polynomials $B_{n,k} = B_{n,k}(x_1, ..., x_{n-k+1})$ defined by the generating function

$$\frac{1}{k!} \left(\sum_{m\geq 1} x_m \frac{z^m}{m!} \right)^k = \sum_{n\geq k} B_{n,k}(x_1, ..., x_{n-k+1}) \frac{z^n}{n!}. \tag{21}$$

The polynomials $B_{n,k}$ have integral coefficients; their combinatorial formula is

$$B_{n,k}(x_1, ..., x_{n-k+1}) = \frac{n!}{k!} \sum_{s_n(k)} \binom{k}{k_1, ..., k_{n-k+1}} \prod_{r=1}^{n-k+1} \left(\frac{x_r}{r!} \right)^{k_r}. \tag{22}$$

Another reformulation of $f^\alpha(z)$ by means of these polynomials is given in [6] and [7] by the following expression:

$$f^\alpha(z) = a_0^\alpha + \sum_{n\geq 1} \sum_{k=1}^{n} (\alpha)_k a_0^{\alpha-k} B_{n,k}(1!a_1, ..., (n-k+1)!a_{n-k+1}) \frac{z^n}{n!}. \tag{23}$$

The Stirling numbers of second kind $S_2(n,k)$ are defined by the generating function

$$\frac{1}{k!} (e^z - 1)^k = \sum_{n\geq 0} S_2(n,k) \frac{z^n}{n!}; \tag{24}$$

since we have

$$e^z - 1 = \sum_{n\geq 1} \frac{z^n}{n!},$$

then

$$S_2(n,k) = B_{n,k}(1,...,1).$$

Consequently, these numbers admit the following formulation:

$$S_2(n,k) = \frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}j^n. \tag{25}$$

The closed formula of the generalized Euler numbers involves the stirling numbers of second kind as it is explained by the following theorem.

Theorem 2. *The generalized Euler numbers take the form*

$$E_n^{(\alpha)} = \sum_{k=1}^{n}(-\alpha)_k\left(\frac{1}{2}\right)^k S_2(n,k) \quad n \geq 1. \tag{26}$$

Therefore, the Euler numbers E_n are given by the following formula.

$$E_n = \sum_{k=1}^{n} k!\left(-\frac{1}{2}\right)^k S_2(n,k) \quad n \geq 1, \tag{27}$$

with the initial terms $E_0^{(\alpha)} = E_0 = 1$.

Proof. We know that

$$\frac{e^n+1}{2} = 1 + \sum_{n\geq 1}\frac{1}{2}\frac{z^n}{n!}$$

and

$$B_{n,k}\left(\frac{1}{2},...,\frac{1}{2}\right) = \left(\frac{1}{2}\right)^k B_{n,k}(1,...,1) = \left(\frac{1}{2}\right)^k S_2(n,k).$$

Then by means of Equation (23) we deduce that

$$\left(\frac{2}{e^z+1}\right)^\alpha = 1 + \sum_{n\geq 1}\sum_{k=1}^{n}(-\alpha)_k\left(\frac{1}{2}\right)^k S_2(n,k)\frac{z^n}{n!}.$$

Furthermore $E_0^{(\alpha)} = 1$ and for $n \geq 1$;

$$E_n^{(\alpha)} = \sum_{k=1}^{n}(-\alpha)_k\left(\frac{1}{2}\right)^k S_2(n,k).$$

Letting $\alpha = 1$ in Equation (26), the second identity is immediate. □

Replacing $E_{n-k-r}^{(\alpha)}$ by its value in equation (18) to obtain the following proposition.

Proposition 1. *Let α be a complex number and r a positive integer. Then we have*

$$A_n^{r,\alpha}(x) = (n)_r x^{n-r} + (n)_r \sum_{k=0}^{n-r-1} \sum_{j=1}^{n-r-k} (-\alpha)_j \left(\frac{1}{2}\right)^j \binom{n-r}{k} S_2(n-r-k,j)\, x^k \quad (28)$$

and

$$A_n^{r,\alpha} = (n)_r \sum_{j=1}^{n-r} (-\alpha)_j \left(\frac{1}{2}\right)^j S_2(n-r,j)\,. \quad (29)$$

We replace n by $n+r$ in Equation (28) to obtain the following evaluation of the x^n.

Corollary 1. *For every positive integer n, we have*

$$x^n = \frac{1}{(n+r)_r} A_{n+r}^{r,\alpha}(x) - \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} (-\alpha)_j \left(\frac{1}{2}\right)^j \binom{n}{k} S_2(n-k,j)\, x^k. \quad (30)$$

We end this section with a theorem which shows the multiplication formula satisfied by the polynomials $A_n^{r,\alpha}(x)$.

Theorem 3. *Let α be a complex number and m an odd positive integer. Then the following identity holds:*

$$\sum_{k=0}^{m-1} (-1)^k A_n^{r,\alpha}\left(\frac{x+k}{m}\right) = \sum_{k=r}^{n} \binom{n}{k} \frac{1}{m^{k-r}} E_{n-k}^{(\alpha-1)} A_k^{(r)}(x). \quad (31)$$

Proof. It follows from the generating function of the polynomials $A_n^{r,\alpha}(x)$ that

$$\begin{aligned}
\sum_{n\geq 0} \sum_{k=0}^{m-1} (-1)^k A_n^{r,\alpha}\left(\frac{x+k}{m}\right) \frac{z^n}{n!} &= \sum_{k=0}^{m-1} (-1)^k \sum_{n\geq 0} A_n^{r,\alpha}\left(\frac{x+k}{m}\right) \frac{z^n}{n!} \\
&= \sum_{k=0}^{m-1} (-1)^k z^r \left(\frac{2}{e^z+1}\right)^\alpha \exp\left(\frac{x+k}{m} z\right) \\
&= z^r \left(\frac{2}{\exp z+1}\right)^\alpha \exp(xz/m)\, \frac{1-(-exp(z/m))^m}{1+\exp(z/m)} \\
&= 2z^r \left(\frac{2}{\exp z+1}\right)^{\alpha-1} \frac{\exp(xz/m)}{1+\exp(z/m)} \\
&= \frac{2z^r \exp(xz/m)}{1+\exp(z/m)} \left(\frac{2}{\exp z+1}\right)^{\alpha-1}.
\end{aligned}$$

But we have

$$\begin{aligned}
\frac{2z^r \exp(xz/m)}{1+\exp(z/m)} \left(\frac{2}{\exp z+1}\right)^{\alpha-1} &= \left(\sum_{n\geq 0} \frac{1}{m^{n-r}} A_n^{(r)}(x) \frac{z^n}{n!}\right) \left(\sum_{n\geq 0} E_n^{(\alpha-1)} \frac{z^n}{n!}\right) \\
&= \sum_{n\geq 0} \sum_{k=r}^{n} \binom{n}{k} \frac{1}{m^{k-r}} E_{n-k}^{(\alpha-1)} A_k^{(r)}(x),
\end{aligned}$$

and the desired result follows. □

Knowing that $E_0^{(0)} = 1$ and $E_n^{(0)} = 0$ for $n \geq 1$ and taking $\alpha = 1$, the last identity becomes the multiplication formula for $A_n^{(r)}(x)$ already established in [2]:

$$\sum_{k=0}^{m-1} (-1)^k A_n^{(r)} \left(\frac{x+k}{m} \right) = \frac{1}{m^{n-r}} A_n^{(r)}(x), \; m \text{ odd}. \tag{32}$$

3. Explicit Formula of Euler-Genocchi Polynomials

The projection of these results on the Euler-Genocchi polynomials serves to establish the following reformulation.

Corollary 2. *We have $A_n^{(r)}(x) = 0$ for $n < r$, $A_r^{(r)}(x) = r!$ and for $n > r$:*

$$A_n^r(x) = (n)_r x^{n-r} + \sum_{k=0}^{n-r-1} \sum_{j=1}^{n-r-k} j!(n)_r \left(-\frac{1}{2}\right)^j \binom{n-r}{k} S_2(n-r-k,j)x^k. \tag{33}$$

Proof. Substitute $\alpha = -1$ into Equation (28) to obtain

$$A_n^{(r)}(x) = (n)_r x^{n-r} + \sum_{k=0}^{n-r-1} \sum_{j=1}^{n-r-k} (-1)_j (n)_r \left(\frac{1}{2}\right)^j \binom{n-r}{k} S_2\left(n-r-k,j\right) x^k.$$

The proof concludes by observing that $(-1)_k = (-1)^k k!$. □

Corollary 3. *We have $A_n^{(r)} = 0$ for $n < r$, $A_r^{(r)} = (r)_r = 1$ and for $n > r$:*

$$A_n^r = (n)_r \sum_{j=1}^{n-r} j! \left(-\frac{1}{2}\right)^j S_2(n-r,j). \tag{34}$$

Proof. We take $x = 0$ in identity (33) and use the convention $0^0 = 1$ to deduce that $k = 0$ and the sum reduces to Equation (34). □

Remark. From Corollary 2 we conclude that

$$x^n = \frac{1}{(n+r)_r} A_{n+r}^{(r)}(x) - \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} j! \left(-\frac{1}{2}\right)^j \binom{n}{k} S_2(n-k,j)x^k, \tag{35}$$

which improves the identity in [2]:

$$x^n = \frac{1}{2(n+r)_r} \left[A_{n+r}^{(r)}(x) + \sum_{k=0}^{n} \binom{n+r}{k+r} A_{k+r}^{(r)} x^k \right].$$

Taking $r \in \{0,1\}$ in Equation (33) we get the following closed formulae of the polynomials $E_n(x)$ and $G_n(x)$.

Corollary 4. *We have* $E_0(x) = 1$ *and for* $n \geq 1$*;*

$$E_n(x) = x^n + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} j! \left(-\frac{1}{2}\right)^j \binom{n}{k} S_2(n-k,j) x^k. \tag{36}$$

Otherwise we have $G_0(x) = 0$*,* $G_1(x) = 1$ *and for* $n \geq 2$*,*

$$G_n(x) = nx^{n-1} + n \sum_{k=0}^{n-2} \sum_{j=1}^{n-1-k} j! \left(-\frac{1}{2}\right)^j \binom{n-1}{k} S_2(n-k-1,j) x^k. \tag{37}$$

According to identities (36) and (37) we have

$$x^n = E_n(x) - \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} j! \left(-\frac{1}{2}\right)^j \binom{n}{k} S_2(n-k,j) x^k \tag{38}$$

and

$$x^n = \frac{1}{n+1} x G_{n+1}(x) - \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} j! \left(-\frac{1}{2}\right)^j \binom{n}{k} S_2(n-k+1,j) x^k. \tag{39}$$

These relations improve identities (4) and (5) established in [2]:

$$x^n = \frac{1}{2} \left[E_n(x) + \sum_{k=0}^{n} \binom{n}{k} E_k(x) \right]$$

and

$$x^n = \frac{1}{2(n+1)} \left[G_{n+1}(x) + \sum_{k=0}^{n} \binom{n+1}{k+1} G_{k+1}(x) \right].$$

4. The Alternate Power Sums

The alternate power sum $T_k(n)$ is defined by

$$T_k(n) = \sum_{j=0}^{n} (-1)^j j^k \tag{40}$$

and generated by the function

$$\frac{1 - (-1)^{n+1} e^{(n+1)z}}{e^z + 1} = \sum_{k \geq 0} T_k(n) \frac{z^n}{n!}. \tag{41}$$

Numbers $T_k(n)$ are connected to Euler polynomials via the relation

$$T_k(n) = \frac{(-1)^n E_k(n+1) + E_k}{2} \tag{42}$$

Another expression of $T_k(n)$ by means of Stirling numbers of second kind is given by the following theorem.

Theorem 4. *We have $T_k(0) = 2$ and for $n \geq 1$ we obtain*

$$\begin{aligned} T_k(n) = \frac{1}{2}(-1)^n(n+1)^k + (1+(-1)^n)\sum_{j=0}^{k}(-1)^j j! 2^{-1-j} S_2(k,j) \\ +(-1)^n \sum_{j=1}^{k-1}\sum_{i=1}^{j}(-1)^i i! 2^{-1-i} S_2(j,i)(n+1)^{k-j} \end{aligned} \tag{43}$$

Proof. We have

$$1-(-1)^{n+1}e^{(n+1)z} = 1+(-1)^n+(-1)^n\sum_{k\geq 1}(n+1)^k \frac{z^k}{k!}.$$

Let b_k be the sequence defined by

$$b_0 = 1+(-1)^n \text{ and } b_k = (-1)^n (n+1)^k, \ k \geq 1.$$

Then

$$1-(-1)^{n+1}e^{(n+1)z} = \sum_{k\geq 0} b_k \frac{t^k}{k!}.$$

Since we have

$$\frac{1}{e^z+1} = \frac{1}{2}\left(\frac{2}{e^z+1}\right) = \frac{1}{2} + \sum_{k\geq 1}\sum_{j=1}^{k}(-1)^j j! \left(\frac{1}{2}\right)^{j+1} S_2(k,j)\frac{z^k}{k!},$$

we can write

$$\frac{1}{e^z+1} = \sum_{k\geq 0} a_k \frac{t^k}{k!},$$

with

$$a_k = \sum_{j=1}^{k}(-1)^j j! \left(\frac{1}{2}\right)^{j+1} S_2(k,j), \ k \geq 1$$

and $a_0 = \frac{1}{2}$. The Cauchy product of the generating functions $1-(-1)^{n+1}e^{(n+1)z}$ and $\frac{1}{e^z+1}$ implies that

$$\frac{1-(-1)^{n+1}e^{(n+1)z}}{e^z+1} = \sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j} a_j b_{k-j}\frac{z^k}{k!}.$$

Then

$$T_k(n) = \sum_{j=0}^{k} \binom{k}{j} a_j b_{k-j} = \frac{1}{2}(-1)^n (n+1)^k + (1+(-1)^n) \sum_{j=1}^{k} (-1)^j j! 2^{-1-j} S_2(k,j)$$
$$+(-1)^n \sum_{j=1}^{k-1} \sum_{i=1}^{j} (-1)^i i! 2^{-1-i} (n+1)^{k-j} S_2(j,i).$$

□

Corollary 5. *According to the parity of the argument, we have*

$$T_k(2n) = \frac{1}{2}(2n+1)^k + 2\sum_{j=0}^{k} (-1)^j j! 2^{-1-j} S_2(k,j)$$
$$+\sum_{j=1}^{k-1} \sum_{i=1}^{j} (-1)^i i! 2^{-1-i} S_2(j,i) (2n+1)^{k-j}$$

and

$$T_k(2n+1) = -(n+1)^k - \sum_{j=1}^{k-1} \sum_{i=1}^{j} (-1)^i i! 2^{k-j-i-1} S_2(j,i) (n+1)^{k-j}.$$

Proposition 2. *The expression of $A_n^{(r)}(x)$, $n \geq r$, by means of the alternate power sums at even arguments is given by the following identity:*

$$A_n^{(r)}(x) = (n)_r \sum_{l=0}^{n-r} \binom{n-r}{l} T_l(m-1) x^{n-r-l} \tag{44}$$
$$+ (n)_r \sum_{k=0}^{n-r-1} \sum_{j=1}^{n-r-k} \sum_{l=0}^{k} j! \left(-\frac{1}{2}\right)^j \binom{n-r}{k} \binom{k}{l} m^{n-r-k} S_2(n-k,j) T_l(m-1) x^{k-l}.$$

Proof. From Equation (35) we deduce for $n \geq r$ that

$$A_n^{(r)}(x) = (n)_r x^{n-r} + (n)_r \sum_{k=0}^{n-r-1} \sum_{j=1}^{n-r-k} j! \left(-\frac{1}{2}\right)^j \binom{n-r}{k} S_2(n-r-k,j) x^k.$$

Then

$$A_n^{(r)}\left(\frac{x+i}{m}\right) = (n)_r \left(\frac{x+i}{m}\right)^{n-r}$$
$$+ (n)_r \sum_{k=0}^{n-r-1} \sum_{j=1}^{n-r-k} j! \left(-\frac{1}{2}\right)^j \binom{n-r}{k} S_2(n-r-k,j) \left(\frac{x+i}{m}\right)^k.$$

The passage to the sum gives

$$\begin{aligned}\sum_{i=0}^{m-1}(-1)^i A_n^{(r)}\left(\frac{x+i}{m}\right) &= (n)_r\sum_{i=0}^{m-1}(-1)^i\left(\frac{x+i}{m}\right)^{n-r}\\ &+ (n)_r\sum_{k=0}^{n-r-1}\sum_{j=1}^{n-r-k} j!\left(-\frac{1}{2}\right)^j\binom{n-r}{k}\\ &\times S_2(n-k,j)\sum_{i=0}^{m-1}(-1)^i\left(\frac{x+i}{m}\right)^k.\end{aligned}$$

But

$$\sum_{i=0}^{m-1}(-1)^i\left(\frac{x+i}{m}\right)^k=\frac{1}{m^k}\sum_{i=0}^{m-1}(-1)^i\sum_{l=0}^{k}\binom{k}{l}i^l x^{k-l}=\frac{1}{m^k}\sum_{l=0}^{k}\binom{k}{l}T_l(m-1)x^{k-l}.$$

Then

$$\begin{aligned}\sum_{i=0}^{m-1}(-1)^i A_n^{(r)}\left(\frac{x+i}{m}\right) &= (n)_r\frac{1}{m^{n-r}}\sum_{l=0}^{n-r}\binom{n-r}{l}T_l(m-1)x^{n-r-l}+\\ &\quad (n)_r\sum_{k=0}^{n-r-1}\sum_{j=1}^{n-r-k} j!\left(-\frac{1}{2}\right)^j\binom{n-r}{k}\\ &\times S_2(n-k,j)\frac{1}{m^k}\sum_{l=0}^{k}\binom{k}{l}T_l(m-1)x^{k-l}\end{aligned}$$

and the desired result follows. □

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