ON CERTAIN DIOPHANTINE EQUATIONS INVOLVING TRIANGULAR NUMBERS

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Abstract

By the theory of the Pell equation, we study the polynomial solutions and positive integer solutions of certain Diophantine equations involving triangular numbers.

1. Introduction

A triangular number is a positive integer of the form

$$t_n = \binom{n}{2} = \frac{n(n-1)}{2},\ n \geq 2,\ n \in \mathbb{Z}.$$

It is a classical result that all positive integer solutions of the Pythagorean equation

$$X^2 + Y^2 = Z^2 \tag{1.1}$$

are given by

$$X = 2kuv,\ Y = k(u^2 - v^2),\ Z = k(u^2 + v^2),$$

where k is a positive integer and $u,\ v$ are co-prime positive integers of different parity with $u > v$. The solution (X, Y, Z) is called a Pythagorean triple.

In 1962, Sierpiński [6] showed that Equation (1.1) has infinitely many positive integer solutions $X = t_x,\ Y = t_y,\ Z = z$ with $\gcd(t_x,\ t_y) > 1$. In other words, he proved that the Diophantine equation

$$z^2 = t_x^2 + t_y^2 \tag{1.2}$$

has infinitely many positive integer solutions. As pointed out by Sierpiński [6], Schinzel showed that the set of positive integer solutions of Equation (1.2) with $\gcd(t_x,\ t_y) = 1$ is infinite. In 2008, Ulas [8] proved that Equation (1.2) has infinitely many polynomial solutions $x(t), y(t), z(t) \in \mathbb{Z}[t]$. By the theory of the Pell equation, we give another proof of Ulas' result in the following theorem.

Theorem 1.1. *Equation* (1.2) *has infinitely many polynomial solutions* $x(t), y(t), z(t) \in \mathbb{Z}[t]$.

In 2010, Ulas and Togbé [7] considered the rational solutions of the Diophantine equations

$$z^2 = f(x)^2 \pm f(y)^2, \tag{1.3}$$

where $f(x)$ are quadratic and cubic polynomials. In 2010, He, Togbé and Ulas [3] further investigated the integer solutions of Equation (1.3) for some special polynomials $f(x)$. They gave infinitely many integer solutions of the Diophantine equation

$$z^2 = (x^2 + a)^2 - (y^2 + a)^2$$

for some special values of a. When $a = -1$, the above Diophantine equation reduces to

$$Z^2 = t_X^2 - t_Y^2,$$

where $Z = z/8,\ X = (x-1)/2,\ Y = (y-1)/2$. Then there are infinitely many Pythagorean triangles (a Pythagorean triangle is a right triangle with integer side lengths) with a leg and the hypotenuse that are triangular numbers. We shall give a new proof of this result.

Theorem 1.2. *The Diophantine equation*

$$t_z^2 = x^2 + t_y^2 \tag{1.4}$$

has infinitely many polynomial solutions $x(t), y(t), z(t) \in \mathbb{Z}[t]$.

In 2004, Rakaczki [5] studied the integer solutions of the Diophantine equation

$$F\left(\binom{x}{n}\right) = b\binom{y}{m}, x \geq n, y \geq m, \tag{1.5}$$

where $F(x)$ is a polynomial with integer coefficients, m (≥ 2) and n are positive integers, and b is a non-zero integer.

Motivated by Equations (1.2) and (1.5), we consider the positive integer solutions of the general Diophantine equation

$$z^2 = at_x^2 + bt_xt_y + ct_y^2, \tag{1.6}$$

where a, b, c are integers with $b^2 - 4ac \neq 0$. When $a = 1, b = 0, c = \pm 1$, Equation (1.6) reduces to Equations (1.2) and (1.4). By the theory of the Pell equation, we

$d(t)$	$x(t), y(t)$
t^2-1	$(t,1)$
$t^2 \pm 2$	$(t^2 \pm 1, t)$
$9t^2 \pm 8t+2$	$((9t \pm 4)^2+1, 3(9t \pm 4))$
$49t^2 \pm 20t+2$	$((49t \pm 10)^2-1, 7(49t \pm 10))$
$t(t^3 \pm 2)$	$(t^3 \pm 1, t)$
$t(r^2t \pm 1), r \in \mathbb{Z}^+$	$(2r^2t \pm 1, 2r)$
$t(r^2t \pm 2), r \in \mathbb{Z}^+$	$(r^2t \pm 1, r)$

Table 1: An integer solution of the Pell equation $x^2 - d(t)y^2 = 1$

get the following results. In order to illustrate Theorem 1.3, we give an integer solution of the Pell equation $x^2 - d(t)y^2 = 1$ in Table 1.

For some special values of a, b, c, we have the following theorem.

Theorem 1.3. *Let* $a = 1,\ b = 0,\ c+1 = d(t)$*, or* $a = 1,\ b = 2,\ c+3 = d(t)$*, or* $a = 1,\ b \neq 2$ ($b+2 > 0$ *is not a perfect square*), $c = 1$*. Then Equation* (1.6) *has infinitely many positive integer solutions.*

For general values of a, b, c, we obtain the following result.

Theorem 1.4. *Let* $a = (2t+1)^2(r^2(2t+1)^2 - s),\ b = -2r^2(2t+1)^2 + s,\ c = r^2$*, or* $a = (2t+1)^2(r^2(2t+1)^2 - 2s(2t^2+2t+1)),\ b = -2r^2(2t+1)^2 + s(8t^2+8t+3),\ c = r^2 - s$*, where* r, s, t *are positive integers. If* $st(t+1)$ *is not a perfect square, then Equation* (1.6) *has infinitely many positive integer solutions.*

Lastly, we investigate the positive integer solutions of the related Diophantine equation

$$t_z = at_x^2 + bt_xt_y + ct_y^2, \tag{1.7}$$

where a, b, c are integers with $a^2 + b^2 + c^2 \neq 0$. When $a = 1, b = c = 0$, the only positive integer solutions of Equation (1.7) are $(z, x) = (1,1), (2,2), (9,4)$ (see [2, 4]).

By the same method of Theorem 1.4, we have the following result.

Theorem 1.5. *Let* $a = \frac{1}{2}(2t+1)^2(sr^2(r+1)^2(2t+1)^2 - 2)s,\ b = -(sr^2(r+1)^2(2t+1)^2 - 1)s,\ c = \frac{1}{2}s^2r^2(r+1)^2$*, where* r, s, t *are positive integers. If* $2st(t+1)$ *is not a perfect square, then Equation* (1.7) *has infinitely many positive integer solutions.*

2. Proofs of the Theorems

Proof of Theorem 1.1. Let $y = 2tx + 1 - 2t$, where t is a parameter. Then Equation (1.2) can be written as the following equation:

$$\frac{1}{4}(x-1)^2\left((16t^4+1)x^2 + 16(-2t^4+t^3)x + 16t^4 - 16t^3 + 4t^2\right) = z^2.$$

Consider

$$(16t^4+1)x^2 + 16(-2t^4+t^3)x + 16t^4 - 16t^3 + 4t^2 = s^2,$$

and put

$$X = (16t^4+1)x + 8(-2t^4+t^3), \quad Y = s.$$

We get the Pell equation

$$X^2 - (16t^4+1)Y^2 = -4t^2(2t-1)^2. \tag{2.1}$$

Equation (2.1) has a solution

$$(X^{'}, Y^{'}) = \left(8t^3(2t-1), 2t(2t-1)\right),$$

and $(X^{''}, Y^{''}) = (32t^4+1, 8t^2)$ is a solution of the Pell equation

$$X^2 - (16t^4+1)Y^2 = 1.$$

An infinite number of solutions of Equation (2.1) are given by

$$\begin{aligned} X_n + Y_n\sqrt{16t^4+1} = &\left(8t^3(2t-1) + 2t(2t-1)\sqrt{16t^4+1}\right) \\ &\times\left(32t^4+1+8t^2\sqrt{16t^4+1}\right)^n, \ n \geq 0, \end{aligned}$$

which leads to

$$X_n = (32t^4+1)X_{n-1} + 8t^2(16t^4+1)Y_{n-1}, \quad Y_n = 8t^2X_{n-1} + (32t^4+1)Y_{n-1}.$$

So

$$\begin{cases} X_n = 2(32t^4+1)X_{n-1} - X_{n-2}, & X_0 = 8t^3(2t-1), \\ & X_1 = 8t^3(2t-1)(64t^4+3), \\ Y_n = 2(32t^4+1)Y_{n-1} - Y_{n-2}, & Y_0 = 2t(2t-1), \\ & Y_1 = 2t(2t-1)(64t^4+1). \end{cases}$$

Using the relation $X_n = 2(32t^4+1)X_{n-1} - X_{n-2}$ twice, we get

$$X_{n+1} = 2\left(2(32t^4+1)^2 - 1\right)X_{n-1} - X_{n-3}.$$

Replacing n by $2n$, we have the relation

$$X_{2n+1} = 2\left(2(32t^4+1)^2 - 1\right)X_{2n-1} - X_{2n-3},$$

which holds for $n \geq 2$. From $X = (16t^4+1)x + 8(-2t^4+t^3),\ Y = s$, we have

$$\begin{aligned} x_{2n+1} = &\ 2\left(2(32t^4+1)^2-1\right)x_{2n-1} - x_{2n-3} + 256t^4(-16t^4+8t^3), \\ s_{2n+1} = &\ 2\left(2(32t^4+1)^2-1\right)s_{2n-1} - s_{2n-3}, \end{aligned}$$

where

$$\begin{aligned} x_1 = &\ 32t^3(2t-1),\ x_3 = 64t^3(2t-1)(8t^2-4t+1)(8t^2+4t+1)(32t^4+1), \\ s_1 = &\ 2t(2t-1)(64t^4+1),\ s_3 = 2t(2t-1)(262144t^{12}+20480t^8+384t^4+1). \end{aligned}$$

Thus, Equation (1.2) has infinitely many polynomial solutions

$$\begin{aligned} &x_{2n+1} \in \mathbb{Z}[t], \\ &y_{2n+1} = 2tx_{2n+1} + 1 - 2t \in \mathbb{Z}[t], \\ &z_{2n+1} = \frac{1}{2}(x_{2n+1}-1)s_{2n+1} \in \mathbb{Z}[t], \end{aligned}$$

where $n \geq 0$. □

Example 2.1. When $n = 0$, Equation (1.2) has a polynomial solution

$$\begin{aligned} x_1(t) = &\ 32t^3(2t-1), \\ y_1(t) = &\ 128t^5 - 64t^4 - 2t + 1, \\ z_1(t) = &\ t(2t-1)(8t^2-4t+1)(8t^2+4t+1)(64t^4-32t^3-1). \end{aligned}$$

Proof of Theorem 1.2. Let $z = 2ty + 1$, where t is a parameter. Then Equation (1.4) is equivalent to

$$\frac{1}{4}y^2\left((16t^4-1)y^2 + (16t^3+2)y + 4t^2 - 1\right) = x^2.$$

Letting $(16t^4-1)y^2 + (16t^3+2)y + 4t^2 - 1 = s^2$, then

$$\left((16t^4-1)y + 8t^3 + 1\right)^2 - (16t^4-1)s^2 = 4t^2(2t+1)^2.$$

Putting $X = (16t^4-1)y + 8t^3 + 1,\ Y = s$, we get the Pell equation

$$X^2 - (16t^4-1)Y^2 = 4t^2(2t+1)^2. \tag{2.2}$$

Note that $(X', Y') = (8t^3(2t+1), 2t(2t+1))$ is a solution of Equation (2.2) and $(X'', Y'') = (4t^2, 1)$ is a solution of the Pell equation

$$X^2 - (16t^4-1)Y^2 = 1.$$

An infinite number of solutions of Equation (2.2) are given by

$$\begin{aligned} X_n + Y_n\sqrt{16t^4-1} = &\left(8t^3(2t+1) + 2t(2t+1)\sqrt{16t^4-1}\right) \\ &\times \left(4t^2 + \sqrt{16t^4-1}\right)^n,\ n \geq 0. \end{aligned}$$

In a similar way as in the proof of Theorem 1.1, we can get infinitely many polynomial solutions. □

Proof of Theorem 1.3. 1) When $a = 1, b = 0$, let $y = x + 1$. Then Equation (1.6) becomes

$$\frac{1}{4}x^2\left((c+1)x^2 + (2c-2)x + c + 1\right) = z^2.$$

Letting $(c+1)x^2 + (2c-2)x + c + 1 = s^2$, we have

$$((c+1)x + (c-1))^2 - (c+1)s^2 = -4c.$$

If $c + 1 = d(t) = t^2 - 1$ (the proofs of the remaining cases in Table 1 are similar to this one), then

$$\left((t^2-1)x + t^2 - 3\right)^2 - (t^2-1)s^2 = -4t^2 + 8.$$

Putting $X = (t^2-1)x + t^2 - 3,\ Y = s$, we get the Pell equation

$$X^2 - (t^2-1)Y^2 = -4t^2 + 8. \tag{2.3}$$

Let us observe that the pair $(X^{'}, Y^{'}) = (-2, 2)$ is a solution of Equation (2.3). Moreover, the pair $(X^{''}, Y^{''}) = (t, 1)$ solves the Pell equation $X^2 - (t^2-1)Y^2 = 1$. An infinite number of positive integer solutions of Equation (2.3) are given by

$$X_n + Y_n\sqrt{t^2-1} = \left(-2 + 2\sqrt{t^2-1}\right)\left(t + \sqrt{t^2-1}\right)^n,\ n \geq 0.$$

The remaining part of the proof is similar to the earlier ones for Theorems 1.1 and 1.2.

2) When $a = 1, b = 2$, put $y = x + 1$. Then Equation (1.6) can be reformulated in the form

$$\frac{1}{4}x^2\left((c+3)x^2 + (2c-2)x + c - 1\right) = z^2.$$

Taking $(c+3)x^2 + (2c-2)x + c - 1 = s^2$, we obtain Pell equation

$$((c+3)x + (c-1))^2 - (c+3)s^2 = -4c + 4.$$

If $c + 3 = d(t) = t^2 - 1$ (the proofs of other cases in Table 1 are similar to this one), we have

$$\left((t^2-1)x + t^2 - 5\right)^2 - (t^2-1)s^2 = -4t^2 + 20.$$

Letting $X = (t^2-1)x + t^2 - 5,\ Y = s$, then

$$X^2 - (t^2-1)Y^2 = -4t^2 + 20. \tag{2.4}$$

It is easy to see that the pair $(X^{'}, Y^{'}) = (-4, 2)$ is a solution of Equation (2.4), and the pair $(X^{''}, Y^{''}) = (t, 1)$ solves the Pell equation $X^2 - (t^2-1)Y^2 = 1$. So an infinite number of positive integer solutions of Equation (2.4) are given by

$$X_n + Y_n\sqrt{t^2-1} = \left(-4 + 2\sqrt{t^2-1}\right)\left(t + \sqrt{t^2-1}\right)^n,\ n \geq 0.$$

The remaining part of the proof is similar to the earlier ones.

3) When $a=1, b\neq 2, c=1$, take $y=x-1$. Then Equation (1.6) is equal to

$$\frac{1}{4}(x-1)^2\left((b+2)x^2-2(b+2)x+4\right)=z^2.$$

Let us consider

$$(b+2)x^2-2(b+2)x+4=s^2,$$

which is equivalent to the Pell equation

$$X^2-(b+2)Y^2=b^2-4, \tag{2.5}$$

where

$$X=(b+2)x-(b+2), \quad Y=s.$$

If $b+2>0$ is not a perfect square, then the Pell equation

$$X^2-(b+2)Y^2=1$$

has infinitely many positive integer solutions. Let (u,v) be the least positive integer solution of $X^2-(b+2)Y^2=1$. And $(X_0,Y_0)=(b+2,2)$ is a positive integer solution of Equation (2.5). An infinite number of positive integer solutions of Equation (2.5) are given by

$$X_n+Y_n\sqrt{b+2}=\left(b+2+2\sqrt{b+2}\right)\left(u+v\sqrt{b+2}\right)^n, \ n\geq 0.$$

The remaining part of the proof is similar to the earlier ones. □

Proof of Theorem 1.4. 1) When $a=(2t+1)^2(r^2(2t+1)^2-s),\ b=-2r^2(2t+1)^2+s,\ c=r^2$, let $y=(2t+1)x-t$. Then Equation (1.6) reduces to

$$\frac{1}{4}t(t+1)\left(sx^2-sx+r^2t(t+1)\right)=z^2.$$

Letting

$$Z=4z, \quad X=2x-1,$$

we get the Pell equation

$$Z^2-st(t+1)X^2=4r^2t^2(t+1)^2-st(t+1). \tag{2.6}$$

If $st(t+1)$ is not a perfect square, then the Pell equation

$$Z^2-st(t+1)X^2=1$$

has infinitely many positive integer solutions. Let (u,v) be the least positive integer solution of $Z^2-st(t+1)X^2=1$. Note that $(Z_0,X_0)=(2rt(t+1),1)$ is a positive integer solution of Equation (2.6). An infinite number of positive integer solutions of Equation (2.6) are given by

$$Z_n + X_n\sqrt{st(t+1)} = \left(2rt(t+1) + \sqrt{st(t+1)}\right)\left(u + v\sqrt{st(t+1)}\right)^n,\ n \geq 0,$$

which leads to

$$Z_n = uZ_{n-1} + vst(t+1)X_{n-1}, \quad X_n = vZ_{n-1} + uX_{n-1}.$$

Thus,

$$\begin{cases} Z_n = 2uZ_{n-1} - Z_{n-2}, & Z_0 = 2rt(t+1),\ Z_1 = t(t+1)(2ru+sv), \\ X_n = 2uX_{n-1} - X_{n-2}, & X_0 = 1,\ X_1 = 2rt(t+1)v + u. \end{cases}$$

From

$$z = \frac{Z}{4}, \quad x = \frac{X+1}{2},$$

we have

$$\begin{cases} z_n = 2uz_{n-1} - z_{n-2}, & z_0 = \dfrac{rt(t+1)}{2},\ z_1 = \dfrac{(2ru+sv)t(t+1)}{4}, \\ x_n = 2ux_{n-1} - x_{n-2} - u + 1, & x_0 = 1,\ x_1 = rt(t+1)v + \dfrac{u+1}{2}. \end{cases}$$

According to the above relations and $2|t(t+1)$, we have

$$z_{2n} \in \mathbb{Z}^+, \quad x_{2n} \in \mathbb{Z}^+,\ n \geq 1.$$

Thus, for $a = (2t+1)^2(r^2(2t+1)^2 - s),\ b = -2r^2(2t+1)^2 + s,\ c = r^2$, if $st(t+1)$ is not a perfect square, Equation (1.6) has infinitely many positive integer solutions

$$(x, y, z) = (x_{2n}, (2t+1)x_{2n} - t, z_{2n}),$$

where $n \geq 1$.

2) We can obtain the result in a similar way like in case 1). □

Example 2.2. When $r = 1, s = 1$, we have

$$(a, b, c) = \left(4t(t+1)(2t+1)^2, -8t^2 - 8t - 1, 1\right).$$

For a given positive integer t, Equation (1.6) has infinitely many positive integer solutions (x_n, y_n, z_n), which satisfy

$$\begin{cases} x_n = 2(2t+1)x_{n-1} - x_{n-2} - 2t, & x_0 = 1,\ x_1 = (t+1)(2t+1), \\ y_n = (2t+1)x_n - t, & \\ z_n = 2(2t+1)z_{n-1} - z_{n-2}, & z_0 = \dfrac{t(t+1)}{2},\ z_1 = t(t+1)^2. \end{cases}$$

Proof of Theorem 1.5. 1) When $a = \frac{1}{2}(2t+1)^2(sr^2(r+1)^2(2t+1)^2-2)s,\ b = -(sr^2(r+1)^2(2t+1)^2-1)s,\ c = \frac{1}{2}s^2r^2(r+1)^2$, let $y = (2t+1)x - t$. Then Equation (1.7) becomes

$$2st(t+1)x^2 - 2st(t+1)x + r^2t^2(t+1)^2(r+1)^2s^2 + 1 = (2z-1)^2.$$

Taking

$$Z = 4z - 2, \quad X = 2x - 1,$$

we get the Pell equation

$$Z^2 - 2st(t+1)X^2 = 4\left(rst(t+1)(r+1)+1\right)^2 - 2st(t+1)(2r+1)^2. \tag{2.7}$$

If $2st(t+1)$ is not a perfect square, then the Pell equation

$$Z^2 - 2st(t+1)X^2 = 1$$

has infinitely many positive integer solutions. Let (u,v) be the least positive integer solution of $Z^2 - 2st(t+1)X^2 = 1$. And $(Z_0, X_0) = (2(rst(t+1)(r+1)+1), 2r+1)$ is a positive integer solution of Equation (2.7). An infinite number of positive integer solutions of Equation (2.7) are given by

$$Z_n + X_n\sqrt{2st(t+1)} = \left(2(rst(t+1)(r+1)+1) + (2r+1)\sqrt{2st(t+1)}\right) \times \left(u + v\sqrt{2st(t+1)}\right)^n,\ n \geq 0.$$

The remaining part of the proof can be obtained in a way similar to that of Theorem 1.4. □

3. A Remark and Question on Equation (1.7)

Taking $(a,b,c) = (1,2,1)$ in Equation (1.7), we have

$$t_z = (t_x + t_y)^2.$$

Considering $t_z = w^2$, we obtain

$$Z^2 - 2W^2 = 1,$$

where

$$Z = 2z - 1, \quad W = 2w.$$

From

$$Z_n + W_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n,\ n \geq 1,$$

we get

$$Z_n = \frac{\varepsilon^n + \bar{\varepsilon}^n}{2}, \quad W_n = \frac{\varepsilon^n - \bar{\varepsilon}^n}{2\sqrt{2}},$$

where $\varepsilon = 3 + 2\sqrt{2}$, $\bar{\varepsilon} = 3 - 2\sqrt{2}$. Thus,

$$z_n = \frac{\varepsilon^n + \bar{\varepsilon}^n + 2}{4}, \quad w_n = \frac{\varepsilon^n - \bar{\varepsilon}^n}{4\sqrt{2}}.$$

It is easy to show that z_n and w_n are positive integers. Then we need to study the positive integer solutions of

$$w_n = t_x + t_y.$$

Solving it for y, we have

$$y = \frac{1 + \sqrt{-(2x-1)^2 + 8w_n + 2}}{2}. \tag{3.1}$$

By some numerical calculations, we get several positive integer solutions of Equation (3.1) in Table 2 in the range $1 \le n \le 100$ and $1 \le x \le 10^7$.

n	(x, y)
1	$(2, 1)$
2	$(3, 3), (4, 1)$
8	$(61, 684), (459, 511)$
10	$(678, 3942), (723, 3934)$
12	$(12726, 19530), (15831, 17110)$
14	$(23496, 133812), (61491, 121147)$
16	$(530868, 587532)$
17	$(28864, 1911458), (598862, 1815453)$

Table 2: Some positive integer solutions of Equation (3.1)

Maybe we could get more positive integer solutions in the large range. However, we were not able to give a positive answer to the following question.

Question 3.1. Are there finitely many positive integer solutions of Equation (3.1)?

A similar remark can be made for the case

$$t_z = (at_x + bt_y)^2.$$

There are some papers (see [1] and the related references) that studied the X-coordinates of the Pell equations as a special number or the sum (or product) of two special numbers, and obtained some finiteness results. But it seems that the methods are not applicable here for our case.

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