PERRON NUMBERS AND POSITIVE MATRICES OF MINIMAL ORDER

Anne Bertrand-Mathis[1]
Laboratoire de mathématiques et applications, Université de Poitiers, Site du Futuroscope - Téléport 2, Poitiers, France
anne.bertrand@math.univ-poitiers.fr

Florent Nguema Ndong
Département de Mathématiques et Informatique, Université des Sciences et Techniques de Masuku, Franceville, Gabon
florentnn@yahoo.fr

Received: 2/26/18, Revised: 2/27/20, Accepted: 4/20/21, Published: 5/4/21

Abstract

A Perron number is an algebraic integer $\lambda > 1$ whose conjugates μ_i satisfy $|\mu_i| < \lambda$. We prove that given a Perron number λ, there is an integer h_0 such that for $h \geq h_0$ there exists a primitive integer matrix B_h whose spectrum includes only λ^h and its conjugates, and whose order is equal to the degree of λ. We also prove that if λ is a Pisot number of degree 2 or 3, then there exists an integer primitive matrix of order 2 or 3 whose Perron eigenvalue is λ, and that if a Perron number λ is a Parry number without pirate value, then the trace of λ is equal to $[\lambda]$ or $[\lambda] + 1$.

1. Introduction

A real matrix B is *primitive* if it is nonnegative and if there exists an integer k such that all entries of B^k are strictly positive (hence for $h \geq k$, B^h is also strictly positive).

Perron's theorem asserts that a primitive matrix B admits a real eigenvalue $\lambda > 0$ such that every other eigenvalue μ satisfies $|\mu| < \lambda$; λ is said to be the *Perron eigenvalue* or *strictly dominant eigenvalue* of B. In this paper, we are interested in matrices with coefficients in $\mathbb{N}$ (integer matrices).

Lind [9] defines a *Perron number of degree d* to be any algebraic integer $\lambda > 0$, that is a root of an irreducible polynomial $P = X^d + c_1 X^{d-1} + \cdots + c_d$, where all c_i belong to $\mathbb{Z}$, and where all other zeroes $\mu_2, \cdots, \mu_d$ of P satisfy $|\mu_i| < \lambda$. The zeroes of P are the algebraic conjugates of λ. He proved the following theorem.

[1]Corresponding author

Theorem 1 (Lind [9]). Given a Perron number λ, there is a primitive integer matrix B whose Perron eigenvalue is λ. It is also possible to find such a matrix with entries 0 or 1.

See also [4] for a proof via language theory. Of course the order of B cannot be strictly smaller than d, but in Lind's construction the order of B is often greater than the degree of λ and B admits eigenvalues which are hence not conjugates of λ. In this case the irreducible polynomial of the Perron eigenvalue λ divides the characteristic polynomial of B and the algebraic degree d of λ is smaller than the order of B, so the following question arises.

Question 1. Given a Perron number of degree d, can we find a primitive matrix B of order d, with entries in $\mathbb{N}$, for which λ is an eigenvalue?

Of course, λ is the Perron eigenvalue of B. This is not always the case, and following Boyle and Handelman [6] and the work of Kim, Ormes and Roush [8], we recall necessary conditions for the existence of such a matrix. The following question also arises.

Question 2. Given a Perron number λ, if there is no such matrix, let C and D be two positive integer matrices of minimal order having λ as eigenvalue; do they have the same spectrum?

In what follows, we shall prove Theorem 2 and Propositions 3 and 4. Recall that Lind [9] proved that given a Perron number λ and an integer k, λ and λ^k have the same degree. (Proposition 5 in [9] asserts that λ belongs to $\mathbb{Q}\left(\lambda^k\right)$ so λ and λ^k have the same degree.)

Theorem 2. *Let λ be a Perron number and let d be the common degree of λ and its powers λ^k. Then there exists an integer k and a primitive integer matrix B of order d whose spectrum consists of λ^k and its conjugates (so λ^k is the Perron eigenvalue of B). We can choose k such that, for each $h \geq k$, λ^h is the eigenvalue of some primitive integer matrix of order d.*

The *trace* $\operatorname{tr}\lambda$ of an algebraic integer λ is the sum of its conjugates (including λ itself); it is equal to $-c_1$, the coefficient of X^{d-1} of the minimal polynomial. If λ is the eigenvalue of a nonnegative integer matrix of order d the trace of this matrix is the trace of λ and has to be nonnegative. A *Pisot number* is an algebraic integer $\beta > 1$ whose conjugates μ_i satisfy $|\mu_i| < 1$.

Proposition 1.
(1) A Perron number of degree 2 *is the Perron eigenvalue of some primitive integer matrix of order* 2.
(2) A Pisot number of degree 2 *(resp.* 3*) is the Perron value of some primitive*

integer matrix of order 2 *(resp.* 3*).*
(3) For each $d \geq 3$ *there exist Perron numbers of degree* d *whose traces are negative and which are not eigenvalues of a primitive integer matrix of order* d*.*

McKee, Rowlinson, and Smyth [12] proved the existence of Pisot numbers of any desired traces (even negative). McKee and Smyth [11] found a Pisot number with negative trace of degree $d = 16$.

Example 1. The zero $\lambda > 1$ of the polynomial $X^3 - X - 1$ (λ is the plastic number) is a Perron number of degree 3, it is also the dominating root of the polynomial $X^5 - X^4 - 1 = (X^3 - X - 1)(X^2 - X + 1)$. It is the Perron eigenvalue of the matrices $U = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, which are both primitive.

Example 2. The matrix $W = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is primitive with Perron value $\frac{3+\sqrt{5}}{2}$.

Now let us explain why we are interested in integer primitive matrices of minimal order.

Given a labelled directed graph G on h vertices, the *adjacency matrix* A of G is the (h, h) matrix where the entry $a_{i,j}$ is the number of labelled edges from vertex i to vertex j, all entries being nonnegative integers. The eigenvalues of the adjacency matrix are said to be *eigenvalues of the graph*.

By a *path of length* n in a graph G we mean a sequence $a_1, \cdots, a_n$ of labelled edges such that the terminal vertex of a_i is the initial vertex of a_{i+1}, $1 \leq i < n$. If the terminal vertex of a_n is the initial vertex of a_1, then the path is a *loop of length* n. A *loop* $a_1, \cdots, a_n$ *is minimal* unless there exists a loop $b_1, \cdots, b_m$ with $m < n$ such that $a_1, \cdots, a_n$ is obtained by concatenating $b_1, \cdots, b_m$ n/m times. Let w_n be the number of paths with length n and suppose that the matrix is primitive, then $\lim_{n\to\infty} \frac{\log w_n}{n} = \ln \lambda$.

Question 1 becomes: *given a Perron number* λ *of degree* d, *is it possible to find a graph on* d *vertices with eigenvalue* λ, *or an automaton with* d *states and eigenvalue* λ*?*

Symbolic Dynamical Systems. Let S be a set of r symbols endowed with the discrete topology. Then the space $S^{\mathbb{Z}}$ of sequences $(x_n)_{n\in\mathbb{Z}} = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$ on S endowed with the product topology is a compact set; the shift map $\sigma : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is defined by setting $(\sigma x)_{n\in\mathbb{Z}} = (x_{n+1})_{n\in\mathbb{Z}}$. If $Y \subseteq S^{\mathbb{Z}}$ is compact, nonempty and σ-invariant (*i.e.,* $\sigma^{-1}(Y) = Y$), then (Y, σ) is called a *symbolic dynamical system.* A general dynamical system is a space with an invariant transformation. Symbolic systems are not anecdotal ones. Hadamard noticed that many dynamical systems can be viewed as symbolic dynamical systems.

Example 3. Let α be an irrational rotation of the the unit circle $[0, 2\pi[$; given a point y, and $n \in \mathbb{Z}$ let $x_n = a$ if $\alpha^n(y) = y + n\alpha \in [0, \pi[$ modulo 2π and $x_n = b$ if $\alpha^n(y)$ is in $[\pi, 2\pi[$; if we know the sequence $(x_n)_{n\in\mathbb{Z}}$, we know the point y, so the rotation can be seen as a symbolic system. Such a situation occurs frequently.

The infinite paths $(a_n)_{n\in\mathbb{Z}}$ of a graph G with a matrix A of order d defines a symbolic dynamical system associated with the graph G; if A is primitive the system is what we call a mixing Markov Shift with d states and $\ln \lambda$ denotes its entropy. So Question 1 becomes: *given a Perron number λ of degree d, does there exist a symbolic dynamical system with entropy $\ln \lambda$ associated with a graph with only d vertices?*

Example 4. The dynamical system $X = (x_n)_{n\in\mathbb{Z}}$ where $x_n \in \{0, 1, 2\}$ and where the word 22 never appears is a system with two states and entropy $\ln \frac{3+\sqrt{5}}{2}$ related to the matrix W of Example 2.

We are also interested in substitutions. A *substitution of h letters $b_1, \cdots, b_h$* is a map φ from the set $\{b_1, \cdots, b_h\}$ into the set of finite words on these letters, extended by concatenation to finite words ($\varphi(uv)$ is the word $\varphi(u)\varphi(b)$). The adjacency matrix B is the nonnegative matrix whose entry $a_{i,j}$ is the number of letters b_i contained in $\varphi(b_j)$; we can easily associate a substitution to each nonnegative matrix. If for some k all $\varphi^k(b_i)$ contains all b_j, the matrix is a primitive matrix and the substitution is said to be primitive. In this case the number l_k of letters contained in $\varphi^k(b_i)$ satisfies $\lim_{k\to\infty} \frac{\log l_k}{k} = \ln \lambda$, where λ is the Perron eigenvalue of B, so λ is said to be an eigenvalue of the substitution. If there is a letter b_i such that $\varphi(b_i)$ begins with b_i then the word $\varphi^k(b_i)$ is the beginning of $\varphi^{k+1}(b_i)$ and the substitution admits a fixed point $\varphi^\infty(b_i)$ (such a letter exists if the trace of B is positive).

Question 1 becomes: *given a Perron number λ of degree d, can we find a primitive substitution on d letters with eigenvalue λ? And what about a fixed point?*

If β is a Pisot number of degree d, does β admits a substitution on d letters with eigenvalue β? (such a substitution is called a Pisot substitution).

Remark. The answer to the last question is no for the general case, since there are Pisot numbers with negative trace $[10, 11]$.

Example 5. The substitution $a \to ab,\ b \to c,\ c \to d,\ d \to e,\ e \to a$ is a primitive substitution with a fixed point $abcdeaab\cdots$ with the matrix V of Example 1.

2. Boyle-Handelman's Spectral Conjecture

We say that a $k-$tuple $\triangle = (d_1, \cdots, d_k)$ of nonzero complex numbers is the *nonzero spectrum* of a matrix A if for some $m \geq 0$, the characteristic polynomial of A is $X_B(t) = t^m \prod_{i=1}^{k}(t - d_i)$. The eigenvalues of A are $d_1, \cdots, d_k$ and m zeroes when m is positive. We set $\triangle^n := (d_1^n, \cdots, d_k^n)$. We denote *trace of* $\triangle$, denoted by $\mathrm{tr}\,\triangle$, the sum of the entries of $\triangle$. If a matrix A has nonzero spectrum $\triangle$, then tr $A^n = \mathrm{tr}(\triangle^n)$. We say that $\triangle$ has a *Perron value* (denoted by $\lambda_\triangle$) if there exists an index i such that $d_i > |d_j|$ for $j \neq i$, and we set $\lambda_\triangle = d_i$.

Consider a nonnegative adjacency matrix A of a graph G. The number of loops of length n is the sum of the numbers of minimal loops of length m where m runs through the set of divisors of n (including 1 and n). Suppose that the decomposition into prime factors of n is $n = p_1^{h_1} \cdots p_r^{h_r}$. Then the number of minimal loops of length n is equal to the following expression called the *nth trace of A* in [6]:

$$\mathrm{tr}_n A := \sum_{d|n} \mu(d)\, \mathrm{tr}\, A^{n/d},$$

where d belongs to $[1, 2, \cdots, n]$, and where μ denotes the Mobius function: $\mu(d) = (-1)^k$ if d is square-free and has k distinct prime divisors and $\mu(d) = 0$ if d has a square divisor.

If $\triangle$ is the nonzero spectrum of a nonnegative integer matrix A, then for each n the $\mathrm{tr}_n A$ has to be nonnegative.

The following conjecture is due to Boyle and Handelman.

Boyle-Handelman Spectral Conjecture (Integers Case) [6]: *a k-tuple* $\triangle = (d_1, \cdots, d_k)$ *is the nonzero spectrum of some primitive nonnegative matrix with entries in* $\mathbb{N}$ *if and only if* $\triangle$ *has a Perron value, the coefficients of* $\prod_{i=1}^{k}(t - d_i)$ *are in* $\mathbb{Z}$*, and* $\mathrm{tr}_n \triangle \geq 0$ *for every positive integer n.*

Kim, Ormes and Roush [8] proved this case of the conjecture. The order of the matrix that they furnish is not always equal to the number of entries d in $\triangle$. So a Perron number λ with nonnegative n^{th} net traces is the eigenvalue of an integer nonnegative matrix whose spectrum contains λ, its algebraic conjugates and perhaps some zeroes.

We want to get rid of these zeroes; we shall also reformulate this conjecture: given an algebraic number λ of degree d with conjugates $\lambda = \mu_1, \mu_2, \cdots, \mu_d$, we set $\triangle = \{\lambda_1, \mu_2, \cdots, \mu_d\}$; for $n = p_1^{h_1} \cdots p_r^{h_r}$ as above let $\mathrm{tr}_n \lambda$ denote the n^{th} net trace of the set $\triangle = \{\lambda_1, \mu_2, \cdots, \mu_d\}$:

$$\mathrm{tr}_n \lambda := \sum_{k=0}^{r} \sum_{i_1 < \cdots < i_k} (-1)^k\, \mathrm{tr}\left(\lambda^{\frac{n}{p_{i_1} \cdots p_{i_k}}}\right) = \mathrm{tr}\left(\lambda^n \prod_{i=1,\cdots,r}\left(1 - \frac{1}{\lambda^{p_i}}\right)\right).$$

Conjecture. A Perron number λ of degree d is the eigenvalue of some positive matrix of order d with entries in $\mathbb{N}$ if and only if $\mathrm{tr}_n\lambda \geq 0$ for all positive integer n.

We shall prove Theorem 3 which trivially implies Theorem 2.

Theorem 3. *Let $\triangle$ be a $d-$tuple of nonzero complex numbers admitting a Perron value; then there exists h_0 such that for each $h \geq h_0$, $\triangle^h$ is the spectrum of some primitive integer matrix B of order d. The matrix can be chosen from the two possibilities:* $B = \begin{bmatrix} 0 & 0 & \dots & a_d \\ 1 & 0 & \dots & \vdots \\ \vdots & \ddots & \dots & \vdots \\ 0 & \dots & 1 & a_1 \end{bmatrix}$ *or* $B = \begin{bmatrix} b & 0 & 0 & \dots & a_d \\ 1 & 0 & 0 & \dots & a_{d-1} \\ 0 & 1 & 0 & \dots & \vdots \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & 1 & a_1 \end{bmatrix}$.

3. Proof of Theorem 3

Lemma 1 [3]. (Companion matrix of certain finite or infinite sequence) *Let $a_1, \cdots, a_k$ be integers with all $a_i \geq 0$ and $a_k > 0$; then the companion matrix* $B\left((a_1 \cdots a_k)\right) := \begin{bmatrix} 0 & 0 & \dots & a_k \\ 1 & 0 & \dots & \vdots \\ \vdots & \ddots & \dots & \vdots \\ 0 & \dots & 1 & a_1 \end{bmatrix}$ *of the polynomial $X^k - a_1X^{k-1} - \cdots - a_k$ is primitive and $X^k - a_1X^{k-1} - \cdots - a_k$ is its characteristic polynomial.*

Let b be a number greater than 0 and let $a_1, \cdots, a_{k-1}, a_k$, be a sequence of positive numbers such that $a_k > 0$; let $(a_n)_{n\geq 1} = a_1, \cdots, a_{k-1}, a_k, a_kb, a_kb^2, a_kb^3, \cdots$ be the infinite sequence with all $a_i \geq 0$, $a_kb > 0$ and for $r \geq 0$ $a_{k+r} = b^r a_k$. Then the "companion matrix" $B\left((a_n)_{n\geq 1}, b\right) := \begin{bmatrix} b & 0 & 0 & \dots & a_k \\ 1 & 0 & 0 & \dots & a_{k-1} \\ 0 & 1 & 0 & \dots & \vdots \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & 1 & a_1 \end{bmatrix}$ *of the pair $\left((a_n)_{n\geq 1}, b\right)$ is primitive and if a number λ satisfies*

$$1 = \frac{a_1}{\lambda} + \cdots + \frac{a_{k-1}}{\lambda^{k-1}} + \frac{a_k}{\lambda^k} + \frac{a_kb}{\lambda^{k+1}} + \frac{a_kb^2}{\lambda^{k+2}} + \frac{a_kb^3}{\lambda^{k+3}} + \cdots,$$

then λ is a zero of the polynomial

$$X^k - (a_1X^{k-1} + \cdots + a_{k-1}X + a_k) - b(X^{k-1} - (a_1X^{k-2} + \cdots + a_{k-1})).$$

which is also the characteristic polynomial of B.

The proof is straightforward and can be found in [3].

Lemma 2. *If a real number $\lambda > 1$ satisfies an equation of the form*

$$\lambda^k = a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_k,$$

or

$$1 = \frac{a_1}{\lambda} + \cdots + \frac{a_{k-1}}{\lambda^{k-1}} + \frac{a_k}{\lambda^k} + \frac{a_k b}{\lambda^{k+1}} + \frac{a_k b^2}{\lambda^{k+2}} + \frac{a_k b^3}{\lambda^{k+3}} + \cdots,$$

where the a_i are nonnegative integers, a_k is greater than 0 and b is greater than 0, then λ is the eigenvalue of a primitive matrix of order k with entries in $\mathbb{N}$.

Proof. Immediate from Lemma 1. □

The following lemma is inspired by Soittola [5].

Lemma 3. *Let $c_1, c_2, \cdots, c_k$ and b be real numbers ($b \neq 0$) and let $\lambda > |b|$ satisfies $\lambda^k = c_1\lambda^{k-1} + c_2\lambda^{k-2} + \cdots + c_k$ (1); then λ satisfies*

$$1 = \frac{a_1}{\lambda} + \cdots + \frac{a_{k-1}}{\lambda^{k-1}} + \frac{a_k b}{\lambda} + \frac{a_k b^2}{\lambda^2} + \frac{a_k b^3}{\lambda^3} + \cdots$$

$$= \frac{a_1}{\lambda} + \cdots + \frac{a_{k-1}}{\lambda^{k-1}} + a_k\left(\frac{b}{\lambda} + \frac{b^2}{\lambda^2} + \frac{b^3}{\lambda^3} + \cdots\right),$$

where a_i and b_i satisfy conditions (2):

$a_1 = c_1 - b,$
$a_2 = b(c_1 - b) + c_2,$
$a_3 = b^2(c_1 - b) + bc_2 + c_3,$
$\vdots$
$a_{k-1} = b^{k-2}(c_1 - b) + b^{k-3}c_2 + b^{k-4}c_3 + \cdots + bc_{k-2} + c_{k-1},$
$a_k = b^{k-1}(c_1 - b) + b^{k-2}c_2 + \cdots + bc_{k-1} + c_k,$
$a_{k+r} = b^{k+r-1}(c_1 - b) + b^{k+r-2}c_2 + \cdots + b^r c_k,$

(for $h \geq k$ we have $a_{h+1} = ba_h$, so that $a_{k+r} = b^r a_k$.)

Proof. Replacing in (1) $c_1\lambda^{k-1}$ by $(c_1 - b)\lambda^{k-1} + b\lambda^{k-1}$, then $b\lambda^{k-1}$ by

$$b\left(c_1\lambda^{k-2} + c_2\lambda^{k-3} + \cdots + \frac{c_k}{\lambda}\right),$$

we obtain $\lambda^k = (c_1 - b)\lambda^{k-1} + c_2\lambda^{k-2} + \cdots + c_k + b\lambda^{k-1}$
$= (c_1 - b)\lambda^{k-1} + c_2\lambda^{k-2} + \cdots + c_k$

$$+bc_1\lambda^{k-2} + bc_2\lambda^{k-3} + \cdots + b\tfrac{c_k}{\lambda}.$$

Then replace $bc_1\lambda^{k-2}$ by $b(c_1-b)\lambda^{k-2} + b^2\lambda^{k-2}$ to obtain

$$\lambda^k = (c_1-b)\lambda^{k-1} + c_2\lambda^{k-2} + \cdots + c_k + \\ + b(c_1-b)\lambda^{k-2} + bc_2\lambda^{k-3} + \cdots + b\tfrac{c_k}{\lambda} + b^2\lambda^{k-2}.$$

Now replace $b^2\lambda^{k-2}$ by $b^2((c_1-b)\lambda^{k-3} + \cdots + \frac{c_k}{\lambda^2}) + b^3\lambda^{k-3}$, so that

$$\lambda^k = (c_1-b)\lambda^{k-1} + c_2\lambda^{k-2} + c_3\lambda^{k-3} + \cdots + c_k + \\ + b(c_1-b)\lambda^{k-2} + bc_2\lambda^{k-3} + .. + b\tfrac{c_k}{\lambda} \\ + b^2(c_1-b)\lambda^{k-3} + +b^2c_2\lambda^{k-4} + \cdots + b^2\tfrac{c_k}{\lambda^2} + b^3\lambda^{k-3}.$$

Iterating the process, we obtain

$$\lambda^k = a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_{k-1}\lambda + a_k + \frac{a_{k+1}}{\lambda} + \frac{a_{k+2}}{\lambda^2} + \cdots,$$

where the numbers a_i satisfy (2). □

Lemma 4. *Let $\triangle = (d_1, \cdots, d_k)$ be a Perron $k-$tuple, $\lambda = \lambda_\triangle = d_1$ its Perron value, and let $\triangle^m$ denote the $k-$tuple $\triangle^m = (d_1^m, \cdots, d_k^m)$.*

Then there exists an integer h_0 such that for $h \geq h_0$ we can find nonnegative integers $a_1, \cdots, a_k, b$ satisfying $a_k b \neq 0$, and

$$1 = \frac{a_1}{\lambda^h} + \frac{a_2}{\lambda^{2h}} + \cdots + \frac{a_{k-1}}{\lambda^{(k-)h}} + \frac{a_k}{\lambda^{kh}} + \frac{a_k b}{\lambda^{(k+1h)}} + \frac{a_k b^2}{\lambda^{(k+2)h}} + \frac{a_k b^3}{\lambda^{(k+3)h}} + \cdots.$$

Proof. Let $\triangle = (d_1, \cdots, d_k)$ be a Perron $k-$tuple. The symmetric functions $\sigma_1, \cdots,$ σ_k of $\triangle$ are the k numbers $\sigma_i = \sum_{1\leq l_1<l_2<\cdots<l_i\leq k} d_{l_1}d_{l_2}\cdots d_{l_i}$, $i = 1, \cdots, k$. If the coefficients of $\prod_{i=1}^k (t - d_i)$ are in $\mathbb{Z}$, all the symmetric functions are in $\mathbb{Z}$ and the d_i are the zeroes of the polynomial

$$X^k - \sigma_1 X^{k-1} + \sigma_2 X^{k-2} - \cdots + (-1)^k \sigma_k.$$

Let $\sigma_1^{(h)}, \cdots, \sigma_d^{(h)}$ denote the symmetric functions of $\triangle^h = \left(d_1^h, \cdots, d_k^h\right)$. Define $\mu = \lambda^h$; then μ is a zero of the polynomial

$$X^k - \sigma_1^{(h)} X^{k-1} + \sigma_2^{(h)} X^{k-2} - \cdots + (-1)^k \sigma_k^{(h)}.$$

Take $(c_1, \cdots, c_k) = (\sigma_1^{(h)}, \cdots, \sigma_d^{(h)})$ in Lemma 3 and take for b the integer part of $\frac{\sigma_1^{(h)}}{2}$; b is about $\frac{\lambda^h}{2}$ and if h is large enough, then b is strictly positive.

The symmetric function $\sigma_j^{(h)}$ is the sum of the term $\lambda^h A_j$, where A_j is the sum of at most $k!$ terms which are each products of $j-1$ factors $d_{i_1}^h \cdots d_{i_{j-1}}^h$, and of at most $k!$ terms which are each products of j factors $d_{i_1}^h \cdots d_{i_j}^h$, all d_{i_r} being smaller than λ. As $d_i < \lambda$ for $i > 1$, $\lim_{h\to\infty} \frac{d_i^h}{\lambda^h} = 0$.

The term a_1 obtained in Lemma 3 is equal to $\sigma_1^{(h)} - b = \sigma_1^{(h)} - \left\lfloor \frac{\sigma_1^{(h)}}{2} \right\rfloor$(here $\lfloor y \rfloor$ denotes the integer part of y). Hence, if h is large enough, $a_1 = \sigma_1^{(h)} - b$ is a positive integer. For $u = 1, \cdots, k-1$, $(\sigma_1^{(h)} - b)b^u = \left(\sigma_1^{(h)} - \left\lfloor \frac{\sigma_1^{(h)}}{2} \right\rfloor\right) \left\lfloor \frac{\sigma_1^{(h)}}{2} \right\rfloor^u$ is equivalent to $\frac{\lambda^{h(1+u)}}{2^h}$. As $d_i < \lambda$, $\lim_{h\to\infty} \frac{d_i^h}{\lambda^h} = 0$, for $i > 1$, so $a_2, \cdots, a_{k-1}$ are also positive integers for h large enough. Then all the a_i's and b's in condition (2) of Lemma 3 are positive. Furthermore,

$$1 = \frac{a_1}{\mu} + \frac{a_2}{\mu^2} + \cdots + \frac{a_{k-1}}{\mu^{k-1}} + \frac{a_k}{\mu^k} + \frac{a_k b}{\mu^{k+1}} + \frac{a_k b^2}{\mu^{(k+2)}} + \frac{a_k b^3}{\mu^{(k+3)}} + \cdots .$$

As $\mu = \lambda^h$, this shows Lemma 4.

□

Proof of Theorem 3 and Theorem 2. Given a $d-$tuple $\triangle$ of nonzero complex numbers admitting a Perron value $\lambda = d_1$, apply Lemma 4 and Lemma 2 to obtain Theorem 3. Theorem 2 is an immediate consequence of Theorem 3.

4. Proof of Proposition 1

Proof of Assertion (1) (see also[6]). Let λ be a Perron number of degree 2; then $\lambda > 1$ is the largest root of an integer polynomial, $X^2 - sX + p$, so $k = s^2 - 4p$ is > 0. If $s = 2h$ the matrix $\begin{pmatrix} h & 1 \\ h^2 - p & h \end{pmatrix}$ is a primitive matrix of which λ is an eigenvalue. If $s = 2h+1$ then $4p < (h+1/2)^2$ so $p \leq h^2+h+1/4$. If $p = h^2+h$, the eigenvalues are $\lambda = (h+1)$ and $\lambda' = h$ whose degree is 1 and not 2; so $p < h^2 + h$ and the matrix $\begin{pmatrix} h+1, & 1 \\ h^2+h-p, & h \end{pmatrix}$ is suitable.

Proof of Assertion (2). We have to examine many cases. We recall that all symmetric functions of λ belong to $\mathbb{Z}$. Let β be a Pisot number of degree 3 (the case $d = 2$ is included in the Perron case). The numbers $\sigma_1, \sigma_2, \sigma_3$ are all integers.

1. *Case* $\beta \leq \frac{1+\sqrt{5}}{2}$: the list of small Pisot numbers has only four entries of degree $2, 3$ or 4, roots of $x^2 - x - 1$, $x^3 - x - 1$, $x^3 - x^2 - 1$ and $x^4 - x^3 - 1$, and the result is true since the companion matrices of theses minimal polynomials are primitive.

2. *Case* $\beta > \frac{1+\sqrt{5}}{2}$. Let x_1, x_2 be the conjugates of β with modulus less than 1. The minimal polynomial of β is $X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3$ where $\sigma_1 = \operatorname{tr}\beta = \beta + x_1 + x_2$, $\sigma_2 = \beta(x_1 + x_2) + x_1 x_2$, $\sigma_3 = x_1 x_2 \beta$. Of course $|x_1 x_2| < 1$ and $|x_1| + |x_2| < 2$.

We shall examine different cases concerning the integer $\operatorname{tr}\beta$.

2.1. `Case` $\operatorname{tr}\beta < 0$: this is impossible. If $\operatorname{tr}\beta = x_1 + x_2 + \beta < 0$, as it belongs to $\mathbb{Z}$, $x_1 + x_2 + \beta \leq -1$ and we should have $x_1 + x_2 \leq -1 - \beta < -2$.

2.2. `Case` $\operatorname{tr}\beta = 0$: if $\operatorname{tr}\beta = 0$ then $x_1 + x_2 = -\beta < -1$, so x_1 and x_2 are both negative or complex conjugates with negative real parts and $|x_1| + |x_2| < 2$ so β is less than 2. Then $\sigma_3 = x_1 x_2 \beta$ is positive and $\sigma_3 \leq \beta < 2$, so $\sigma_3 = 1$. As

$\beta\,(x_1+x_2) = -\beta^2 < -1$ and $x_1x_2 \le 1$, σ_2 is less than 0. Then β is a root of equation $X^3 = |\sigma_2|X+1$, and the companion matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & |\sigma_2| \\ 0 & 1 & 0 \end{bmatrix}$ is primitive.

2.3. `Case` $\operatorname{tr}\beta = 1$: x_1 and x_2 cannot both be positive or have positive real parts since $\operatorname{tr}\beta$ would be greater than 1.

2.3.1. `Case` $\operatorname{tr}\beta = 1$ *and* x_1 *and* x_2 *are negative or have negative real parts*: σ_3 is positive and as $\beta > \frac{1+\sqrt{5}}{2}$, $\beta(\beta-1) > 1$, then $\sigma_2 = \beta(1-\beta)+x_1x_2 < -1+1 < 0$. So the equation $\beta^3 = \beta^2 + |\sigma_2|x + \sigma_3$ gives a nonnegative primitive companion matrix.

2.3.2. `Case` $\operatorname{tr}\beta = 1$ *and* x_1 *is negative and* x_2 *positive*: as $1 = x_1+x_2+\beta$, x_1+x_2 is negative and $\beta < 2$; σ_2 is the sum of two strictly negative terms so it is strictly negative, and $|\sigma_2| \ge 1$; σ_3 is also negative, β is at most 2 so $\sigma_3 = -1$. We have $\beta^3 = \beta^2 + |\sigma_2|\beta - 1$.

Apply Lemma 3 with $b = 1$ to this equation to obtain

$$1 = \frac{0}{\beta} + \frac{|\sigma_2|}{\beta^2} + \frac{|\sigma_2|-1}{\beta^3} + \frac{|\sigma_2|-1}{\beta^4} + \cdots .$$

This series with $a_1 = 0, a_2 = |\sigma_2|,\ a = |\sigma_2|-1$ and $b = 1$ give us a primitive matrix with order 3 $\begin{bmatrix} 1 & 0 & (|\sigma_2|-1) \\ 1 & 0 & |\sigma_2| \\ 0 & 1 & 0 \end{bmatrix}$ (Lemma 1). Note that $\sigma_2 - 1 \ne 0$, otherwise the degree of β would not be 3 .

2.4. `Case` $\operatorname{tr}\beta \ge 2$.

2.4.1. `Case` $\operatorname{tr}\beta \ge 2$ *and* $\operatorname{tr}\beta < \beta$: then $x_1+x_2 = (\sigma_2-\beta) < 0$, x_1 and x_2 cannot both be positive or have positive real parts.

2.4.1.1. `Case` $\operatorname{tr}\beta \ge 2$, $\operatorname{tr}\beta < \beta$ *and both* x_i *are less than* 0 *or have negative real part*: σ_3 is greater than 0; $\sigma_2 = \beta\,(x_1+x_2)+x_1x_2$ is an integer and is the sum of a positive term $x_1x_2 < 1$ and a negative term: σ_2 is less than or equal to 0 so $\beta^3 = \sigma_1\beta^2 + |\sigma_2|\beta + \sigma_3$. All coefficients are positive and we can apply Lemma 2 to obtain the result.

2.4.1.2 `Case` $\operatorname{tr}\beta \ge 2$, $\operatorname{tr}\beta < \beta$ *and* $x_1 < 0$, $x_2 > 0$: then $x_1x_2 < 0$ and $x_1+x_2 < 0$; hence $\sigma_2 < 0$ and $\sigma_3 < 0$. Apply the method of Lemma 3 with $b = 1,\ c_1 = \sigma_1,\ c_2 = |\sigma_2|,\ c_3 = \sigma_3$: we need only to prove that $a_1 = \sigma_1 - 1$, $a_2 = |\sigma_2| + \sigma_1 - 1,\ a_3 = \sigma_3 + |\sigma_2| + \sigma_1 - 1$ are all greater than or equal to 0 and that a_3 is greater than 0. It is clear that a_1, a_2 are nonnegative; $a_3 = \beta x_1x_2 + \beta|x_1+x_2| + \beta - x_1 - x_2$; as the x_i have modulus less than 1, $\beta x_1x_2 + \beta$ is greater than 0 and $\beta|x_1+x_2| - x_1 - x_2$ is positive since $\beta > 1$ so $a_3 > 0$.

2.4.2. `Case` $\operatorname{tr}\beta \ge 2$ *and* $\operatorname{tr}\beta > \beta$: the x_i are not both less than 0.

2.4.2.1. `Case` $\operatorname{tr}\beta \ge 2$, $\operatorname{tr}\beta > \beta$, $x_1 > 0$ *and* $x_2 < 0$: then $\sigma_3 < 0$ and $|\sigma_3| < \beta$; the function $f\,(x) = x^3 - \sigma_1x^2 + \sigma_2x - \sigma_3$ has to be negative for $x = 1$, so $1 - \sigma_1 + \sigma_2 - \sigma_3 < 0$ and $1 + \sigma_2 + |\sigma_3| < \sigma_1$: apply Lemma 2 with $b = \sigma_2 + \sigma_3$ to the equation $x^3 = \sigma_1x^2 - \sigma_2x + \sigma_3$ to obtain positive a_i and conclude using Lemma 1.

2.4.2.2. `Case` $\operatorname{tr}\beta \geq 2$, $\operatorname{tr}\beta > \beta$ *and the* x_i *are both greater than* 0 : then $\sigma_3 > 0$. We are looking for an integer $b > 0$ such that $a_1 = \sigma_1 - b > 0$, and $a_2 = b\,(\sigma_1 - b) - \sigma_2 \geq 0$ (so necessarily $a_3 = b^2\,(\sigma_1 - b)) - b\sigma_2 + \sigma_3 > 0$. We shall examine the two possibilities: $x_1 + x_2 < 1$ and $x_1 + x_2 > 1$.
2.4.2.2.1. If $0 < x_1 + x_2 < 1$, $\sigma_1 = \lfloor\beta\rfloor + 1$, σ_3 is positive and $\sigma_2 = \beta\,(x_1 + x_2) + x_1x_2$ is the sum of a term smaller than β and a term smaller than 1, so $\sigma_2 \leq \lfloor\beta\rfloor + 1 = \sigma_1$.

If $\sigma_1 \geq \sigma_2 + 1$ apply Lemma 2 to the equation $\lambda^3 = \sigma_1\lambda^2 - \sigma_2\lambda + \sigma_3$ with $b = 1$ to obtain the result.

If $\sigma_1 = \sigma_2$, $\sigma_2 = \beta\,(x_1 + x_2) + x_1x_2 = \beta + x_1 + x_2$. As $x_1 + x_2 < 1$ this means that $x_1x_2 > x_1 + x_2$; if x_1 and x_2 are real, setting $a = x_1 + x_2$ we obtain $x_1(a - x_1) > a$, which is impossible with $a > 1$ and positive x_i. If they are complex numbers the discriminant of the derivative gives $\sigma_1 < 3$ so $\sigma_1 = 1$ (impossible since $\sigma_1 > \beta$) or $\sigma_1 = 2$, hence $\sigma_3 = 1$. But 1 is a root of the polynomial $X^3 - 2X^2 + 2X - 1$, which is not the irreducible polynomial of a Pisot number of degree 3.
2.4.2.2.2 *Case* $x_1 + x_2 > 1$: as we suppose that $\beta > \frac{1+\sqrt{5}}{2}$, if $x_1 + x_2 > 1$, then $\beta + 1 > 2$ and $\sigma_1 \geq 3$.

Case $\sigma_1 = 3$: $\beta \in \left[\frac{1+\sqrt{5}}{2}, 2\right]$, $x_1 + x_2 < 3 - \frac{1+\sqrt{5}}{2}$ which is at most 1.4; then $\sigma_2 \leq \beta(1.4) + x_1x_2 \leq 3.8$ and $\sigma_2 \leq 3$; but $\sigma_2 = \beta(x_1 + x_2) + x_1x_2 \geq 2$.

As $\beta < 2$, σ_3 is less than 2, it is equal to 1.

If $\sigma_2 = 3$ the equation is $X^3 - 3X^2 + 3X - 1$; this is impossible because the polynomial is not irreducible.

If $\sigma_2 = 3$ the equation becomes $X^3 - 3X^2 + 2X - 1$, and we apply Lemma 3 with $b = 1$.

If $\sigma_1 \geq 4$, $\beta > 2$ and $x_1 + x_2 = 1 + a$ where $a < 1$; $\sigma_1 = \beta + 1 + a$ and $\sigma_2 = \beta\,(1 + a) + x_1x_2$; $2(\sigma_1 - 2) = 2\beta + 2a - 2$; $2(\sigma_1 - 2) - \sigma_2 = \beta\,(1 - a) - 2\,(1 - a) - x_1x_2$. As $\beta > 2$, $\beta\,(1 - a) - 2\,(1 - a)$ is positive, the other term is negative, so $2(\sigma_1 - 2) > \sigma_2$, and we apply Lemma 3 with $b = 2$.

Proof of Assertion (3). All powers of a Pisot number of degree d are Pisot numbers of the same degree d and for each $d \geq 2$ there exists a Pisot number φ of degree d [2]. Let $d > 2 \in \mathbb{N}$; consider a power φ^n of φ and an integer a such that $\varphi^n - a > a + 1$ and $\varphi^n - da + d < 0$ ($a \in \,]\varphi^n/d - 1, \varphi^n/2 + 1[$). Then the $d - 1$ conjugates α_i of φ^n have modulus < 1, so the trace $\varphi^n + \sum \alpha_i - ad$ of $\varphi^n - a$ is negative and $\varphi^n - a > |\alpha_i - a|$, hence $\varphi^n - a$ is a Perron number with negative trace and degree d.

Example 6. The Tribonacci number φ, root of $X^3 - X^2 - X - 1$, is approximately 1.8393 and has degree 3; φ^5 is almost 38.7 and has two conjugates α and β of modulus less than 1, hence $\alpha - 14$ and $\beta - 14$ have modulus less than 15 and since $|varphi^5 - 14| > 15$, $\xi = \varphi^5 - 14$ is a Perron number, its trace is $\xi + \alpha + \beta - 3.14$ which is $< 38.7 + 2 - 42$, and is negative.

5. About Parry numbers

Given a real number $\beta > 1$, we know that 1 admits a so-called $\beta-$*expansion*: $1 = \sum_{k\geq 1} \frac{d_k}{\beta^k}$ where $d_k \leq \lfloor \beta \rfloor$ for all k, all the d_k are nonnegative integers and for every $h \geq 1$ we have $\frac{d_h}{\beta^h} + \frac{d_{h+1}}{\beta^{h+1}} + \cdots < \frac{1}{\beta^{h-1}}$; *in all cases* $d_1 = \lfloor \beta \rfloor$ [14]. If the sequence $(d_k)_{k\geq 1}$ is ultimately periodic β is called a *Parry number*, and if the sequence ends by some zeroes β is a *simple Parry number* and we say that the expansion of 1 is finite. The Parry numbers are Perron numbers.

Example 7. The expansion of $\frac{3+\sqrt{5}}{2}$ is $21111\cdots$, $\frac{3+\sqrt{5}}{2}$ is a Parry number. The plastic number γ (*i.e.,* the greatest zero of $X^3 - X - 1$ and smallest Pisot number) admits the expansion 10001 and is a simple Parry number.

We define the *Parry polynomial* of a simple Parry number with finite expansion $d_1 \cdots d_r$ as $X^r - d_1 X^{r-1} - \cdots - d_r$. Given a Parry number β with expansion $(d_k)_{k\geq 1} = d_1 \cdots d_s b_1 \cdots b_t b_1 \cdots b_t$, let s be the length of the preperiod, and let t be the length of the period. The polynomial $X^{s+t} - d_1 X^{s+t-1} - \cdots - d_s X^t - b_1 X^{t-1} - \cdots - b_t - \left(X^s - d_1 X^{s-1} - \cdots - d_s\right)$ is called the Parry polynomial of β; β is a zero of the Parry polynomial. If the minimal polynomial of β is equal to the Parry polynomial, we say that β *has no complementary (or pirate) value*, if the degree of the Parry polynomial is greater than the degree of the minimal polynomial we say that β *has pirate values (or complementary values).*

Example 8. The plastic number has the expansion 10001, $r = 5$ and its minimal polynomial is $(X^3 - X - 1)$, its Parry polynomial is $X^5 - X^4 - 1 = (X^3 - X - 1)(X^2 - X + 1)$, the roots of $(X^2 - X + 1)$ are pirate values. If $\beta = \frac{3+\sqrt{5}}{2}$ the expansion is $211111\cdots$ $(s = t = 1)$, the Parry polynomial is equal to the minimal polynomial, there are no pirate values.

Given a Parry number β the β-expansion $(d_k)_{k\geq 0}$ of 1 provides a primitive integer matrix M_β whose spectrum contains β, its conjugates and the pirate values if there are any (see [3]).

Looking at the second term of the Parry polynomial and using the equality $d_1 = \lfloor \beta \rfloor$ we obtain the following result.

Lemma 5. *Let β be a Parry number, $(d_k)_{k\geq 1}$ the $\beta-$expansion of 1, $P(X) = X^l - m_1 X^{l-1} - \cdots - m_l$ the Parry polynomial of β. Suppose that the $\beta-$expansion is finite or admits a period of length $t > 1$; then the term m_1 of the Parry polynomial is equal to d_1, i.e., to $\lfloor \beta \rfloor$.*

Suppose that the expansion admits a period of length $t = 1$; then $m_1 = d_1 + 1 = \lfloor \beta \rfloor + 1$ (in this case the β expansion of 1 take the form $d_1 \cdots d_s bbbbb \cdots$).

Looking at the roots of the Parry polynomial we obtain the following result.

Lemma 6. *Let β a Parry number of degree d, let $\alpha_2, \cdots, \alpha_d$ be its algebraic conjugates and let $\gamma_1, \cdots, \gamma_n$ be the pirate conjugates if there are any. Then $\beta + \alpha_2 + \cdots + \alpha_d + \gamma_1 + \cdots + \gamma_n = \lfloor \beta \rfloor$ except in the case $t = 1$ where it is equal to $\lfloor \beta \rfloor + 1$.*

Lemmas 5 and 6 immediately imply Proposition 2.

Proposition 2. Let β be a Parry number, let $(d_k)_{k \geq 1} = d_1 \cdots d_s b_1 \cdots b_t b_1 \cdots b_t \cdots$ be his expansion, $\alpha_2, \cdots, \alpha_d$ be its algebraic conjugates and $\gamma_1, \cdots, \gamma_r$ be its pirate values if there are some.

(1) If β is a simple Parry number without pirate values, then $\operatorname{tr} \beta = \lfloor \beta \rfloor < \beta$, and $\alpha_2 + \cdots + \alpha_d$ is negative and belongs to $]-1, 0[$.

If β is simple and if there are pirate values then $\alpha_2 + \cdots + \alpha_d + \gamma_1 + \cdots + \gamma_r \in \,]-1, 0[$.

(2) If β is a nonsimple Parry number without pirate values, and if the period t is at least 2, then $\operatorname{tr} \beta = \lfloor \beta \rfloor < \beta$ and $\alpha_2 + \cdots + \alpha_d \in \,]-1, 0[$; if the period t is 1, $\operatorname{tr} \beta = \lfloor \beta \rfloor + 1 > \beta$, and $\alpha_2 + \cdots + \alpha_d \in \,]0, 1[$.

If β is a nonsimple Parry number with some pirate values and if the period is $t \geq 2$ then $\alpha_2 + \cdots + \alpha_d + \gamma_1 + \cdots + \gamma_r \in \,]-1, 0[$, if the period is 1 then $\alpha_2 + \cdots + \alpha_d + \gamma_1 + \cdots + \gamma_r \in \,]0, 1[$.

(3) A Parry number such that $|\alpha_2 + \cdots + \alpha_d| > 1$ has always pirate values.

Proposition 3. Let β be a Parry number and let $\alpha_2, \cdots, \alpha_d$ be its algebraic conjugates.

(1) If β is not a Pisot number, the set of integers n such that β^n has pirate values has positive density.

(2) Let β be a Pisot number of degree d, and let H be the set of integers n such that $\alpha_2^n + \cdots + \alpha_d^n \in \,]0, 1[$. Then the set H has a positive density.

If $n \in H$ and if β^n does not have pirate value, then β^n is not simple. The period t is equal to 1 and the β^n expansion of 1 looks like $d_1 \cdots d_{d-1} bbbbb \cdots$.

If $n \in H$ and if β^n has pirate values and is simple, then the sum of the pirate values is equal to -1.

If $n \in H$ and if β^n has pirate values and is not simple, then the sum of the pirate values is 0 if the period is $t = 1$ and -1 if the period is $t \geq 2$.

In [3] one can find more details concerning totally real Pisot numbers.

Proof of Assertion (1). Let β be a Perron number of degree d, let $\alpha_2, \cdots, \alpha_d$ be its conjugates, $\theta_2, \cdots, \theta_d$ their arguments, and $\theta_2^{(n)}, \cdots, \theta_d^{(n)}$ the arguments of $\alpha_2^n, \cdots, \alpha_d^n$, all represented by numbers in $[-\pi, \pi[$. For $\varepsilon > 0$, the set H_ε such that $\theta_2^{(n)}, \cdots, \theta_d^{(n)}$ all belong to the interval $[-\varepsilon, \varepsilon]$ has positive density ([7], Th.201). Choosing a small ε, all $\alpha_2^n, \cdots, \alpha_d^n$ shall have positive real parts. If one of the $|\alpha_{i_0}|$ is greater than or equal to 1, taking a small ε, there are arbitrarily large $n \in H_\varepsilon$ such that tr $\beta^n = \beta^n + \alpha_{i_0}^n + \sum_{i \neq i_0} \alpha_i^n > [\beta^n] + 1$; because of Assertion (3) of

Proposition 2, β^n admits pirate values. If all the $|\alpha_i|$ are less than or equal to 1, either β is a Pisot number, or it is a Salem number (that is, one of the α_i is $\frac{1}{\beta}$ and the others conjugates are complex numbers of modulus exactly one). But if β is a Salem number for large enough n belonging to H_ε, tr $\beta^n > [\beta^n] + 1$. As H_ε has positive density Assertion (1) is true.

Proof of Assertion (2). Take for H the set $\{n \in H_\varepsilon; \alpha_2^n, \cdots, \alpha_d^n < 1\}$; then H and H_ε have the same density. Now use Proposition 2: suppose that $n \in H$, the sum $\alpha_2^n + \cdots + \alpha_d^n$ is positive and smaller than 1 so we have tr $\beta^n > \beta^n$, hence tr $\beta^n = \lfloor\beta^n\rfloor + 1$ and (2.1) implies that β^n cannot be simple without pirate values. If β^n is simple with pirate values let $\gamma_1^{(n)}, \cdots, \gamma_r^{(n)}$ denote the pirate values of β^n; from 2.1 we know that $\alpha_2^n + \cdots + \alpha_d^n + \gamma_1^{(n)} + \cdots + \gamma_r^{(n)}$ belongs to $]-1, 0[$ hence as $\gamma_1^{(n)} + \cdots + \gamma_r^{(n)}$ is an integer and $\alpha_2^n + \cdots + \alpha_d^n$ is smaller than 1, $\gamma_1^{(n)} + \cdots + \gamma_r^{(n)} = -1$. If β^n is not simple and do not admit pirate values 2.2 gives the period $t = 1$, the expansion of 1 in the base β^n looks like $d_1 \cdots d_{d-1} bbbbb \cdots$. If β^n is not simple and admits pirate values, in the case where $t = 1$, we get $\alpha_2^n + \cdots + \alpha_d^n + \gamma_1^{(n)} + \cdots + \gamma_r^{(n)} \in]0, 1[$; as $\gamma_1^{(n)} + \cdots + \gamma_d^{(n)}$ is an integer and $\alpha_2^n + \cdots + \alpha_d^n$ is smaller than 1, we get $\gamma_1^{(n)} + \cdots + \gamma_r^{(n)} = 0$. In the case where $t \geq 2$, $\alpha_2^n + \cdots + \alpha_d^n + \gamma_1^{(n)} + \cdots + \gamma_r^{(n)}$ belongs to $]-1, 0[$ and $\gamma_1^{(n)} + \cdots + \gamma_r^{(n)} = -1$.

Acknowledgement. We thank the referee and the editor for many useful remarks.

References

[1] J.-P. Allouche and J. O. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press (2003).

[2] M.-J. Bertin, A. Decomps, M. Grandet, M. Pathiaux and J.-P. Schreiber, *Pisot and Salem Numbers*, Birkhäuser, Bâle (1992).

[3] A. Bertrand-Mathis, Nombres de Pisot, matrices primitives et bêta-conjugués, *J. Théor. Nombres Bordeaux* **24** (2012), 57-72.

[4] A. Bertrand-Mathis, Nombres de Perron et questions de rationalité, *Journées Arithmétiques 1989*, Astérisque **198-200** (1991), 67-76 .

[5] J. Berstel and C. Reutenauer, *Les séries rationnelles et leurs langages*, Masson, Paris (1984).

[6] M. Boyle and D. Handelman, The spectra of nonnegative matrices via symbolic dynamics, *Ann. of Math.* **133** (1991), 249-316.

[7] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford (1938).

[8] K. H. Kim, N. S. Ormes and F. W. Roush, The spectra of nonnegative integer matrices *via* formal power series, *J. Amer. Math. Soc.* **13** (2000), 773-806.

[9] D. Lind, The entropies of topological Markov shifts and a related class of algebraic integers, *Ergodic Theory Dynam. Systems* **4** (1984), 283-300.

[10] J. F. McKee, Families of Pisot numbers with negative trace, *Acta Arith.* **93** (2000), 373-385.

[11] J. F. McKee and C. J. Smyth, Salem numbers and Pisot numbers via interlacing, *Canad. J. Math.* **64** (2012), 345-367.

[12] J. F. McKee, P. Rowlinson and C. J. Smyth, Salem numbers and Pisot numbers from stars, in *Number Theory in Progress* Vol. 1, De Gruyter, Berlin (1999), 309-319.

[13] C. Mauduit, Caractérisation des ensembles normaux substitutifs, *Invent. Math.* **95** (1999), 133-147.

[14] W. Parry, On the β-expansion of real numbers, *Acta Math. Acad. Sci. Hungar.* **8** (1960), 401-416.