# ON CARMICHAEL AND POLYGONAL NUMBERS, BERNOULLI POLYNOMIALS, AND SUMS OF BASE-P DIGITS 

Bernd C. Kellner<br>Göppert Weg 5, Göttingen, Germany<br>bk@bernoulli.org<br>Jonathan Sondow<br>209 West 97th Street, New York, New York<br>jsondow@alumni.princeton.edu

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#### Abstract

We give a new characterization of the set $\mathcal{C}$ of Carmichael numbers in the context of $p$-adic theory, independently of the classical results of Korselt and Carmichael. The characterization originates from a surprising link to the denominators of the Bernoulli polynomials via the sum-of-base- $p$-digits function. More precisely, we show that such a denominator obeys a triple-product identity, where one factor is connected with a $p$-adically defined subset $\mathcal{S}$ of the squarefree integers that contains $\mathcal{C}$. This leads to the definition of a new subset $\mathcal{C}^{\prime}$ of $\mathcal{C}$, called the "primary Carmichael numbers". Subsequently, we establish that every Carmichael number equals an explicitly determined polygonal number. Finally, the set $\mathcal{S}$ is covered by modular subsets $\mathcal{S}_{d}(d \geq 1)$ that are related to the Knödel numbers, where $\mathcal{C}=\mathcal{S}_{1}$ is a special case.


## 1. Introduction

A composite positive integer $m$ is called a Carmichael number if the congruence

$$
\begin{equation*}
a^{m-1} \equiv 1 \quad(\bmod m) \tag{1.1}
\end{equation*}
$$

holds for all integers $a$ coprime to $m$ (see [11, Sec. A13], [24, Chap.2, Sec.IX]). Clearly, if $m$ were a prime, then this congruence would be valid by Fermat's little theorem.

Let "number" mean "positive integer" unless otherwise specified, and let $p$ always denote a prime. A first result on Carmichael numbers is the following criterion (for a proof, see [6] or [8, p. 134]).

Theorem 1.1 (Korselt's criterion [20] (1899)). A composite number $m$ is a Carmichael number if and only if $m$ is squarefree and every prime divisor $p$ of $m$ satisfies $p-1 \mid m-1$.

Korselt did not give any examples of such numbers, while Carmichael succeeded in determining the first ones, e.g.,

$$
561=3 \cdot 11 \cdot 17, \quad 1105=5 \cdot 13 \cdot 17, \quad \text { and } \quad 1729=7 \cdot 13 \cdot 19
$$

Apparently unaware of Korselt's result, Carmichael showed the following properties.
Theorem 1.2 (Carmichael [3,4] (1910,1912)). Every Carmichael number m is odd and squarefree and has at least three prime factors. If $p$ and $q$ are prime divisors of $m$, then

$$
\text { (i) } \quad p-1 \mid m-1, \quad \text { (ii) } \quad p-1 \left\lvert\, \frac{m}{p}-1\right., \quad \text { (iii) } \quad p \nmid q-1 \text {. }
$$

An easy consequence of part (ii) is that (see [6])

$$
\begin{equation*}
p<\sqrt{m} \tag{1.2}
\end{equation*}
$$

Denote the set of Carmichael numbers by

$$
\mathcal{C}=\{561,1105,1729,2465,2821,6601,8911,10585,15841, \ldots\}
$$

In 1994 Alford, Granville, and Pomerance [1] proved that $\mathcal{C}$ is infinite, i.e., infinitely many Carmichael numbers exist. More precisely, they showed that if $C(x)$ denotes the number of Carmichael numbers less than $x$, then $C(x)>x^{2 / 7}$ for sufficiently large $x$. This was improved by Harman [13] in 2008 to

$$
C(x)>x^{1 / 3} \quad \text { for all large } x
$$

In the other direction, Erdős [9] in 1956 improved a result of Knödel [19] to show that

$$
C(x)<x^{1-c \log \log \log x / \log \log x} \quad \text { for all large } x
$$

where $c>0$ is a constant. For which estimate is closer to the true asymptotic for $C(x)$, see Granville and Pomerance's discussion in [10] (see also [24, Chap. 4, Sec. VIII]).

The purpose of the present paper is to give a new characterization of the Carmichael numbers in the context of $p$-adic theory, independently of the results of Korselt and Carmichael in Theorems 1.1 and 1.2. The characterization originates from a surprising link to the denominators of the Bernoulli polynomials via the sum-of-base- $p$-digits function $s_{p}$.

The link is introduced in Sections 2 and 3. Section 2 also introduces a $p$-adically defined set of squarefree integers $\mathcal{S} \supset \mathcal{C}$, and the subset of "primary Carmichael numbers" $\mathcal{C}^{\prime} \subset \mathcal{C}$. Section 4 establishes that every Carmichael number equals an explicitly determined polygonal number.

Subsequently, Sections 5, 6, and 7 contain the postponed proofs of the results in Sections 2, 3, and 4, respectively.

Finally, in Section 8 the set $\mathcal{S}$ is covered by modular subsets $\mathcal{S}_{d}$ for $d=1,2,3, \ldots$, providing a modular generalization of $\mathcal{C}=\mathcal{S}_{1}$. It turns out that each $\mathcal{S}_{d}$ is contained in a certain superset $\widehat{\mathcal{K}}_{d}$ of the so-called $d$-Knödel numbers $\mathcal{K}_{d}$.

## 2. Carmichael Numbers and Squarefree Integers

Define $\mathbb{S}$ to be the set of squarefree integers greater than 1 :

$$
\mathbb{S}=\{2,3,5,6,7,10, \ldots\}
$$

Denoting by $s_{p}(n)$ the sum of the base-p digits of $n$, we further define two subsets of $\mathbb{S}$, namely,

$$
\mathcal{S}:=\left\{m \in \mathbb{S}: p \mid m \Longrightarrow s_{p}(m) \geq p\right\}
$$

and

$$
\mathcal{C}^{\prime}:=\left\{m \in \mathbb{S}: p \mid m \Longrightarrow s_{p}(m)=p\right\}
$$

Note that $\mathcal{C}^{\prime}$ is a subset of $\mathcal{S}$. One computes that

$$
\mathcal{S}=\{231,561,1001,1045,1105,1122,1155,1729,2002, \ldots\}
$$

and

$$
\mathcal{C}^{\prime}=\{1729,2821,29341,46657,252601,294409,399001, \ldots\}
$$

We will show that $\mathcal{C}^{\prime} \subset \mathcal{C}$ (see Theorem 2.1). If $m \in \mathcal{C}^{\prime}$, then $s_{p}(m)=p$ for all primes $p \mid m$, so we call $m$ a primary Carmichael number (hence the notation $\mathcal{C}^{\prime}$, meaning "C prime"). The first one is 1729 , Ramanujan's famous "taxicab" number, defined by him as "the smallest number expressible as the sum of two [positive] cubes in two different ways" (see [12, p. 12]). The first primary Carmichael number not congruent to 1 modulo 4 is

$$
1152271 \equiv 3 \quad(\bmod 4)
$$

while the first element of $\mathcal{C}^{\prime}$ with more than three prime factors is

$$
10606681=31 \cdot 43 \cdot 73 \cdot 109
$$

We can now state our first main results. The following one extends parts of Theorem 1.2 to a larger set.

Theorem 2.1. There are the strict inclusions

$$
\mathcal{C}^{\prime} \subset \mathcal{C} \subset \mathcal{S} \subset \mathbb{S}
$$

Moreover, for any $m \in \mathcal{S}$ each prime factor $p$ satisfies the property (1.2) that $p<\sqrt{m}$. In particular, $m$ must have at least three (respectively, four) prime factors, if $m$ is odd (respectively, even).

Theorem 2.1 leads to a new criterion for the Carmichael numbers.
Theorem 2.2. We have the characterization

$$
\mathcal{C}=\left\{m \in \mathcal{S}: p \mid m \Longrightarrow s_{p}(m) \equiv 1 \quad(\bmod p-1)\right\} .
$$

In other words, an integer $m>1$ is a Carmichael number if and only if $m$ is squarefree and each of its prime divisors $p$ satisfies both

$$
s_{p}(m) \geq p \quad \text { and } \quad s_{p}(m) \equiv 1 \quad(\bmod p-1)
$$

From this characterization it follows directly that $m$ is odd and has at least three prime factors, each less than $\sqrt{m}$.

Unlike the criterion of Korselt, that in Theorem 2.2 does not assume compositeness. Indeed, all results of Theorems 2.1 and 2.2 are deduced only from properties of the function $s_{p}$. In this vein, we can even sharpen the consequence of Theorems 2.1 and 2.2 that $p<\sqrt{m}$ if $p \mid m$.

Theorem 2.3. For certain subsets $\mathcal{T} \subseteq \mathcal{S}$, we have the sharp estimate

$$
p \leq \alpha_{\mathcal{T}} \sqrt{m} \quad(m \in \mathcal{T}, p \mid m)
$$

with

$$
\alpha_{\mathcal{T}}=1 / \sqrt{2-\frac{1}{q}}=\left\{\begin{array}{lll}
0.7237 \ldots, & q=11, & \text { if } \mathcal{T}=\mathcal{S} \\
0.7177 \ldots, & q=17, & \text { if } \mathcal{T}=\mathcal{C} \\
0.7071 \ldots, & q=66337, & \text { if } \mathcal{T}=\mathcal{C}^{\prime}
\end{array}\right.
$$

and

$$
\alpha_{\mathcal{T}}=1 / \sqrt{3-\frac{1}{q}}=0.5789 \ldots, \quad q=61, \quad \text { if } \mathcal{T}=\mathcal{S}_{\text {even }}
$$

where $\mathcal{S}_{\text {even }}:=\{m \in \mathcal{S}: m$ is even $\}$.
Interestingly, to achieve the nontrivial bounds in Theorem 2.3, in each of the sets $\mathcal{S}, \mathcal{C}$, and $\mathcal{C}^{\prime}$ we find certain polygonal numbers, as discussed in Section 4 and Table 4.1.

It is not obvious from its definition that the set $\mathcal{S}$ is infinite. However, that is an immediate corollary of Theorem 2.1 and the existence of infinitely many Carmichael numbers. An independent proof showing directly that $\mathcal{S}$ is infinite, without involving the set $\mathcal{C}$, would certainly be of interest.

Corollary 2.4. The set $\mathcal{S}$ is infinite.
If one could show that $\mathcal{C}^{\prime}$ is infinite, this would give not only a new proof of the infinitude of Carmichael numbers, but also another proof that $\mathcal{S}$ is infinite.

Let $C^{\prime}(x)$ and $S(x)$ count the numbers of elements of $\mathcal{C}^{\prime}$ and $\mathcal{S}$ less than $x$, respectively. Table 2.1 reports the slow but steady increase in size of $C^{\prime}(x)$ compared to $C(x)$ and $S(x)$.

| $x$ | $C^{\prime}(x)$ | $C(x)$ | $S(x)$ |
| :---: | ---: | ---: | ---: |
| $10^{3}$ | 0 | 1 | 2 |
| $10^{4}$ | 2 | 7 | 57 |
| $10^{5}$ | 4 | 16 | 636 |
| $10^{6}$ | 9 | 43 | 7048 |
| $10^{7}$ | 19 | 105 | 75150 |
| $10^{8}$ | 51 | 255 | 801931 |
| $10^{9}$ | 107 | 646 | 8350039 |
| $10^{10}$ | 219 | 1547 | 86361487 |

Table 2.1: Distributions of $C^{\prime}(x), C(x)$, and $S(x)$.

For the values of $C\left(10^{n}\right)$ up to $n=16$ and $n=21$, as well as a more detailed analysis of their distribution, see [10] and Pinch [22], respectively. The primary Carmichael numbers with more than three prime factors seem to occur rarely. Indeed, up to $10^{10}$ there are only five elements of $\mathcal{C}^{\prime}$ with four (but not more) prime factors.

Granville and Pomerance [10] gave a precise conjecture that Carmichael numbers with exactly three prime factors should satisfy

$$
C_{3}(x)=O\left(x^{1 / 3} / \log ^{3} x\right)
$$

Heath-Brown [14] showed the upper bound $C_{3}(x)=O\left(x^{7 / 20+\varepsilon}\right)$ for any fixed $\varepsilon>0$.

## 3. Bernoulli Numbers and Polynomials

The Bernoulli polynomials are defined by the generating function

$$
\frac{t e^{x t}}{e^{t}-1}=\sum_{n \geq 0} B_{n}(x) \frac{t^{n}}{n!} \quad(|t|<2 \pi)
$$

where

$$
B_{n}(x)=\sum_{k=0}^{n}\binom{n}{k} B_{k} x^{n-k} \quad(n \geq 0)
$$

and $B_{k}=B_{k}(0) \in \mathbb{Q}$ is the $k$ th Bernoulli number.

For $n \geq 1$ denote by $\mathbf{D}_{n}, \mathbb{D}_{n}$, and $\mathfrak{D}_{n}$ the denominators (see [17])

$$
\begin{aligned}
\mathbf{D}_{n} & :=\operatorname{denom}\left(B_{n}\right)=2,6,1,30,1,42,1,30,1,66, \ldots \\
\mathbb{D}_{n} & :=\operatorname{denom}\left(B_{n}(x)-B_{n}\right)=1,1,2,1,6,2,6,3,10,2, \ldots, \\
\mathfrak{D}_{n} & :=\operatorname{denom}\left(B_{n}(x)\right)=2,6,2,30,6,42,6,30,10,66, \ldots
\end{aligned}
$$

The denominators of the Bernoulli numbers are well known by the von StaudtClausen theorem of 1840 (see [5,26]) to be

$$
\mathbf{D}_{n}=\left\{\begin{array}{cl}
2, & \text { if } n=1  \tag{3.1}\\
1, & \text { if } n \geq 3 \text { is odd } \\
\prod_{p-1 \mid n} p, & \text { if } n \geq 2 \text { is even }
\end{array}\right.
$$

The initial connection between the Carmichael numbers and the denominators of the Bernoulli numbers and polynomials results from the known relations

$$
\begin{equation*}
m \in \mathcal{C} \Longrightarrow m\left|\mathbf{D}_{m-1}\right| \mathfrak{D}_{m-1} \tag{3.2}
\end{equation*}
$$

The first relation, $m \in \mathcal{C} \Rightarrow m \mid \mathbf{D}_{m-1}$, actually holds as an equivalence: An odd composite number $m$ is a Carmichael number if and only if $m$ divides $\mathbf{D}_{m-1}$ (see Pomerance, Selfridge, and Wagstaff [23, p. 1006]). The equivalence follows easily from Korselt's criterion and the von Staudt-Clausen theorem. The second relation, $\mathbf{D}_{m-1} \mid \mathfrak{D}_{m-1}$, is easily seen, since even

$$
\begin{equation*}
\mathfrak{D}_{n}=\operatorname{lcm}\left(\mathbb{D}_{n}, \mathbf{D}_{n}\right) \tag{3.3}
\end{equation*}
$$

holds for all $n \geq 1$ (cf. [16, Thm. 4]). Now the sum-of-base- $p$-digits function $s_{p}$ comes into play, as follows.

The authors [15-17] have recently shown that the denominators of the Bernoulli polynomials $B_{n}(x)-B_{n}$ (which have no constant term) are given by the remarkable formula

$$
\begin{equation*}
\mathbb{D}_{n}=\prod_{s_{p}(n) \geq p} p \tag{3.4}
\end{equation*}
$$

in which the product is finite since $s_{p}(n)=n$ if $p>n$. Moreover, the following relation, supplementary to (3.3), holds for $n \geq 1$ (see [17]):

$$
\begin{equation*}
\mathfrak{D}_{n}=\operatorname{lcm}\left(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\right) \tag{3.5}
\end{equation*}
$$

where $\operatorname{rad}(n):=\prod_{p \mid n} p$.
In particular, $\mathbf{D}_{n}, \mathbb{D}_{n}$, and $\mathfrak{D}_{n}$ are squarefree. Furthermore, these denominators obey the following properties (see [17]):

$$
\begin{array}{ll}
\mathbb{D}_{n}=\operatorname{lcm}\left(\mathbb{D}_{n+1}, \operatorname{rad}(n+1)\right), & \text { if } n \geq 3 \text { is odd } \\
\mathfrak{D}_{n}=\operatorname{lcm}\left(\mathfrak{D}_{n+1}, \operatorname{rad}(n+1)\right), & \text { if } n \geq 2 \text { is even } \tag{3.7}
\end{array}
$$

and (see [15])

$$
\begin{equation*}
\operatorname{rad}(n+1) \mid \mathbb{D}_{n}, \quad \text { if } n+1 \text { is composite. } \tag{3.8}
\end{equation*}
$$

To substantiate the relationship between the Carmichael numbers and the Bernoulli polynomials, we introduce for $n \geq 1$ the decomposition

$$
\begin{equation*}
\mathbb{D}_{n}=\mathbb{D}_{n}^{\top} \cdot \mathbb{D}_{n}^{\perp} \tag{3.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbb{D}_{n}^{\top}:=\prod_{\substack{p \mid n \\ s_{p}(n) \geq p}} p \quad \text { and } \quad \mathbb{D}_{n}^{\perp}:=\prod_{\substack{p \nmid n \\ s_{p}(n) \geq p}} p \tag{3.10}
\end{equation*}
$$

Additionally, we define the complementary number to $\mathbb{D}_{n}^{\top}$ for $n \geq 1$ as

$$
\begin{equation*}
\mathbb{D}_{n}^{\top^{\star}}:=\prod_{\substack{p \mid n \\ s_{p}(n)<p}} p \tag{3.11}
\end{equation*}
$$

which satisfies the relation

$$
\begin{equation*}
\operatorname{rad}(n)=\mathbb{D}_{n}^{\top} \cdot \mathbb{D}_{n}^{\top^{\star}} \tag{3.12}
\end{equation*}
$$

As an application of these definitions, the next theorem gives a complete description of the structure of the denominator $\mathfrak{D}_{n}$ of the Bernoulli polynomial $B_{n}(x)$ in terms of a decomposition of $\mathfrak{D}_{n}$ into three factors. (The result may be compared to the von Staudt-Clausen theorem in (3.1), which describes the structure of the denominator of the Bernoulli number $B_{n}$.) Furthermore, we obtain for all squarefree numbers $m>1$ a generalization of (3.2), when omitting its middle term $\mathbf{D}_{m-1}$.

Theorem 3.1. For $n \geq 1$ the denominator $\mathfrak{D}_{n}$ of the Bernoulli polynomial $B_{n}(x)$ splits into the triple product

$$
\mathfrak{D}_{n}=\mathbb{D}_{n+1}^{\perp} \cdot \mathbb{D}_{n+1}^{\top} \cdot \mathbb{D}_{n+1}^{\top^{\star}}
$$

Moreover,

$$
m \in \mathbb{S} \Longleftrightarrow m \mid \mathfrak{D}_{m-1}
$$

The interplay of the three factors of $\mathfrak{D}_{n}$ instantly yields the two relations

$$
\begin{equation*}
\mathfrak{D}_{n}=\mathbb{D}_{n+1}^{\perp} \cdot \operatorname{rad}(n+1)=\mathbb{D}_{n+1} \cdot \mathbb{D}_{n+1}^{\top^{\star}} \tag{3.13}
\end{equation*}
$$

Explicit product formulas for $\mathfrak{D}_{n}$, in the contexts of (3.3) and (3.13), are given in [16, Thm. 4] and [17, Cor. 1], respectively.

We can now state our second main result. It establishes a fundamental relationship between the Bernoulli polynomials and the Carmichael numbers, since $\mathcal{C} \subset \mathcal{S}$ by Theorem 2.1.

Theorem 3.2. The following claims are true:
(i) The sequence $\left(\mathbb{D}_{n}^{\top}\right)_{n \geq 1}$ contains all elements of $\mathcal{S}$. More precisely,

$$
m \in \mathcal{S} \cup\{1\} \Longleftrightarrow m=\mathbb{D}_{m}^{\top} \Longleftrightarrow m \mid \mathbb{D}_{m}
$$

(ii) If $m+1$ is composite, then $\operatorname{rad}(m+1) \mid \mathbb{D}_{m}^{\perp}$.

Corollary 3.3. The sequence $\left(\mathbb{D}_{n}^{\top}\right)_{n \geq 1}$ contains all the Carmichael numbers. More precisely,

$$
m \in \mathcal{C} \Longrightarrow m=\mathbb{D}_{m}^{\top}, \quad m\left|\mathbb{D}_{m}, \quad m\right| \mathbb{D}_{m-1}^{\perp}, \quad \text { and } \quad m \mid \mathbb{D}_{m-1}
$$

but the converse does not hold.
Remark 3.4. The sequence $\left(\mathbb{D}_{n}^{\perp}\right)_{n \geq 1}$ very rarely intersects the Carmichael numbers. Indeed, the only example below $10^{6}$ is $\mathbb{D}_{198}^{\perp}=2465 \in \mathcal{C}$.
Remark 3.5. Comparing the definitions of the complementary numbers $\mathbb{D}_{n}^{\top^{\star}}$ and the set $\mathcal{S}$, one immediately observes that the sequence $\left(\mathbb{D}_{n}^{\top^{\star}}\right)_{n \geq 1}$ cannot contain any elements of $\mathcal{S}$, and thus none of the Carmichael numbers. Interestingly, it turns out that $\left(\mathbb{D}_{n}^{\top \star}\right)_{n \geq 1}$ is connected with the quotients $\mathbb{D}_{n} / \mathbb{D}_{n+1}$ and $\mathfrak{D}_{n} / \mathfrak{D}_{n+1}$ (as introduced in [17]), which are integral for odd and even indices $n$ by (3.6) and (3.7), respectively.

Remark 3.6. It was actually the observation of the unexpected relationship

$$
m \in \mathcal{C} \Longrightarrow m=\mathbb{D}_{m}^{\top}
$$

as stated in Corollary 3.3, which led to the new characterization of the Carmichael numbers via the sum-of-base-p-digits function $s_{p}$, given in Theorem 2.2.

## 4. Polygonal Numbers

Surprisingly, the polygonal numbers (see [2, Chap. XVIII] and [7, pp. 38-42]) are connected with the Carmichael numbers and the set $\mathcal{S}$.

Initially, we consider the following polygonal numbers for $n \geq 1$ :

$$
\mathbf{P}_{n}=n(3 n-1) / 2, \quad \mathbf{H}_{n}=n(2 n-1), \quad \mathbf{O}_{n}=n(3 n-2),
$$

which are the $n$th pentagonal, hexagonal, and octagonal numbers, respectively. They satisfy an important property when $n=p$ is an odd prime:

$$
s_{p}\left(\mathbf{H}_{p}\right)=s_{p}\left(\mathbf{O}_{p}\right)=p \quad \text { and } \quad s_{p}\left(2 \mathbf{P}_{p}\right)=p+1
$$

To establish a connection between the set $\mathcal{S}$ and the polygonal numbers, we first introduce some definitions. Define $P(n)$ to be the greatest prime factor of $n$ if $n \geq 2$, and set $P(1):=1$. Also, denote the (double-shifted) $p$-adic value of $n$ by

$$
\ell(n):=\left\lfloor\frac{n}{P(n)^{2}}\right\rfloor=\min _{p \mid n}\left\lfloor\frac{n}{p^{2}}\right\rfloor .
$$

We shall use the abbreviation $\ell=\ell(n)$ later on, if there is no ambiguity in context. Finally, we need Legendre's formula (see [25, Sec. 5.3, p. 241]), which gives the $p$-adic valuation $\mathrm{v}_{p}$ of a factorial by

$$
\begin{equation*}
\mathrm{v}_{p}(n!)=\frac{n-s_{p}(n)}{p-1} \tag{4.1}
\end{equation*}
$$

For simplicity, twice a polygonal number will be called a quasi polygonal number. The next theorem shows the special cases when $m \in \mathcal{S}$ equals a (quasi) polygonal number $\mathbf{H}_{p}, \mathbf{O}_{p}$, or $2 \mathbf{P}_{p}$ with $p=P(m)$, the classification being determined by the parameter $\ell(m)$.

Theorem 4.1. Let $m \in \mathcal{S}$, and set $p=P(m)$ and $\ell=\ell(m)$. Then the following statements hold:
(i) We have $\ell \geq 1$.
(ii) There is the equivalence

$$
\ell=1 \Longleftrightarrow m=\mathbf{H}_{p} \text { is a hexagonal number. }
$$

(iii) There is the equivalence

$$
\ell=2 \Longleftrightarrow m= \begin{cases}\mathbf{O}_{p}, & \text { if } s_{p}(m)=p \\ 2 \mathbf{P}_{p}, & \text { if } s_{p}(m)>p\end{cases}
$$

In particular, for $m \in \mathcal{C}^{\prime}$ we have $\ell=2$ if and only if $m=\mathbf{O}_{p}$ is an octagonal number.

As needed later, Table 4.1 reports the first occurrences of the polygonal numbers $\mathbf{H}_{p}$ and $\mathbf{O}_{p}$ in each of the sets $\mathcal{S}, \mathcal{C}$, and $\mathcal{C}^{\prime}$, as well as the first occurrence of $2 \mathbf{P}_{p}$ in $\mathcal{S}$. In contrast to the relatively small values in Table 4.1, the exceptionally large number 8801128801 , which is indeed the least hexagonal number in $\mathcal{C}^{\prime}$, could be found only by a computer search.

| set | $m$ | factors | $p=P(m)$ | $\ell(m)$ | number |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{S}$ | 231 | $3 \cdot 7 \cdot 11$ | 11 | 1 | $\mathbf{H}_{11}$ |
| $\mathcal{C}$ | 561 | $3 \cdot 11 \cdot 17$ | 17 | 1 | $\mathbf{H}_{17}$ |
| $\mathcal{C}^{\prime}$ | 8801128801 | $181 \cdot 733 \cdot 66337$ | 66337 | 1 | $\mathbf{H}_{66337}$ |
| $\mathcal{S}$ | 1045 | $5 \cdot 11 \cdot 19$ | 19 | 2 | $\mathbf{O}_{19}$ |
| $\mathcal{C}$ | 2465 | $5 \cdot 17 \cdot 29$ | 29 | 2 | $\mathbf{O}_{29}$ |
| $\mathcal{C}^{\prime}$ | 2821 | $7 \cdot 13 \cdot 31$ | 31 | 2 | $\mathbf{O}_{31}$ |
| $\mathcal{S}$ | 11102 | $2 \cdot 7 \cdot 13 \cdot 61$ | 61 | 2 | $2 \mathbf{P}_{61}$ |

Table 4.1: First occurrences of (quasi) polygonal numbers $\mathbf{H}_{p}$ and $\mathbf{O}_{p}$ in $\mathcal{S}, \mathcal{C}$, and $\mathcal{C}^{\prime}$, as well as $2 \mathbf{P}_{p}$ in $\mathcal{S}$.

To generalize the results, we further consider the polygonal numbers of rank $r \geq 3$, also called $r$-gonal numbers, namely,

$$
\mathbf{G}_{n}^{r}=\frac{1}{2}\left(n^{2}(r-2)-n(r-4)\right),
$$

where $\mathbf{P}_{n}=\mathbf{G}_{n}^{5}, \mathbf{H}_{n}=\mathbf{G}_{n}^{6}$, and $\mathbf{O}_{n}=\mathbf{G}_{n}^{8}$. Note though that an $r$-gonal number can also be an $r^{\prime}$-gonal number with $r \neq r^{\prime}$; for instance, $\mathbf{G}_{n}^{6}=\mathbf{G}_{2 n-1}^{3}$ for $n \geq 1$. Note also that $\mathbf{G}_{1}^{r}=1$ and $\mathbf{G}_{2}^{r}=r$ cover for $r \geq 3$ all positive integers except 2 . For that reason, our results on polygonal numbers $\mathbf{G}_{n}^{r}$ will implicitly involve only those with $n \geq 3$. Clearly, for fixed $n \geq 3$ the sequence of such numbers $\left(\mathbf{G}_{n}^{r}\right)_{r \geq 3}$ is strictly increasing.

To extend the results of Theorem 4.1, one may ask which $m \in \mathcal{S}$ are equal to a (quasi) polygonal number $\mathbf{G}_{p}^{r}$ or $2 \mathbf{G}_{p}^{r}$ for $p=P(m)$ and some $r \geq 3$. The example

$$
m=2145=\mathbf{H}_{33}=3 \cdot 5 \cdot 11 \cdot 13 \in \mathcal{S} \backslash \mathcal{C}
$$

shows that $m$ is indeed a polygonal number, but $m$ is not of the form $\mathbf{G}_{p}^{r}$ with $p=P(m)=13$, as verified by the consecutive values

$$
\mathbf{G}_{13}^{29}=2119<m=2145<2197=\mathbf{G}_{13}^{30} .
$$

The next theorem clarifies this situation by showing that an element $m \in \mathcal{S}$ equals a (quasi) polygonal number $\mathbf{G}_{p}^{r}$ or $2 \mathbf{G}_{p}^{r}$ with $p=P(m)$ if and only if $s_{p}(m)$ satisfies certain conditions.

Theorem 4.2. Let $m \in \mathcal{S}$, and set $p=P(m)$ and $\ell=\ell(m)$. Further define the integers $\eta \geq 1$ and $0 \leq \mu<p-1$ satisfying

$$
\begin{equation*}
s_{p}(m)=\eta(p-1)+\mu . \tag{4.2}
\end{equation*}
$$

The number $m$ equals a (quasi) polygonal number $\mathbf{G}_{p}^{r}$ or $2 \mathbf{G}_{p}^{r}$ for some $r \geq 3$ if and only if $\mu$ can be written as

$$
\begin{equation*}
\mu=d+e \frac{p-1}{2} \quad \text { with } \quad(d, e) \in\{(1,0),(1,1),(2,0)\} \tag{4.3}
\end{equation*}
$$

Then in these cases we have that

$$
\begin{equation*}
m=d \cdot \mathbf{G}_{p}^{r} \quad \text { with } \quad r=\frac{2}{d}\left(\ell+\mathrm{v}_{p}(\ell!)+\eta+d\right)+e \tag{4.4}
\end{equation*}
$$

As an application we obtain the following corollary for the Carmichael numbers.
Corollary 4.3. All Carmichael numbers are polygonal numbers. More precisely, if $m \in \mathcal{C}, p=P(m)$, and $\ell=\ell(m)$, then

$$
\begin{equation*}
m=\mathbf{G}_{p}^{r} \quad \text { with } \quad r=2\left(\ell+\mathrm{v}_{p}(\ell!)+\eta+1\right) \tag{4.5}
\end{equation*}
$$

where $\eta \geq 1$ is the integer satisfying $s_{p}(m)=\eta(p-1)+1$. In particular, relation (4.5) holds with $\eta=1$ for all primary Carmichael numbers $m \in \mathcal{C}^{\prime}$.

The first few numbers satisfying the conditions of Theorem 4.1 are listed in Table 4.2. Additional numbers below 7000 satisfying the conditions of Theorem 4.2, but not covered by Theorem 4.1, are listed in Table 4.3. As a special case, the taxicab number 1729 equals the 12 -gonal number $\mathbf{G}_{19}^{12}$. Regarding Corollary 4.3, the first element of $\mathcal{C}$ with $\eta=2$ is

$$
1050985=5 \cdot 13 \cdot 19 \cdot 23 \cdot 37=\mathbf{G}_{37}^{1580}
$$

| $m$ | $p=P(m)$ | $s_{p}(m)$ | $\ell(m)$ | number |
| ---: | :---: | :---: | :---: | :---: |
| $231 \in \mathcal{S}$ | 11 | 11 | 1 | $\mathbf{H}_{11}$ |
| $561 \in \mathcal{C}$ | 17 | 17 | 1 | $\mathbf{H}_{17}$ |
| $1045 \in \mathcal{S}$ | 19 | 19 | 2 | $\mathbf{O}_{19}$ |
| $2465 \in \mathcal{C}$ | 29 | 29 | 2 | $\mathbf{O}_{29}$ |
| $2821 \in \mathcal{C}^{\prime}$ | 31 | 31 | 2 | $\mathbf{O}_{31}$ |
| $3655 \in \mathcal{S}$ | 43 | 43 | 1 | $\mathbf{H}_{43}$ |
| $5565 \in \mathcal{S}$ | 53 | 53 | 1 | $\mathbf{H}_{53}$ |
| $8911 \in \mathcal{C}$ | 67 | 67 | 1 | $\mathbf{H}_{67}$ |
| $10585 \in \mathcal{C}$ | 73 | 73 | 1 | $\mathbf{H}_{73}$ |
| $11102 \in \mathcal{S}$ | 61 | 62 | 2 | $2 \mathbf{P}_{61}$ |

Table 4.2: The first (quasi) polygonal numbers $\mathbf{H}_{p}, \mathbf{O}_{p}$, and $2 \mathbf{P}_{p}$ in $\mathcal{S}$.

| $m$ | $p=P(m)$ | $s_{p}(m)$ | $\ell(m)$ | number |
| :---: | :---: | :---: | :---: | :---: |
| $1105 \in \mathcal{C}$ | 17 | 17 | 3 | $\mathbf{G}_{17}^{10}$ |
| $1122 \in \mathcal{S}$ | 17 | 18 | 3 | $2 \mathbf{H}_{17}$ |
| $1729 \in \mathcal{C}^{\prime}$ | 19 | 19 | 4 | $\mathbf{G}_{19}^{12}$ |
| $3458 \in \mathcal{S}$ | 19 | 20 | 9 | $2 \mathbf{G}_{19}^{12}$ |
| $3570 \in \mathcal{S}$ | 17 | 18 | 12 | $2 \mathbf{G}_{17}^{15}$ |
| $5005 \in \mathcal{S}$ | 13 | 13 | 29 | $\mathbf{G}_{13}^{66}$ |
| $5642 \in \mathcal{S}$ | 31 | 32 | 5 | $2 \mathbf{O}_{31}$ |
| $6118 \in \mathcal{S}$ | 23 | 24 | 11 | $2 \mathbf{G}_{23}^{14}$ |
| $6545 \in \mathcal{S}$ | 17 | 17 | 22 | $\mathbf{G}_{17}^{50}$ |
| $6601 \in \mathcal{C}$ | 41 | 41 | 3 | $\mathbf{G}_{41}^{10}$ |
| $6734 \in \mathcal{S}$ | 37 | 38 | 4 | $2 \mathbf{G}_{37}^{7}$ |

Table 4.3: Additional (quasi) polygonal numbers $\mathbf{G}_{p}^{r}$ and $2 \mathbf{G}_{p}^{r}$ in $\mathcal{S}$ below 7000 .

## 5. Proofs of Theorems 2.1, 2.2, and 2.3

Recall the definitions and notation of Section 4. From Legendre's formula (4.1) one easily sees that

$$
\begin{equation*}
n \equiv s_{p}(n) \quad(\bmod p-1) \tag{5.1}
\end{equation*}
$$

Proof of Theorem 2.1. By the definitions and the computed examples, we immediately obtain the strict inclusions $\mathcal{C}^{\prime} \subset \mathcal{S} \subset \mathbb{S}$.

Given $m \in \mathcal{S}$, we first show that $p \mid m$ implies $p<\sqrt{m}$. As $m$ is squarefree, we can write

$$
\begin{equation*}
\frac{m}{p}=a_{0}+a_{1} p \tag{5.2}
\end{equation*}
$$

with $1 \leq a_{0} \leq p-1$ and $a_{1} \geq 0$. Since

$$
\begin{equation*}
a_{0}+s_{p}\left(a_{1}\right)=s_{p}(m / p)=s_{p}(m) \geq p \tag{5.3}
\end{equation*}
$$

we infer that $a_{1} \geq 1$. Consequently, we obtain $a_{0}+a_{1} p>p$, implying that $\sqrt{m}>p$. As a result, $m$ must have at least three prime factors.

Now let $m$ be even. Suppose to the contrary that in this case $m$ has only three prime factors. Hence we have

$$
\begin{equation*}
m=2 q p \quad \text { with } \quad p>q \tag{5.4}
\end{equation*}
$$

where $p$ and $q$ are odd primes. By $s_{p}(2 q)=s_{p}(m) \geq p$, we infer that $2 q \geq p$. Together with (5.4) we then obtain that $2 p>2 q>p$. Using (5.2), we conclude
that $m / p=2 q=a_{0}+a_{1} p$ with $a_{1}=1$. Since $a_{0}=2 q-p<p-1$, it follows that $s_{p}(m)<p$, giving a contradiction. Thus, if $m$ is even, then $m$ must have at least four prime factors.

Next, we show that $\mathcal{C}$ is equal to the set

$$
\tilde{\mathcal{S}}:=\left\{m \in \mathcal{S}: p \mid m \Longrightarrow s_{p}(m) \equiv 1 \quad(\bmod p-1)\right\} .
$$

Resolving the definition of $\mathcal{S}$, for $m \in \tilde{\mathcal{S}}$ we have the condition

$$
\begin{equation*}
p \mid m \Longrightarrow \quad s_{p}(m) \geq p \quad \text { and } \quad s_{p}(m) \equiv 1 \quad(\bmod p-1) \tag{5.5}
\end{equation*}
$$

Moreover, applying (5.1) then yields

$$
\begin{equation*}
m \equiv s_{p}(m) \equiv 1 \quad(\bmod p-1) \tag{5.6}
\end{equation*}
$$

For any $n \in \mathbb{S}$, we have $n>1$ is squarefree, so

$$
\begin{equation*}
s_{p}(n)=1 \Longleftrightarrow n=p \tag{5.7}
\end{equation*}
$$

By (5.6) and (5.7), condition (5.5) implies that

$$
\begin{equation*}
m \text { is composite, and } \quad p|m \Longrightarrow p-1| m-1 \tag{5.8}
\end{equation*}
$$

Thus $m$ satisfies Korselt's criterion. Hence, we conclude that $\tilde{\mathcal{S}} \subseteq \mathcal{C}$.
Conversely, any $m \in \mathcal{C}$ satisfies (5.8). In view of (5.1), we then have $s_{p}(m) \equiv 1$ $(\bmod p-1)$. Since $m$ is squarefree and composite, from (5.7) we deduce that $s_{p}(m) \geq$ $p$. This implies that (5.5) holds, so $m \in \tilde{\mathcal{S}}$ and consequently $\mathcal{C} \subseteq \tilde{\mathcal{S}}$, proving that $\mathcal{C}=\tilde{\mathcal{S}}$.

Now, if $m \in \mathcal{C}^{\prime}$, then (5.5) holds, so $m \in \mathcal{C}$. Considering the computed examples again, we finally deduce that $\mathcal{C}^{\prime} \subset \mathcal{C} \subset \mathcal{S} \subset \mathbb{S}$. This completes the proof of the theorem.

Proof of Theorem 2.2. The first statement is the equality $\mathcal{C}=\tilde{\mathcal{S}}$, established in the proof of Theorem 2.1. Since $m \in \tilde{\mathcal{S}}$ if and only if (5.5) holds, the second statement then follows.

Moreover, Theorem 2.1 also implies by $\mathcal{C} \subset \mathcal{S}$ that any $m \in \mathcal{C}$ has at least three prime factors, each satisfying $p<\sqrt{m}$. As $m$ is composite and squarefree, an odd prime $p$ divides $m$. Using (5.1), we then get relation (5.6), so $p-1 \mid m-1$, whence $m$ is odd.

Proof of Theorem 2.3. Consider a non-empty subset $\mathcal{T} \subseteq \mathcal{S}$ and define

$$
\alpha_{\mathcal{T}}:=\sup _{m \in \mathcal{T}} \frac{P(m)}{\sqrt{m}}
$$

where $\alpha_{\mathcal{T}} \leq 1$ by Theorem 2.1. Clearly, this definition includes for any $m \in \mathcal{T}$ that

$$
\begin{equation*}
p \mid m \quad \Longrightarrow \quad p \leq \alpha_{\mathcal{T}} \sqrt{m} \tag{5.9}
\end{equation*}
$$

but it suffices to study the case where $p=P(m)$ is the greatest prime divisor of $m$. To show that the estimate in (5.9) is sharp, we further have to find an explicit $m^{\prime} \in \mathcal{T}$ such that $\alpha_{\mathcal{T}}=P\left(m^{\prime}\right) / \sqrt{m^{\prime}}$ holds.

Now, let $m \in \mathcal{T}$. In view of (5.2) and (5.3), we obtain by (5.9) that

$$
\begin{equation*}
\frac{1}{\alpha_{\mathcal{T}}^{2}} \leq \frac{m}{P(m)^{2}}=\frac{a_{0}}{P(m)}+a_{1} \tag{5.10}
\end{equation*}
$$

with $1 \leq a_{0} \leq P(m)-1$ and $a_{1} \geq 1$. Thus, we are interested in finding firstly a minimal number $a_{1}$, and secondly a minimal fraction $a_{0} / P(m) \in(0,1)$. If they exist, then $\alpha_{\mathcal{T}}$ is determined.

Next, we assume that there exists an element $m \in \mathcal{T}$ with $a_{1}=1$. (This is true for the sets of interest $\mathcal{T}=\mathcal{S}, \mathcal{C}, \mathcal{C}^{\prime}$.) From now on, let $p=P(m)$. Since $m \in \mathcal{S}$, we have the condition $s_{p}(m)=a_{0}+a_{1} \geq p$, so $a_{0}=p-1$. Then (5.10) becomes

$$
\begin{equation*}
\frac{1}{\alpha_{\mathcal{T}}^{2}} \leq \frac{m}{p^{2}}=\frac{p-1}{p}+1=2-\frac{1}{p} \tag{5.11}
\end{equation*}
$$

Hence, to determine a minimal $\alpha_{\mathcal{T}}$, we also have to determine a minimal $p$ satisfying (5.11). Since $p=P(m)$ and $m=p(2 p-1)$, the factor $p$ strictly increases with $m$. As a consequence, we can identify the aforementioned element $m^{\prime}$ as the minimal element $m^{\prime} \in \mathcal{T}$ for which $a_{1}=1$. Finally, we achieve that

$$
\alpha_{\mathcal{T}}=1 / \sqrt{2-\frac{1}{P\left(m^{\prime}\right)}}
$$

Now we use a link to the polygonal numbers. Since

$$
\begin{equation*}
m^{\prime}=p(2 p-1)=\mathbf{H}_{p} \tag{5.12}
\end{equation*}
$$

we have to find the least hexagonal number $\mathbf{H}_{p}$ in each of the sets $\mathcal{T}=\mathcal{S}, \mathcal{C}, \mathcal{C}^{\prime}$. This is done in Table 4.1, providing the solutions

$$
P\left(m^{\prime}\right)=11,17,66337 \quad \text { for } \quad \mathcal{T}=\mathcal{S}, \mathcal{C}, \mathcal{C}^{\prime}
$$

respectively.
There remains the case when $m \in \mathcal{S}_{\text {even }}$. For this purpose, let $\mathcal{T}=\mathcal{S}_{\text {even }}$ with $m \in \mathcal{T}$ and $p=P(m)$. Note that $p$ is odd, since $m$ is composite. We adapt and reuse the arguments that lead to (5.10) and (5.11). By (5.10) we have to find again a minimal $a_{1} \geq 1$. The case $a_{1}=1$ implies (5.12) and so an odd $m=\mathbf{H}_{p}$ for odd $p$. Therefore, we show that case $a_{1}=2$ works, as follows. By $s_{p}(m)=a_{0}+a_{1} \geq p$,
we obtain two solutions $a_{0}=p-2$ and $a_{0}=p-1$. Since $a_{0}=p-2$ implies $m=p(3 p-2)=\mathbf{O}_{p}$, being odd for odd $p$, there remains the case $a_{0}=p-1$. Then we get $m=p(3 p-1)=2 \mathbf{P}_{p}$, which is always even. Similar to (5.11), we deduce that

$$
\frac{1}{\alpha_{\mathcal{T}}^{2}} \leq \frac{m}{p^{2}}=\frac{p-1}{p}+2=3-\frac{1}{p}
$$

To find the minimal element $m^{\prime} \in \mathcal{T}$ with $a_{1}=2$, we have to find the least quasi pentagonal number $2 \mathbf{P}_{p}$ in $\mathcal{T}$. Table 4.1 shows that $P\left(m^{\prime}\right)=61$. With that we finally obtain

$$
\alpha_{\mathcal{T}}=1 / \sqrt{3-\frac{1}{P\left(m^{\prime}\right)}}
$$

This completes the proof of the theorem.

## 6. Proofs of Theorems 3.1 and 3.2 and Corollary 3.3

Proof of Theorem 3.1. From relations (3.5) and (3.12) we get

$$
\mathfrak{D}_{n}=\operatorname{lcm}\left(\mathbb{D}_{n+1}, \mathbb{D}_{n+1}^{\top} \cdot \mathbb{D}_{n+1}^{\top \star}\right)
$$

and the decomposition (3.9) gives $\mathbb{D}_{n+1}=\mathbb{D}_{n+1}^{\top} \cdot \mathbb{D}_{n+1}^{\perp}$. Since $\mathbb{D}_{n+1}^{\top}, \mathbb{D}_{n+1}^{\perp}$, and $\mathbb{D}_{n+1}^{T^{\star}}$ are pairwise coprime by the definitions in (3.10) and (3.11), the desired triple product formula follows.

If $m \in \mathbb{S}$, then $m=\operatorname{rad}(m)>1$. By relation (3.5), we then have $m \mid \mathfrak{D}_{m-1}$. Conversely, if $m \mid \mathfrak{D}_{m-1}$, then $m>1$ is squarefree, so $m \in \mathbb{S}$. This proves the required equivalence and completes the proof of the theorem.

Proof of Theorem 3.2. We have to show two parts.
(i). It suffices to prove the second statement. The definitions of $\mathcal{S}$ and $\mathbb{D}_{n}^{\top}$ yield immediately that $m \in \mathcal{S} \cup\{1\}$ if and only if $\mathbb{D}_{m}^{\top}=m$. Since $\mathbb{D}_{n}^{\top} \mid \mathbb{D}_{n}$ by (3.9), we have that $\mathbb{D}_{m}^{\top}=m$ implies $m \mid \mathbb{D}_{m}$. Conversely, if $m \mid \mathbb{D}_{m}$, then $\mathbb{D}_{m}^{\top}=m$ by (3.4) and (3.9). This proves (i).
(ii). If $m+1$ is composite, then we have by (3.8) and (3.9) that

$$
\operatorname{rad}(m+1) \mid \mathbb{D}_{m}=\mathbb{D}_{m}^{\top} \cdot \mathbb{D}_{m}^{\perp}
$$

Since $\operatorname{gcd}(m, m+1)=1$, we infer by (3.10) that $\operatorname{rad}(m+1) \mid \mathbb{D}_{m}^{\perp}$, proving (ii).
Proof of Corollary 3.3. The implication follows from Theorem 3.2 parts (i) and (ii), using the strict inclusion $\mathcal{C} \subset \mathcal{S}$ and the compositeness of Carmichael numbers. The converse does not hold, by Theorem 3.2 part (i) and considering $\mathcal{S} \backslash \mathcal{C}$.

## 7. Proofs of Theorems 4.1 and 4.2 and Corollary 4.3

Proof of Theorem 4.1. Fix $m \in \mathcal{S}$ and set $p=P(m)$ and $\ell=\ell(m)$. We have to show three parts.
(i). As $m$ is squarefree, we obtain

$$
\begin{equation*}
\frac{m}{p}=a_{0}+a_{1} p=a_{0}+\ell p \tag{7.1}
\end{equation*}
$$

where $1 \leq a_{0} \leq p-1$ and $a_{1} \geq 0$. The case $a_{1}=0$ would imply $s_{p}(m)=s_{p}(m / p)=$ $a_{0}<p$. Since $s_{p}(m) \geq p$ by $m \in \mathcal{S}$, we must have $\ell=a_{1} \geq 1$. We shall use (7.1) implicitly in the remaining parts.
(ii). If $\ell=1$, then $s_{p}(m)=a_{0}+1 \geq p$, since $m \in \mathcal{S}$. But $a_{0} \leq p-1$, so $a_{0}=p-1$. Thus $m=p(p-1+p)=\mathbf{H}_{p}$. Conversely, if $m=\mathbf{H}_{p}$, then $\ell=a_{1}=1$.
(iii). Assume that $s_{p}(m)=p$. If $\ell=2$, then $a_{0}+2=s_{p}(m)=p$, so $a_{0}=p-2$ and $m=p(p-2+2 p)=\mathbf{O}_{p}$. Conversely, if $m=\mathbf{O}_{p}$, then $\ell=a_{1}=2$.

In particular, it then follows for $m \in \mathcal{C}^{\prime}$ that $\ell=2$ if and only if $m=\mathbf{O}_{p}$, since $s_{p}(m)=p$ by the definition of $\mathcal{C}^{\prime}$.

Assume now that $s_{p}(m)>p$. If $\ell=2$, then $a_{0}+2=s_{p}(m)>p$, so $a_{0}>p-2$. But $a_{0} \leq p-1$, so $a_{0}=p-1$ and $m=p(p-1+2 p)=2 \mathbf{P}_{p}$. Conversely, if $m=2 \mathbf{P}_{p}$, then $\ell=a_{1}=2$. This proves the theorem.

Proof of Theorem 4.2. Fix $m \in \mathcal{S}$ and set $p=P(m)$ and $\ell=\ell(m)$. Since $s_{p}(m) \geq p$, we can determine the integers $\eta \geq 1$ and $0 \leq \mu<p-1$ satisfying (4.2). Again, as in (7.1) we have $m / p=a_{0}+\ell p$, where $1 \leq a_{0} \leq p-1$ and $\ell \geq 1$. Using (4.2) we then obtain

$$
\frac{m}{p}=\eta(p-1)+\mu-s_{p}(\ell)+\ell p
$$

Now let $d \in\{1,2\}$. Resolving the desired equality

$$
d \cdot \mathbf{G}_{p}^{r}=m
$$

yields the equation

$$
\frac{d}{2}\left(p^{2}(r-2)-p(r-4)\right)=p\left(\eta(p-1)+\mu-s_{p}(\ell)+\ell p\right)
$$

with solution

$$
\begin{equation*}
r=\frac{2}{d}\left(\ell+\frac{\ell-s_{p}(\ell)}{p-1}+\eta+d+\frac{\mu-d}{p-1}\right) . \tag{7.2}
\end{equation*}
$$

By Legendre's formula (4.1) we have

$$
\begin{equation*}
\frac{\ell-s_{p}(\ell)}{p-1}=\mathrm{v}_{p}(\ell!) \tag{7.3}
\end{equation*}
$$

Since $d \mid 2$ and $r \in \mathbb{Z}$, formulas (7.2) and (7.3) imply the condition

$$
\begin{equation*}
\tilde{e}:=\frac{2}{d} \cdot \frac{\mu-d}{p-1} \in \mathbb{Z} \tag{7.4}
\end{equation*}
$$

Since $m \in \mathcal{S}$ has at least three prime factors by Theorem 2.1, we have $p=$ $P(m) \geq 5$. This and the fact that $0 \leq \mu<p-1$ allow us to continue deriving solutions of (7.4) for $\mu$, as follows.

In case $d=2$, we infer that $\mu=2$ and $\tilde{e}=0$. In case $d=1$, we get the solutions $\mu=1$ and $\tilde{e}=0$, as well as $\mu=1+(p-1) / 2$ and $\tilde{e}=1$. One easily observes that all solutions of (7.4) for $\mu, d$, and $\tilde{e}$ coincide with condition (4.3) when taking $e=\tilde{e}$. Finally, relation (4.4) with $e=\tilde{e}$ follows from (7.2) by considering (7.3) and (7.4). This completes the proof of the theorem.

Proof of Corollary 4.3. If $m \in \mathcal{C}$, then $s_{p}(m)=\eta(p-1)+1$ with $\eta \geq 1$ by Theorem 2.2. In particular, if $m \in \mathcal{C}^{\prime}$, then $\eta=1$ by definition of the set $\mathcal{C}^{\prime}$. Since $\mathcal{C} \subset \mathcal{S}$ by Theorem 2.1, relation (4.5) follows by applying Theorem 4.2 with parameters $(d, e)=(1,0)$.

## 8. Modular Properties of the Set $\mathcal{S}$

Define for a positive integer $d$ the subset $\mathcal{S}_{d}$ of $\mathcal{S}$ by

$$
\mathcal{S}_{d}:=\left\{m \in \mathcal{S}: p \mid m \quad \Longrightarrow \quad s_{p}(m) \equiv d \quad(\bmod p-1)\right\}
$$

Theorem 2.2 shows that $\mathcal{S}_{1}=\mathcal{C}$. Thus the sets $\mathcal{S}_{d}$ can be viewed as a generalization, with the Carmichael numbers as a special case. The first terms of the sets $S_{d}$ for $d=1,2,3$ are (compare Table 8.1)

$$
\begin{aligned}
& \mathcal{S}_{1}=\{561,1105,1729,2465,2821,6601,8911,10585,15841, \ldots\} \\
& \mathcal{S}_{2}=\{1122,3458,5642,6734,11102,13202,17390,17822, \ldots\} \\
& \mathcal{S}_{3}=\{3003,3315,5187,7395,8463,14763,19803,26733, \ldots\}
\end{aligned}
$$

Let $\varphi$ denote Euler's totient function. The Carmichael function $\lambda$ (see [3]) is defined for $m=p_{1}^{e_{1}} \cdots p_{k}^{e_{k}}$ with $p_{1}<\cdots<p_{k}$ by

$$
\lambda(m)=\operatorname{lcm}\left(\lambda\left(p_{1}^{e_{1}}\right), \ldots, \lambda\left(p_{k}^{e_{k}}\right)\right)
$$

where $\lambda\left(p^{e}\right)=\delta \varphi\left(p^{e}\right)$ with $\delta=\frac{1}{2}$ if $p=2$ and $e \geq 3$, otherwise $\delta=1$.
For positive integers $m$ the Carmichael function $\lambda$ has the property that

$$
\begin{equation*}
a^{\lambda(m)} \equiv 1 \quad(\bmod m) \tag{8.1}
\end{equation*}
$$

holds for all integers $a$ coprime to $m$, where $\lambda(m)$ is the smallest possible positive exponent. Since (8.1) generalizes the Euler-Fermat congruence, it follows that $\lambda(m)$ divides $\varphi(m)$. Moreover, for $m \in \mathcal{S}$ we have the relation

$$
\begin{equation*}
\lambda(m)=\operatorname{lcm}\left(p_{1}-1, \ldots, p_{k}-1\right) \tag{8.2}
\end{equation*}
$$

where

$$
\begin{equation*}
m=p_{1} \cdots p_{k} \geq 231 \quad \text { and } \quad k \geq 3 \tag{8.3}
\end{equation*}
$$

Define the function $\rho$ for positive integers $m$ by

$$
\rho(1)=0, \quad \rho(2)=1,
$$

and

$$
\begin{equation*}
\rho(m) \equiv m \quad(\bmod \lambda(m)) \quad(m \geq 3) \tag{8.4}
\end{equation*}
$$

being the least positive residue.
In view of (8.1) and (8.4) the Fermat congruence (1.1) can be restated for $m \geq 1$ in the form

$$
a^{m-\rho(m)} \equiv 1 \quad(\bmod m)
$$

holding for all integers $a$ coprime to $m$. As a special case, one has

$$
\rho(m)=1 \Longleftrightarrow m \text { is prime or } m \in \mathcal{C}
$$

which Carmichael proved with $m \equiv 1(\bmod \lambda(m))$ in place of $\rho(m)=1$.
Moreover, since $\lambda(m)$ is even for $m \geq 3$ by construction, we have the parity relation

$$
\rho(m) \equiv m \quad(\bmod 2) \quad(m \geq 3)
$$

| $m$ | 231 | 561 | 1001 | 1045 | 1105 | 1122 | 1155 | 1729 | 2002 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $\rho(m)$ | 21 | 1 | 41 | 145 | 1 | 2 | 15 | 1 | 22 |
| $\lambda(m)$ | 30 | 80 | 60 | 180 | 48 | 80 | 60 | 36 | 60 |

Table 8.1: First values of $\rho(m)$ and $\lambda(m)$ for $m \in \mathcal{S}$.

Theorem 8.1. If $m \in \mathcal{S}$, then $\rho(m)$ equals the least positive index $d<\lambda(m)$ such that $m \in \mathcal{S}_{d}$. Moreover, we have

$$
m \in \mathcal{S}_{d+j \lambda(m)} \quad\left(j \in \mathbb{Z}_{\geq 0}\right)
$$

Proof. Given $m \in \mathcal{S}$, factor $m=p_{1} \cdots p_{k}$ and consider by (8.3) and (8.4) the congruences

$$
d \equiv m \equiv \rho(m) \quad(\bmod \lambda(m))
$$

From (5.1) and (8.2), we further deduce the system of congruences

$$
d \equiv m \equiv s_{p_{\nu}}(m) \quad\left(\bmod p_{\nu}-1\right) \quad(\nu=1, \ldots, k)
$$

Thus, $d=\rho(m)<\lambda(m)$ is the least positive index such that $m \in \mathcal{S}_{d}$. Moreover, it also follows that $m \in \mathcal{S}_{d+j \lambda(m)}$ for $j \geq 1$.

Define the $d$-Knödel numbers $\mathcal{K}_{d}$ (see [18]) to be the set of composite integers $m>d$ such that

$$
\begin{equation*}
a^{m-d} \equiv 1 \quad(\bmod m) \tag{8.6}
\end{equation*}
$$

holds for all integers $a$ coprime to $m$. (Note that the usual but equivalent definition is further restricted to $1<a<m$.) For example, the 1 -Knödel numbers are the Carmichael numbers: $\mathcal{K}_{1}=\mathcal{C}$. For $d=2,3$ the $d$-Knödel numbers are

$$
\begin{aligned}
& \mathcal{K}_{2}=\{4,6,8,10,12,14,22,24,26,30,34,38,46,56,58,62,74, \ldots\} \\
& \mathcal{K}_{3}=\{9,15,21,33,39,51,57,63,69,87,93,111,123,129,141, \ldots\} .
\end{aligned}
$$

Makowski [21] showed that each of the sets $\mathcal{K}_{d}$ for $d \geq 2$ is infinite. More precisely, for given $d \geq 2$ he proved the existence of infinitely many primes $p>d$ such that (see [24, pp. 125-126])

$$
\begin{equation*}
d p \in \mathcal{K}_{d} \tag{8.7}
\end{equation*}
$$

Our final theorem shows properties of the sets $\mathcal{S}_{d}$, as well as a connection with generalizations of the sets $\mathcal{K}_{d}$. Avoiding the restriction $m>d$ on numbers $m \in \mathcal{K}_{d}$, we define the superset $\widehat{\mathcal{K}}_{d}$ of $\mathcal{K}_{d}$ to be all composites $m>1$ satisfying (8.6) for all $a$ coprime to $m$. Note that $\mathcal{K}_{1}=\widehat{\mathcal{K}}_{1}$ and, in case $d$ is composite, $d \in \widehat{\mathcal{K}}_{d}$.

Theorem 8.2. The following statements hold:
(i) We have $\mathcal{S}_{1}=\mathcal{K}_{1}=\mathcal{C}$ and $\mathcal{S}_{d} \subset \widehat{\mathcal{K}}_{d}$ for $d \geq 2$.
(ii) All elements of $\mathcal{S}_{d}$ have the same parity as d for $d \geq 1$.
(iii) A cover of the set $\mathcal{S}$ is

$$
\mathcal{S}=\bigcup_{d \geq 1} \mathcal{S}_{d}
$$

Proof. We have to show three parts:
(i). We have $\mathcal{S}_{1}=\mathcal{K}_{1}=\mathcal{C}$ by definition. Fix $d \geq 2$. If $m \in \mathcal{S}_{d}$, then Theorem 8.1 implies that $d \equiv \rho(m)(\bmod \lambda(m))$. By (8.3) and (8.4) this translates to $d \equiv m$ $(\bmod \lambda(m))$. Finally, (8.1) and (8.6) imply that $m \in \widehat{\mathcal{K}}_{d}$. This shows that $\mathcal{S}_{d} \subseteq \widehat{\mathcal{K}}_{d}$.

By (8.7) there exists a prime $p>d$ such that $m^{\prime}=d p \in \mathcal{K}_{d} \subseteq \widehat{\mathcal{K}}_{d}$. Since $s_{p}\left(m^{\prime}\right)=d<p$, it follows that $m^{\prime} \notin \mathcal{S}$. This implies that $\mathcal{S}_{d} \neq \widehat{\mathcal{K}}_{d}$, and finally $\mathcal{S}_{d} \subset \widehat{\mathcal{K}}_{d}$.
(ii). Fix $d \geq 1$ and $m \in \mathcal{S}_{d}$. As in part (i) we have $d \equiv \rho(m) \equiv m(\bmod \lambda(m))$. By (8.5) the result follows.
(iii). Set $\mathcal{U}=\bigcup_{d \geq 1} \mathcal{S}_{d}$. Since $\mathcal{S}_{d} \subset \mathcal{S}$ for $d \geq 1$, it follows that $\mathcal{U} \subseteq \mathcal{S}$. By Theorem 8.1 we obtain for any $m \in \mathcal{S}$ an index $d=\rho(m)$ such that $m \in \mathcal{S}_{d}$. As a consequence, $\mathcal{S} \subseteq \mathcal{U}$ and finally $\mathcal{S}=\mathcal{U}$.

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