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A GENERALIZATION OF THE KŐVÁRI–SÓS–TURÁN THEOREM

Jesse Geneson
Department of Mathematics & Statistics, San Jose State University, San Jose, California
jesse.geneson@sjsu.edu

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Abstract

We present a new proof of the Kővári–Sós–Turán theorem that $\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/t})$ for $s, t \geq 2$. We also obtain generalizations of the Kővári–Sós–Turán theorem for d-uniform hypergraphs and d-dimensional 0-1 matrices, some of which we show are sharp. For any d-uniform hypergraph H, let $\mathrm{ex}_d(n, H)$ be the maximum possible number of edges in an H-free d-uniform hypergraph on n vertices. Let $K_{H,t}$ be the $(d+1)$-uniform hypergraph obtained from H by adding t new vertices $v_1, \ldots, v_t$ and replacing every edge e in $E(H)$ with t edges $e \cup \{v_1\}, \ldots, e \cup \{v_t\}$ in $E(K_{H,t})$. For example, if H is the 1-uniform hypergraph on s vertices with s edges, then $K_{H,t} = K_{s,t}$. We prove that $\mathrm{ex}_{d+1}(n, K_{H,t}) = O(\mathrm{ex}_d(n, H)^{1/t} n^{d+1-d/t} + tn^d)$ for any d-uniform hypergraph H with at least two edges such that $\mathrm{ex}_d(n, H) = o(n^d)$. This implies that $\mathrm{ex}_{d+1}(n, K_{H,t}) = O(n^{d+1-1/t})$ for any d-uniform hypergraph H with at least two edges such that $\mathrm{ex}_d(n, H) = O(n^{d-1})$, which implies the Kővári–Sós–Turán theorem in the $d = 1$ case. This also implies that $\mathrm{ex}_{d+1}(n, K_{H,t}) = O(n^{d+1-1/t})$ when H is a d-uniform hypergraph with at least two edges in which all edges are pairwise disjoint, which generalizes an upper bound of Mubayi and Verstraëte. Our bounds for 0-1 matrix Turán problems are analogous, though for 0-1 matrices we show that the bounds are sharp up to a constant factor in some cases.

1. Introduction

The Kővári–Sós–Turán theorem is one of the most famous results in extremal combinatorics [16, 8, 10]. The theorem states that the maximum number of edges in a $K_{s,t}$-free graph of order n is $O(n^{2-1/t})$. There are multiple known proofs of this theorem, including a standard double-counting proof that uses Jensen's inequality, as well as a proof that uses dependent random choice and Jensen's inequality [2].

In this paper, we present a simple new method for proving the bound in the Kővári–Sós–Turán theorem. The method is based on techniques used in Nivasch's bounds on Davenport–Schinzel sequences [19] and Alon et al.'s bounds on interval chains [1]. This new proof gives a simple way to teach the proof of the Kővári–Sós–Turán theorem to students that are not familiar with convexity, and the same method can be used to prove a generalization of the Kővári–Sós–Turán theorem for uniform hypergraphs.

In [19], Nivasch found upper bounds on the maximum lengths of Davenport–Schinzel sequences using two different methods. Both methods gave the same bounds, but the first method was more like the proofs in past papers on Davenport–Schinzel sequences, and the second method was similar to proofs about interval chains in [1]. The second method in [19] was much simpler than the first for proving bounds on Davenport–Schinzel sequences, so we imitate the second method here for graph, hypergraph, and 0-1 matrix Turán problems.

Let $\mathrm{ex}_d(n,H)$ denote the maximum number of edges in an H-free d-uniform hypergraph on n vertices. Let $K_{H,t}$ be the $(d+1)$-uniform hypergraph obtained from H by adding t new vertices $v_1,\dots,v_t$ and replacing every edge e in $E(H)$ with $e\cup\{v_1\},\dots,e\cup\{v_t\}$ in $E(K_{H,t})$. For example, if H is the 1-uniform hypergraph of order s with s edges, then $K_{H,t}=K_{s,t}$. Mubayi and Verstraëte [18] proved that $\mathrm{ex}_3(n,K_{H,t})=O(n^{3-1/t})$ when H is a matching.

In Section 2, we prove that $\mathrm{ex}_{d+1}(n,K_{H,t})=O(\mathrm{ex}_d(n,H)^{1/t}n^{d+1-d/t}+tn^d)$ for any d-uniform hypergraph H with at least two edges such that $\mathrm{ex}_d(n,H)=o(n^d)$, giving an alternative proof of the Kővári–Sós–Turán theorem when H is the 1-uniform hypergraph of order s with s edges. As a corollary, this implies that $\mathrm{ex}_{d+1}(n,K_{H,t})=O(n^{d+1-1/t})$ when H is a d-uniform hypergraph with at least two edges in which all edges are pairwise disjoint, generalizing the upper bound of Mubayi and Verstraëte from [18]. In Section 3, we discuss analogous results about d-dimensional 0-1 matrices that can be proved with similar methods.

2. The Letter Method

In order to illustrate the idea of the letter method, we first reprove the Kővári–Sós–Turán theorem using the letter method, and then we use the method to obtain a generalization. We start by defining a *lettered* graph as the structure obtained from labeling each edge of an ordered graph with a letter such that two edges can be labeled with the same letter only if they have the same greatest vertex. Given a graph H, we say that a lettered graph is H-free if its underlying graph is H-free. For any graph H, let $\mathrm{f}(n,k,H)$ denote the maximum possible number of distinct letters in an H-free lettered graph on n vertices in which every letter occurs at least k times. The next lemma is analogous to inequalities in [19, 4, 11, 13] about sequences and 0-1 matrices, and it is proved similarly.

Lemma 2.1. *For all positive integers n,k and graphs H, we have $\mathrm{ex}(n,H)\le k(\mathrm{f}(n,k,H)+n)$.*

Proof. Start with an H-free graph G of order n with $\mathrm{ex}(n,H)$ edges. Order the vertices of G arbitrarily to create an ordered graph Q. For each vertex v in $V(Q)$ in order from greatest to least, label the unlabeled edges adjacent to v in any order with letters $v_0, v_1, \ldots,$ only using each letter v_i exactly k times and deleting up to $k-1$ remaining edges adjacent to v if k does not divide the total number of edges in which v is the greatest vertex. Observe that the new lettered graph has at most $\mathrm{f}(n,k,H)$ distinct letters with every letter occurring exactly k times, and it is H-free. □

When combined with Lemma 2.1, the next lemma will complete our proof of the Kővári–Sós–Turán theorem using the letter method. We use Stirling's bound in the proof of the next lemma, but it is not actually necessary. We explain after the proof how the use of Stirling's bound can be replaced with an elementary argument.

Lemma 2.2. *For $s,t \geq 2$ and $k = \lceil 2en^{1-1/t}(s-1)^{1/t} \rceil$, we have $\mathrm{f}(n,k,K_{s,t}) = O(t(\frac{n}{s})^{1/t})$.*

Proof. Suppose for contradiction that there exists a $K_{s,t}$-free lettered graph Q on n vertices with $r = \lfloor \frac{t}{e}(\frac{n}{s-1})^{1/t} \rfloor$ distinct letters in which every letter occurs at least k times. Suppose that n is sufficiently large so that $r \geq \frac{t}{2e}(\frac{n}{s-1})^{1/t}$. Without loss of generality, suppose that every letter in Q occurs exactly k times.

For each vertex $z \in V(Q)$, define $\deg_>(z)$ to be the number of edges in $E(Q)$ that contain z and a greater vertex in the ordering. Let p be the number of vertices z of $V(Q)$ with $\deg_>(z) > 0$. The number of t-tuples of edges in $E(Q)$ that have the same least vertex is equal to $\sum_{z:\deg_>(z)\geq t} \binom{\deg_>(z)}{t}$, which is at most $\binom{r}{t}(s-1)$, or else Q would contain a copy of $K_{s,t}$. This follows by the pigeonhole principle, since every t-tuple of edges in $E(Q)$ that have the same least vertex must have different letters on each edge.

Then $kr = \sum_z \deg_>(z)$ and we have the following inequality.

$$
\begin{aligned}
&(t-1)p \geq \\
&\sum_{z:\deg_>(z)<t} \deg_>(z) + \sum_{z:\deg_>(z)\geq t} (t-1) = \\
&\sum_{z:\deg_>(z)<t} \deg_>(z) + \sum_{z:\deg_>(z)\geq t} \deg_>(z) - \sum_{z:\deg_>(z)\geq t} (\deg_>(z) - t + 1) = \\
&\sum_{z} \deg_>(z) - \sum_{z:\deg_>(z)\geq t} (\deg_>(z) - t + 1) = \\
&kr - \sum_{z:\deg_>(z)\geq t} (\deg_>(z) - t + 1).
\end{aligned}
$$

In the following, we have already proved the first inequality. The second inequality follows since $\binom{x}{t} = \frac{\prod_{i=1}^{t}(x+1-i)}{\prod_{i=1}^{t}(t+1-i)} \geq x-t+1$ for $x \geq t$, since the i^{th} term in the numerator is at least the i^{th} term in the denominator for each $i \leq t-1$. The third inequality was explained in the last full paragraph, and the fourth inequality follows from the factorial definition of binomial coefficients. The fifth inequality follows from the definition of r, which implies that $\frac{r^t(s-1)}{t!} \leq \frac{(\frac{t}{e})^t}{t!}n$, and the assumption that n is sufficiently large so that $r \geq \frac{t}{2e}(\frac{n}{s-1})^{1/t}$, which implies that $kr \geq tn$. The last inequality follows from Stirling's bound.

$$\begin{aligned}
(t-1)p &\geq \\
kr - \sum_{z:\deg_>(z)\geq t} (\deg_>(z) - t + 1) &\geq \\
kr - \sum_{z:\deg_>(z)\geq t} \binom{\deg_>(z)}{t} &\geq \\
kr - \binom{r}{t}(s-1) &\geq \\
kr - \frac{r^t(s-1)}{t!} &\geq \\
tn - \frac{(\frac{t}{e})^t}{t!}n &> (t-1)n.
\end{aligned}$$

However $p \leq n$, a contradiction. □

Combining the last two lemmas gives the next theorem.

Theorem 2.3. *For fixed $s, t \geq 2$, we have* $\mathrm{ex}(n, K_{s,t}) = O(s^{1/t}n^{2-1/t} + tn)$.

The use of Stirling's bound in Lemma 2.5 may seem to make the proof non-elementary, but it was unnecessary. All we need is that there exists some constant c such that $t! > \left(\frac{t}{c}\right)^t$ for all $t > 1$, and then we can replace each e in the last proof with c. It is simple to show this by induction, e.g., for $c = 8$. It is clearly true for $t \leq 8$, which covers the base case. For the inductive step, if we assume that $t! > \left(\frac{t}{8}\right)^t$, then we obtain the even case as follows. The first inequality follows since $\prod_{i=t+1}^{2t} i > t^t$. The second inequality follows since $t^t = \left(\frac{t}{4}\right)^t 4^t$ and $t! > \left(\frac{t}{8}\right)^t$ by inductive hypothesis. The equality follows by cancellation and rearranging the terms, and the final inequality follows because $2^t > 1$.

$$
\begin{aligned}
(2t)! &> \\
t^t t! &> \\
\left(\frac{t}{4}\right)^t 4^t \left(\frac{t}{8}\right)^t &= \\
\left(\frac{t}{4}\right)^{2t} 2^t &> \\
\left(\frac{t}{4}\right)^{2t}&.
\end{aligned}
$$

The odd case is obtained similarly. The first equality is immediate, and the first inequality follows since we already proved in the even case that $(2t)! > \left(\frac{t}{4}\right)^{2t} 2^t$. The second inequality follows since $2t+1 > \frac{2t+1}{8}$ and $(1+\frac{1}{2t})^2 < 2$ for $t > 1$. The final equality follows by cancellation and rearranging the terms.

$$
\begin{aligned}
(2t+1)! &= \\
(2t+1)(2t)! &> \\
\left(\frac{t}{4}\right)^{2t} 2^t(2t+1) &> \\
\left(\frac{t}{4}\right)^{2t} \left(1+\frac{1}{2t}\right)^{2t} \left(\frac{2t+1}{8}\right) &= \\
\left(\frac{2t+1}{8}\right)^{2t+1}&.
\end{aligned}
$$

Thus the whole proof is elementary. Now we prove a generalization of the Kővári–Sós–Turán theorem. An *ordered* d-uniform hypergraph is a d-uniform hypergraph with a linear order on the vertices. We define a *lettered* d-uniform hypergraph as the structure obtained from labeling each edge of an ordered d-uniform hypergraph with a letter such that two edges can be labeled with the same letter only if they have the same greatest vertex. Given a d-uniform hypergraph H, we say that a lettered d-uniform hypergraph is H-free if its underlying d-uniform hypergraph is H-free. For any d-uniform hypergraph H, let $\mathrm{f}_d(n, k, H)$ denote the maximum possible number of distinct letters in an H-free lettered d-uniform hypergraph on n vertices in which every letter occurs at least k times.

Lemma 2.4. *For all positive integers n, k and d-uniform hypergraphs H, we have $\mathrm{ex}_d(n, H) \le k(\mathrm{f}_d(n, k, H) + n)$.*

Proof. Start with a d-uniform H-free hypergraph Q of order n with $\mathrm{ex}_d(n, H)$ edges. Order the vertices of Q arbitrarily. For each vertex v in $V(Q)$ in order from greatest to least, label the unlabeled edges adjacent to v in any order with letters $v_0, v_1, \ldots,$ only using each letter v_i exactly k times and deleting up to $k-1$ remaining edges adjacent to v if k does not divide the total number of edges in which v is the greatest vertex. Observe that the new lettered hypergraph has at most $\mathrm{f}_d(n,k,H)$ distinct letters with every letter occurring exactly k times, and it is H-free. $\square$

Lemma 2.5. *For $t \geq 2$, H a d-uniform hypergraph with at least two edges such that $\mathrm{ex}_d(n,H) = o(n^d)$, and $k = \lceil 2en^{d(1-1/t)} \mathrm{ex}_d(n,H)^{1/t} \rceil$, we have $\mathrm{f}_{d+1}(n,k,K_{H,t}) = O(t(\frac{n^d}{\mathrm{ex}_d(n,H)})^{1/t})$.*

Proof. Suppose for contradiction that there exists a $K_{H,t}$-free lettered $(d+1)$-uniform hypergraph Q on n vertices with $r = \lfloor \frac{t}{e}(\frac{n^d}{\mathrm{ex}_d(n,H)})^{1/t} \rfloor$ distinct letters in which every letter occurs at least k times. Suppose that n is sufficiently large so that $r \geq \frac{t}{2e}(\frac{n^d}{\mathrm{ex}_d(n,H)})^{1/t}$. The fact that there is such an n follows because $\mathrm{ex}_d(n,H) = o(n^d)$. Without loss of generality, suppose that every letter in Q occurs exactly k times.

For each d-subset z of $V(Q)$, define $\deg_>(z)$ to be the number of edges in $E(Q)$ that contain all of the vertices in z and a greater vertex in the ordering. Let p be the number of d-subsets z of $V(Q)$ with $\deg_>(z) > 0$. The number of t-tuples of edges in $E(Q)$ that have the same d least vertices is equal to $\sum_{z:\deg_>(z)\geq t} \binom{\deg_>(z)}{t}$, which is at most $\binom{r}{t}\mathrm{ex}_d(n,H)$, or else Q would contain a copy of $K_{H,t}$. This follows by the pigeonhole principle, since every t-tuple of edges in $E(Q)$ that have the same d least vertices must have different letters on each edge.

Then $kr = \sum_z \deg_>(z)$ and the following string of inequalities follows by an analogous argument to the string of inequalities in Lemma 2.2.

$$\begin{aligned}
(t-1)p &\geq \\
kr - \sum_{z:\deg_>(z)\geq t} (\deg_>(z) - t + 1) &\geq \\
kr - \sum_{z:\deg_>(z)\geq t} \binom{\deg_>(z)}{t} &\geq \\
kr - \binom{r}{t}\mathrm{ex}_d(n,H) &\geq \\
kr - \frac{r^t \,\mathrm{ex}_d(n,H)}{t!} &\geq \\
tn^d - \frac{(\frac{t}{e})^t}{t!} n^d &> (t-1)n^d.
\end{aligned}$$

However $p \leq \binom{n}{d}$, a contradiction. $\square$

Theorem 2.6. *For fixed $t \geq 2$ and d-uniform hypergraph H with at least two edges such that $\mathrm{ex}_d(n, H) = o(n^d)$, we have $\mathrm{ex}_{d+1}(n, K_{H,t}) = O(\mathrm{ex}_d(n, H)^{1/t} n^{d+1-d/t} + tn^d)$.*

Corollary 2.7. *If H is a d-uniform hypergraph with at least two edges in which all edges are pairwise disjoint, then $\mathrm{ex}_{d+1}(n, K_{H,t}) = O(n^{d+1-1/t})$.*

Proof. Any d-uniform hypergraph of order n with at least $kd\binom{n}{d-1}$ edges must have at least k disjoint edges, since there can be at most $x\binom{n}{d-1}$ edges that intersect any given set of x distinct vertices. Thus if H is a d-uniform hypergraph in which all edges are pairwise disjoint, then $\mathrm{ex}_d(n, H) = O(n^{d-1})$. So the bound on $\mathrm{ex}_{d+1}(n, K_{H,t})$ follows from Theorem 2.6. □

The last corollary yields the bound of Mubayi and Verstraëte from [18] when $d = 2$.

3. 0-1 Matrices

Using the same method, we can get similar bounds for Turán-type problems on d-dimensional 0-1 matrices. In order to state these results, we introduce more terminology. We say that d-dimensional 0-1 matrix A *contains* d-dimensional 0-1 matrix B if some submatrix of A can be turned into B by changing some nonnegative number of ones to zeroes. Otherwise A *avoids* B. For any d-dimensional 0-1 matrix Q, define $\mathrm{ex}(n, Q, d)$ to be the maximum number of ones in a d-dimensional 0-1 matrix of dimensions $n \times \cdots \times n$ that avoids Q.

As with the case of d-uniform hypergraphs, most of the past research on the topic of d-dimensional 0-1 matrices has focused on the case $d = 2$. We mention several results for $d = 2$ that have been generalized to higher values of d. For example, Klazar and Marcus [15] proved that $\mathrm{ex}(n, P, d) = O(n^{d-1})$ for every d-dimensional permutation matrix P, generalizing the result of Marcus and Tardos [17]. Geneson and Tian [14] sharpened this bound by proving that $\mathrm{ex}(n, P, d) = 2^{O(k)} n^{d-1}$ for d-dimensional permutation matrices P of sidelength k, generalizing a result of Fox [6]. Geneson and Tian also proved that $\mathrm{ex}(n, P, d) = O(n^{d-1})$ for every d-dimensional double permutation matrix P, generalizing the upper bound in [12].

In order to state the next result, we define $Q_{B,t}$ to be the $(d+1)$-dimensional 0-1 matrix obtained from the d-dimensional 0-1 matrix B by stacking t copies of B with the same orientation in the direction of the new dimension. In other words, entry $(x_1, \ldots, x_d, i)$ of $Q_{B,t}$ is 1 if and only if entry $(x_1, \ldots, x_d)$ of B is 1 for each $1 \leq i \leq t$. For example if B is the 1-dimensional matrix of length 4 with all entries equal to 1, then $Q_{B,t}$ is the $4 \times t$ matrix of all ones.

Theorem 3.1. *1. For fixed t and d-dimensional 0-1 matrix B with at least two ones,* $\mathrm{ex}(n, Q_{B,t}, d+1) = O(\mathrm{ex}(n, B, d)^{1/t} n^{d+1-d/t})$.

2. For any d-dimensional 0-1 matrix B with at least two ones such that $\mathrm{ex}(n, B, d) = O(n^{d-1})$, *we have* $\mathrm{ex}(n, Q_{B,t}, d+1) = O(n^{d+1-1/t})$. *In particular,* $\mathrm{ex}(n, Q_{B,2}, d+1) = \Theta(n^{d+1/2})$ *for any d-dimensional 0-1 matrix B with at least two ones such that* $\mathrm{ex}(n, B, d) = O(n^{d-1})$. *Moreover,* $\mathrm{ex}(n, Q_{B,3}, d+1) = \Theta(n^{d+2/3})$ *for any d-dimensional 0-1 matrix B with at least three ones differing in the first coordinate such that* $\mathrm{ex}(n, B, d) = O(n^{d-1})$.

Proof. The proof for the upper bounds is similar to the proof of Lemma 2.5. We define a *lettered* d-dimensional 0-1 matrix as the structure obtained from labeling each one of a d-dimensional 0-1 matrix with a letter such that two ones can be labeled with the same letter only if they have the same last coordinate. Given a d-dimensional 0-1 matrix B, we say that a lettered d-dimensional 0-1 matrix is B-free if its underlying d-dimensional 0-1 matrix is B-free.

For any d-dimensional 0-1 matrix B, let $\mathrm{f}(n, k, B, d)$ denote the maximum possible number of distinct letters in a B-free lettered d-dimensional 0-1 matrix of dimensions $n \times \cdots \times n$ in which every letter occurs at least k times. As in Lemma 2.4, we have $\mathrm{ex}(n, B, d) \leq k(\mathrm{f}(n, k, B, d) + n)$ for all positive integers n, k and d-dimensional 0-1 matrices B. We can start with a B-free d-dimensional 0-1 matrix A with $\mathrm{ex}(n, B, d)$ ones and dimensions $n \times \cdots \times n$. For each $j = 1, \ldots, n$, label the ones with last coordinate j in any order with letters $j_0, j_1, \ldots$, only using each letter j_i exactly k times and deleting up to $k-1$ remaining ones with last coordinate j if k does not divide the total number of ones with last coordinate j. Observe that the new lettered d-dimensional 0-1 matrix has at most $\mathrm{f}(n, k, B, d)$ distinct letters with every letter occurring exactly k times, and it is B-free.

As in Lemma 2.5, we can show for $t \geq 2$ that if B a d-dimensional 0-1 matrix with at least two ones such that $\mathrm{ex}(n, B, d) = o(n^d)$ and $k = \lceil 2en^{d(1-1/t)} \mathrm{ex}(n, B, d)^{1/t} \rceil$, we have $\mathrm{f}(n, k, Q_{B,t}, d+1) = O(t(\frac{n^d}{\mathrm{ex}(n,B,d)})^{1/t})$. Note that $\mathrm{ex}(n, B, d) = o(n^d)$ for all d-dimensional 0-1 matrices B [14]. With $\mathrm{ex}(n, Q_{B,t}, d+1) \leq k(\mathrm{f}(n, k, Q_{B,t}, d+1) + n)$, this will imply that $\mathrm{ex}(n, Q_{B,t}, d+1) = O(\mathrm{ex}(n, B, d)^{1/t} n^{d+1-d/t} + tn^d)$. Since $\mathrm{ex}(n, B, d) \geq n^{d-1}$ for any d-dimensional 0-1 matrix B with at least two ones, this will imply that $\mathrm{ex}(n, Q_{B,t}, d+1) = O(\mathrm{ex}(n, B, d)^{1/t} n^{d+1-d/t})$.

Suppose for contradiction that there exists a $Q_{B,t}$-free lettered $(d+1)$-dimensional 0-1 matrix A of dimensions $n \times \cdots \times n$ with $r = \lfloor \frac{t}{e}(\frac{n^d}{\mathrm{ex}(n,B,d)})^{1/t} \rfloor$ distinct letters in which every letter occurs at least k times. Suppose that n is sufficiently large so that $r \geq \frac{t}{2e}(\frac{n^d}{\mathrm{ex}(n,B,d)})^{1/t}$. The fact that there is such an n follows because $\mathrm{ex}(n, B, d) = o(n^d)$ for all d-dimensional 0-1 matrices B [14]. Without loss of generality, suppose that every letter in A occurs exactly k times.

For each d-tuple $z \in [n]^d$, define $\deg_>(z)$ to be the number of ones in A that have first d coordinates equal to z. Let p be the number of d-tuples $z \in [n]^d$ with

$\deg_>(z) > 0$. The number of t-tuples of ones in A that have the same first d coordinates is equal to $\sum_{z:\deg_>(z)\geq t} \binom{\deg_>(z)}{t}$, which is at most $\binom{r}{t} \operatorname{ex}(n,B,d)$, or else A would contain a copy of $Q_{B,t}$. This follows by the pigeonhole principle, since every t-tuple of ones in A that have the same first d coordinates must have different letters on each one.

Then $kr = \sum_z \deg_>(z)$ and again, the following string of inequalities follows by an analogous argument to the string of inequalities in Lemma 2.2.

$$\begin{aligned}
(t-1)p &\geq \\
kr - \sum_{z:\deg_>(z)\geq t} (\deg_>(z) - t + 1) &\geq \\
kr - \sum_{z:\deg_>(z)\geq t} \binom{\deg_>(z)}{t} &\geq \\
kr - \binom{r}{t} \operatorname{ex}(n,B,d) &\geq \\
kr - \frac{r^t \operatorname{ex}(n,B,d)}{t!} &\geq \\
tn^d - \frac{(\frac{t}{e})^t}{t!} n^d &> (t-1)n^d.
\end{aligned}$$

We get a contradiction from $p \leq n^d$. This gives the upper bounds.

For the lower bounds, let $J_{s,t}$ denote the $s \times t$ matrix of all ones. It is known that $\operatorname{ex}(n, J_{2,2}, 2) = \Theta(n^{3/2})$ and $\operatorname{ex}(n, J_{3,3}, 2) = \Theta(n^{5/3})$ [3, 5, 9, 10]. Suppose that Q is any 2-dimensional 0-1 matrix of dimensions $r \times s$ and R is any d-dimensional 0-1 matrix with first and last dimensions of length r and s respectively such that entry (i,j) of Q is 1 if and only if there exist $x_2, \ldots, x_{d-1}$ such that entry $(i, x_2, \ldots, x_{d-1}, j)$ of R is 1. It is easy to see that $\operatorname{ex}(n,R,d) \geq n^{d-2} \operatorname{ex}(n,Q,2)$ for any such 2-dimensional 0-1 matrix Q and d-dimensional 0-1 matrix R (for a proof, see, e.g., [14]). So if B is a d-dimensional 0-1 matrix with at least two ones with different first coordinates, then $\operatorname{ex}(n, Q_{B,2}, d+1) \geq n^{d-1} \operatorname{ex}(n, J_{2,2}) = \Omega(n^{d+1/2})$ since $Q_{B,2}$ contains a $(d+1)$-dimensional 0-1 matrix R with four ones having (first, last) coordinate pairs $(1,1),(1,2),(2,1),(2,2)$. Similarly if B is a d-dimensional 0-1 matrix with at least three ones differing in the first coordinate, then $\operatorname{ex}(n, Q_{B,3}, d+1) \geq n^{d-1} \operatorname{ex}(n, J_{3,3}) = \Omega(n^{d+2/3})$ since $Q_{B,3}$ contains a $(d+1)$-dimensional 0-1 matrix R' with nine ones having (first, last) coordinate pairs $(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$. □

Permutation matrices and double permutation matrices P with at least three ones are some examples for which Theorem 3.1 gives sharp bounds on $\operatorname{ex}(n, Q_{P,2}, d+1)$ and $\operatorname{ex}(n, Q_{P,3}, d+1)$ up to a constant factor [17, 12, 15, 14].

4. Concluding Remarks

The standard double-counting method used to prove the Kővári–Sós–Turán theorem can also be used to prove the bounds in Theorem 2.6 and 3.1. We did not include this method since it uses convexity, and it gives the same bounds up to a constant factor as the letter method. Dependent random choice can also be used to obtain the same bounds for uniform hypergraphs up to a constant factor when $\mathrm{ex}_d(n, H) = O(n^{d-1})$, and it can be applied to a larger family of hypergraphs that contains the family of $K_{H,t}$, but it gives a worse bound than the letter method when $\mathrm{ex}_d(n, H) = \omega(n^{d-1})$. The next lemma is a generalization of the dependent random choice lemma from [2] and [7]. In the next lemma, we call a vertex v and a d-subset T of vertices of a $(d+1)$-uniform hypergraph G *neighbors* if $v \notin T$ and there is some edge of G that contains v and all of the vertices of T. For each vertex v, set of vertices S, and set of d-subsets of vertices Q, we define $N(v)$ to be the set of d-subsets of vertices that are neighbors with v, we define $N(S)$ to be the set of d-subsets of vertices that are neighbors with every vertex in S, and we define $N(Q)$ to be the set of vertices that are neighbors with every d-subset in Q.

Lemma 4.1. *Let $G = (V, E)$ be a $(d+1)$-uniform hypergraph with $|V| = n$ vertices and $|E| = m$ edges. If there is a positive integer t such that $n\binom{n}{d}^{-t}\left(\frac{m}{n}\right)^t - \binom{n}{r}\left(\frac{x}{\binom{n}{d}}\right)^t \geq a$, then G contains a subset A of at least a vertices such that every r vertices in A have at least x common neighbors among the d-subsets of V.*

Proof. Pick a set T of d-subsets of vertices of V, choosing t d-subsets uniformly at random with repetition. Let $B = N(T)$, and let X be the cardinality of B. Then $\mathbb{E}[X] = \sum_{v\in V}\left(\frac{|N(v)|}{\binom{n}{d}}\right)^t = \binom{n}{d}^{-t}\sum_{v\in V}|N(v)|^t \geq n\binom{n}{d}^{-t}\left(\frac{\sum_{v\in V}|N(v)|}{n}\right)^t \geq n\binom{n}{d}^{-t}\left(\frac{m}{n}\right)^t$, where the second-to-last inequality used Jensen's inequality.

Let Y be the random variable for the number of subsets $S \subset B$ of size r with fewer than x common neighbors among the d-subsets of vertices of V. The probability that an arbitrary r-subset S is a subset of B is $\left(\frac{|N(S)|}{\binom{n}{d}}\right)^t$, so $\mathbb{E}[Y] \leq \binom{n}{r}\left(\frac{x}{\binom{n}{d}}\right)^t$.

Thus by linearity of expectation, $\mathbb{E}[X - Y] \geq n\binom{n}{d}^{-t}\left(\frac{m}{n}\right)^t - \binom{n}{r}\left(\frac{x}{\binom{n}{d}}\right)^t \geq a$. Thus there exists a choice of T for which the corresponding set B of cardinality X satisfies $X - Y \geq a$, so we can remove at most Y vertices from B (one for each subset $S \subset B$ of size r with fewer than x common neighbors among the d-subsets of vertices of V) to produce a new subset A so that all r-subsets of A have at least x common neighbors among the d-subsets of V. $\square$

We can use Lemma 4.1 to get upper bounds for more general families of $(d+1)$-uniform hypergraphs that contain the family $K_{H,t}$. The next theorem describes one such family.

Theorem 4.2. *For any d-uniform hypergraph H, let $K_{H,t,s,r}$ be the $(d+1)$-uniform hypergraph obtained by starting with s vertices $S = \{v_1, \ldots, v_s\}$, making r disjoint copies $H_{T_1}, \ldots, H_{T_r}$ of H for each t-subset T of vertices of S, and replacing each edge e in each H_{T_i} with t edges of the form $e \cup \{u\}$ for each $u \in T$. Then $\mathrm{ex}_{d+1}(n, K_{H,t,s,r}) = O((\mathrm{ex}_d(n, H) + n^{d-1})n^{2-1/t})$ for any d-uniform hypergraph H with at least two edges and any integers $s \geq t \geq 2$ and $r > 0$. For any d-uniform hypergraph H with at least two edges such that $\mathrm{ex}_d(n, H) = O(n^{d-1})$ and any integers $s \geq t \geq 2$ and $r > 0$, we have $\mathrm{ex}_{d+1}(n, K_{H,t,s,r}) = O(n^{d+1-1/t})$.*

Proof. Suppose that H has order k. We use Lemma 4.1 with $a = s$, $r = t$, $x = s\binom{n}{d-1} + \mathrm{ex}_d(n, H) + 1 + \binom{s}{t}rk\binom{n}{d-1}$, and m sufficiently large so that $n\binom{n}{d}^{-t}\left(\frac{m}{n}\right)^t - \binom{n}{r}\left(\frac{x}{\binom{n}{d}}\right)^t \geq a$. If G is a $(d+1)$-uniform hypergraph with n vertices and m edges, then G contains a subset A of at least s vertices such that every t vertices in A have at least $s\binom{n}{d-1} + \mathrm{ex}_d(n, H) + 1 + \binom{s}{t}rk\binom{n}{d-1}$ common neighbors among the d-subsets of vertices of G. Using A, we form a copy of $K_{H,t,s,r}$ in G. We start by letting $S = A$, so each t-subset T of vertices of S has at least $s\binom{n}{d-1} + \mathrm{ex}_d(n, H) + 1 + \binom{s}{t}rk\binom{n}{d-1}$ common neighbors among the d-subsets of vertices of G. Going through the t-subsets of S in any order, we form the copy of $K_{H,t,s,r}$. Let T be the current t-subset in the order as we form the copy of $K_{H,t,s,r}$.

At most $s\binom{n}{d-1}$ of the $s\binom{n}{d-1} + \mathrm{ex}_d(n, H) + 1 + \binom{s}{t}rk\binom{n}{d-1}$ common neighbors of T have nonempty intersection with S, since any given vertex can be an element of at most $\binom{n}{d-1}$ d-subsets of the vertices of G. Moreover H has k vertices and each t-subset of S is combined with r copies of H in $K_{H,t,s,r}$, so at most $(\binom{s}{t} - 1)rk\binom{n}{d-1}$ of the $s\binom{n}{d-1} + \mathrm{ex}_d(n, H) + 1 + \binom{s}{t}rk\binom{n}{d-1}$ common neighbors of T have nonempty intersection with any of the common neighbors of earlier t-subsets that were used in the copy of $K_{H,t,s,r}$. Thus T has at least $\mathrm{ex}_d(n, H) + rk\binom{n}{d-1} + 1$ common neighbors that have no intersection with S or any of the common neighbors used to form an edge in the copy of $K_{H,t,s,r}$ with an earlier t-subset of S. These $\mathrm{ex}_d(n, H) + rk\binom{n}{d-1} + 1$ common neighbors of T that do not intersect S or any of the previously used common neighbors must contain at least r disjoint copies $H_{T_1}, \ldots, H_{T_r}$ of H, so we can form the edges $e \cup \{u\}$ for each $u \in T$ and e in the edge set of H_{T_i}. $\square$

Note that $K_{H,t,t,1} = K_{H,t}$, and that the letter method also works to show that $\mathrm{ex}_{d+1}(n, K_{H,t,t,r}) = O(n^{d+1-1/t})$ for any integers $t \geq 2$, $r > 0$, and d-uniform hypergraph H with at least two edges such that $\mathrm{ex}_d(n, H) = O(n^{d-1})$. It would be interesting to see if the letter method is useful for other Turán-type problems, and what else can be said about $\mathrm{ex}_{d+1}(n, K_{H,t})$ in general.

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References

[1] N. Alon, H. Kaplan, G. Nivasch, M. Sharir, and S. Smorodinsky, Weak ϵ-nets and interval chains, *J. ACM* **55** (2008), #A28.

[2] N. Alon, M. Krivelevich, and B. Sudakov, Turán numbers of bipartite graphs and related Ramsey-type questions, *Combin. Probab. Comput.* **12** (2003), 477–494.

[3] W. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966), 281–285.

[4] J. Cibulka and J. Kynčl, Tight bounds on the maximum size of a set of permutations with bounded VC-dimension, *J. Combin. Theory Ser. A* **119** (2012), 1461–1478.

[5] P. Erdős, A. Rényi, and V. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235.

[6] J. Fox, Stanley-Wilf limits are typically exponential, preprint, 2013. Available at https://arxiv.org/abs/1310.8378

[7] J. Fox and B. Sudakov, Dependent random choice, *Random Structures Algorithms* **38** (2011), 68–99.

[8] Z. Füredi, An upper bound on Zarankiewicz problem, *Combin. Probab. Comput.* **5** (1996), 29–33.

[9] Z. Füredi, New asymptotics for bipartite Turán numbers, *J. Combin. Theory Ser. A* **75** (1996), 141–144.

[10] Z. Füredi and M. Simonovits, The history of degenerate (bipartite) extremal graph problems, *Bolyai Soc. Math. Stud.* **25** (2013), 169–264.

[11] J. Geneson, A Relationship Between Generalized Davenport–Schinzel Sequences and Interval Chains, *Electron. J. Combin.* **22** (2015), #P3.19.

[12] J. Geneson, Extremal functions of forbidden double permutation matrices, *J. Combin. Theory Ser. A* **116** (2009), 1235–1244.

[13] J. Geneson, Forbidden formations in multidimensional 0-1 matrices, *European J. Combin.* **78** (2019), 147–154.

[14] J. Geneson and P. Tian, Extremal functions of forbidden multidimensional matrices, *Discrete Math.* **340** (2017), 2769–2781.

[15] M. Klazar and A. Marcus, Extensions of the linear bound in the Füredi–Hajnal conjecture, *Adv. Appl. Math.* **38** (2007), 258–266.

[16] T. Kővari, V. Sós, and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.* **3** (1954), 50–57.

[17] A. Marcus and G. Tardos, Excluded permutation matrices and the Stanley–Wilf conjecture, *J. Combin. Theory Ser. A* **107** (2004), 153–160.

[18] D. Mubayi and J. Verstraëte, A hypergraph extension of the bipartite Turán problem, *J. Combin. Theory Ser. A* **106** (2004), 237–253.

[19] G. Nivasch, Improved bounds and new techniques for Davenport–Schinzel sequences and their generalizations, *J. ACM* **57** (2010), #A17.