PARTICULAR PELL-FERMAT EQUATIONS REVISITED

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Abstract

In this note we mainly give iterative integer solutions (whenever one trivial solution exists) to the following Pell equation $x^2 - (w^2(2^{n-2}p)^2 + 2^n(2^{n-2}p))y^2 = c$ for some $w, p, c \in \mathbb{Z}^*$. The idea relies on elementary arguments for the equation $x^2 - (w^2p^2 + p)y^2 = c$ as a quadratic polynomial over the rationals and some 2×2 matrix manipulations.

1. Introduction and Preliminaries

The classical Pell-Fermat equation asks to find all integer solutions $(x, y) \in \mathbb{N}^2$ to $x^2 - dy^2 = 1$ for d a non-square integer. The theory has a rich history and one can read for example [1], [3], [4], [6] and [7]. The authors used mainly continued fractions and elaborated techniques; we will give a completely different method for the equation:

$$x^2 - (w^2(2^{n-2}p)^2 + 2^n(2^{n-2}p))y^2 = c \tag{1}$$

using elementary properties of second degree polynomials. As one can notice, any integer $d > 1$ can be written as $w_0^2p_0^2 + sp_0$ where all variables are in $\mathbb{Z}^*$, or $d = (\frac{w_0}{s})^2(sp_0)^2 + sp_0 = w^2p^2 + p$, $w \in \mathbb{Q}^*$, $p \in \mathbb{Z}^*$.

Before going further we illustrate the infinite loop here:

$$x^2 - (w^2p^2 + p)y^2 = c, \tag{2}$$

where all variables can be considered in $\mathbb{Q}^*$. We write $x = wpy + u$; then after replacement we obtain a second degree polynomial in y:

$$-py^2 + 2wpyu + u^2 - c = 0.$$

Calculating its discriminant (which must be a perfect square) we obtain the following

$$\Delta = 4w^2p^2u^2 + 4pu^2 - 4pc = t^2.$$

With $t = 2pl$ and simplifying we arrive at the equation: $u^2(w^2p+1) - c - pl^2 = 0$. Set $l = wu + v$; then we get the polynomial in u: $u^2 - 2pwuv - c - pv^2 = 0$. Lastly $\Delta = 4p^2w^2v^2 + 4c + 4pv^2 = 4k^2$, *i.e.*, we have the similar equation:

$$k^2 - (w^2p^2 + p)v^2 = c.$$

2. Main Result

Keeping with the same previous notations we have the following result.

Proposition 1. *Let (x_0, y_0) be such that $x_0^2 - (w^2p^2 + p)y_0^2 = c$. Then*

$$(x, y) = \begin{cases} x = (2w^2p+1)x_0 + 2w(w^2p^2+p)y_0 \\ y = 2wx_0 + (2w^2p+1)y_0 \end{cases}$$

also verifies Equation (2).

Proof. Set $(k, v) = (x_0, y_0)$, after replacement $u_0 = wpy_0 + x_0$ and $l_0 = wu_0 + y_0$, we get $y = 2(w^2py_0 + wx_0) + y_0$ (or $y = -y_0$) and $x = 2w^3p^2y_0 + 2w^2px_0 + 2wpy_0 + x_0$ (or $x = x_0$). $\square$

If we set $A := \begin{pmatrix} 2w^2p+1 & 2w(w^2p^2+p) \\ 2w & 2w^2p+1 \end{pmatrix}$, $X_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = AX_n$ is an iterative system of solutions starting from (x_0, y_0). It is readily seen that with $n \geq 1$, $X_{n+1} = \begin{pmatrix} 2w^2p+1 & 2w(w^2p^2+p) \\ 2w & 2w^2p+1 \end{pmatrix} \begin{pmatrix} x_n \\ -y_n \end{pmatrix}$

$$X_{n+1} = \begin{pmatrix} 2w^2p+1 & -2w(w^2p^2+p) \\ 2w & -2w^2p-1 \end{pmatrix} \begin{pmatrix} 2w^2p+1 & 2w(w^2p^2+p) \\ 2w & 2w^2p+1 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ -y_{n-1} \end{pmatrix}.$$

Remark 1. It is well known -see for example [2] and [5]- that for $c = 1$ and $w, p \in \mathbb{N}^*$, Equation (2) has the smallest positive integer solution (fundamental) $(x_0, y_0) = (2w^2p+1, 2w)$. All other positive integer solutions are generated by $X_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = AX_n$, $n \geq 0$.

Remark 2. For $|2w^2p+1| > 1$ one can choose the signs of $\begin{pmatrix} \pm x_n \\ \pm y_n \end{pmatrix}$ so that $|x_{n+s+1}| > |x_{n+s}|$ and $|y_{n+s+1}| > |y_{n+s}|$ for every non-negative s. With integer d as $d = w^2p^2 + p$ or $d = (\frac{w_0}{2})^2(2p)^2 + 2p$ and $w, w_0 \in \mathbb{N}^*$, $p \in \mathbb{Z}^*$; the matrix $A \in \mathbb{M}_2(\mathbb{Z})$.

We have by direct computation:

$$A^3 = \begin{pmatrix} 32w^6p^3 + 48w^4p^2 + 18w^2p + 1 & 32w^7p^4 + 64w^5p^3 + 38w^3p^2 + 6wp \\ 32w^5p^2 + 32w^3p + 6w & 32w^6p^3 + 48w^4p^2 + 18w^2p + 1 \end{pmatrix}.$$

Theorem 1. *For $n \geq 2$, let $d = ((\frac{w_0}{2^n})^2(2^{2n-2}p_0)^2 + 2^{2n-2}p_0) = w^2p^2 + p$. Set without loss of generality $w = \frac{w_0}{2^i}$, $p = 2^{2n-2+j}p_0$ with w_0 and p_0 odd and $i \leq n$, $j \geq 0$. If $2(n-i)+j-1 \geq 0$, then $A^{2^{(i-1)}} \in \mathbb{M}_2(\mathbb{Z})$; otherwise $n = i$, $j = 0$ and $A^{3\cdot 2^{(n-2)}} \in \mathbb{M}_2(\mathbb{Z})$.*

Proof. Setting $w = \frac{w_0}{2^i}$ and $p = 2^{2n-2+j}p_0$:

$$A = \begin{pmatrix} 2^{2(n-i)+j-1}w_0^2p_0 + 1 & 2^{i-1}(2^{4(n-i)+2j-2}w_0^3p_0^2 + 2^{2(n-i)+j}w_0p_0) \\ \dfrac{w_0}{2^{i-1}} & 2^{2(n-i)+j-1}w_0^2p_0 + 1 \end{pmatrix}$$

and for $n = i$, $j = 0$

$$A^3 = \begin{pmatrix} \dfrac{w_0^6p_0^3}{2} + 3w_0^4p_0^2 + \dfrac{3^2w_0^2p_0}{2} + 1 & 2^{n-2}(4w_0^5p_0^3 + 6w_0p_0 + \dfrac{w_0^7p_0^4}{2} + \dfrac{19w_0^3p_0^2}{2}) \\ \dfrac{w_0^5p_0^2 + 4w_0^3p_0 + 3w_0}{2^{n-1}} & \dfrac{w_0^6p_0^3}{2} + 3w_0^4p_0^2 + \dfrac{3^2w_0^2p_0}{2} + 1 \end{pmatrix}.$$

It is easy to see that if $(a, b, f) \in \mathbb{Z}^3$ and $M = \begin{pmatrix} a & 2^kb \\ \frac{f}{2^l} & a \end{pmatrix}$ $(k \geq l)$, then we have $M^{2^l} \in \mathbb{M}_2(\mathbb{Z})$. □

Example 1. For $x^2 - 2021y^2 = c^2$, $d = 45^2 - 4 = (\frac{45}{4})^2 4^2 - 4$ so that $B = A^3 = \begin{pmatrix} -4139590049 & 186097479480 \\ 92081880 & -4139590049 \end{pmatrix}$. An infinite family of integer solutions is given by $X_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = BX_n$ with $X_0 = \begin{pmatrix} c \\ 0 \end{pmatrix}$.

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