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NEW LOWER BOUNDS FOR WEAK SCHUR PARTITIONS

Fred Rowley
formerly of *Lincoln College, Oxford, United Kingdom*
fred.rowley@ozemail.com.au

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Abstract

This paper records some apparently new results for the partitioning of integer intervals $[1, n]$ into weakly sum-free subsets. These were produced using a method closely related to that used by Schur in 1917. Weak Schur partitions of unlimited size can be produced in this way, and hence we can deduce new lower bounds for the corresponding weak Schur numbers $WS(r)$. The asymptotic growth rate of the lower bounds, as the number of subsets increases, cannot be less than the same growth rate for strongly sum-free partitions, and so exceeds 3.27. Specific results arising from partitions into a 'small' number of subsets include $WS(6) \geq 642$, $WS(7) \geq 2146$, $WS(8) \geq 6976$, $WS(9) \geq 21848$, and $WS(10) \geq 70778$.

– I dedicate this paper to the memory of my very good friend and colleague, the late Paul A. Stanway, former Exhibitioner of St. John's College, Cambridge.

1. Introduction

If an integer interval $S = [1, n]$ can be partitioned into r disjoint non-empty subsets S_i for $i = 1, 2, \ldots, r$, where no subset contains three (distinct) integers a, b, c, such that $a + b = c$, then each such subset is *(weakly) sum-free* and that partition is a *(weak) Schur (r-)partition.* The order of the set S is clearly n, and is also referred to as the *order of the partition.* An (r-)partition p of $[1, n]$ into r subsets may be denoted by $p(r; n)$.

For any r, $S(r)$ is the maximum value of n such that a Schur partition $p(r; n)$ exists, and $WS(r)$ is the maximum value of n such that a weak Schur partition $q(r; n)$ exists. $S(r)$ is known as the *Schur number*, and $WS(r)$ is known as the *weak Schur number*: and the existence of both is established by Ramsey's Theorem.

This paper concerns itself with the construction of weak Schur partitions. In that context it is natural to ask how the most successful methods used to color linear triangle-free graphs (or equivalently, to construct strong Schur partitions) might be modified so as to permit the construction of larger weak Schur partitions.

The author's previous experience indicates that the most successful constructions for 'small' triangle-free graphs can be characterized as special cases having particular unique attributes. Smaller graphs have typically been derived by a range of exhaustive or partial search strategies: and have then been combined or extended by methods involving various forms of 'compounding'. Compound graphs may be derived using periodically repetitive structures (translations) and/or reflections. These techniques usually succeed by vastly reducing the size of the difference sets derived from the subsets comprised in the coloring.

Any strong Schur partition is also a weak Schur partition, so both the size of any maximal weak partitions into r subsets, and their ultimate growth rate as r increases, cannot be less than in the strong case. Previous papers, including [3], have demonstrated that the ultimate growth rate for strong Schur partitions, as the number of colors r increases, exceeds 3.27. This author has seen evidence that some constructions become much more difficult when the ratio approaches $(3+\sqrt{13})/2$, which is a little over 3.3.

One immediate observation, when attempting to construct weak Schur partitions, is that translations or reflections are much less useful. A pair of positive integers $(x, 2x)$ may be referred to as a *weak pair*. If such a weak pair exists in a single subset S_i in a weak Schur partition, it is clearly not possible in general to feature either of the pairs $(x+a, 2x+a)$ or $(a-x, a-2x)$ in the same subset, since in each case the difference is x, and $x \in S_i$.

Some recent work of this topic has succeeded in increasing the known lower bounds by sidestepping these constraints using various algorithms and search constraints - see, for instance, [2]. So far, however, it has apparently not proved possible to demonstrate in this way an infinite sequence of weak Schur partitions with a growth rate above 3, which does not consist simply of strong partitions.

This paper now provides two such sequences. Although one might rightly say that the partitions in these sequences are 'almost' strong partitions, that may simply indicate that there is room for more imaginative constructions.

In Section 2, it is proved that, starting from a single specific partition, a series of partitions can be constructed, giving improved values for $WS(r)$ applicable for all $r \geq 6$. Numerical lower bounds are shown for $1 \leq r \leq 10$.

In Section 3, some brief conclusions are drawn.

2. Construction of Weak Schur Partitions

Theorem 1. (Construction Theorem). *If there is a strong Schur partition of the integers* $[1, m]$ *into* r *subsets, then there is a weak Schur partition of* $[1, 4m+2]$ *into* $r+1$ *subsets; and a weak Schur partition of* $[1, 13m+8]$ *into* $r+2$ *subsets.*

The theorem depends on two very simple constructions, which are closely related to that used by Schur in [4].

Proof. As stated above, the repetition or reflection of weak pairs $(x, 2x)$ within a prototype partition, into a compound partition, is not useful in the general case. The first construction minimizes this problem by relying on a sequence of partitions, each of which has only the single weak pair $(1, 2)$ in one of its subsets. No other weak pairs are involved.

The first construction takes as its starting point the following weak Schur 2-partition of order 6:

$S_1 = \{1, 2, 6\}$, $S_2 = [3, 5]$.

We assume the existence of a strong Schur partition $q(r; m)$ into subsets Q_i for $i = 1, 2, \ldots, r$.

Next we define $U_{r+1} = \{1, 2\} \cup \{4i + 2 \mid i \in [1, m]\}$. It is simple to verify that, excluding the weak pair $(1, 2)$, the difference between any two members of U_{r+1} is equivalent to either 0 or 1 mod 4, and greater than 3, and so cannot be a member of the same subset.

Then we construct m 'translates' of the set S_2, such that

$T_i = [4i - 1, 4i + 1]$, for $i \in [1, m]$.

It can be seen that within any one of these subsets T_i, the absolute differences are either 1 or 2 and so are members of U_{r+1}. Therefore each subset T_i is sum-free.

We then form the remaining subsets of the new partition by taking the unions of all the subsets T_i whose indices are in the same subset Q_k in the strong r-partition, such that

$U_k = \bigcup_{i \in Q_k} T_i$ for $k = 1, 2, \ldots, r$.

Let us assume that two distinct subsets T_i, T_j (with $i < j$) are included in the same subset of the new weak Schur partition. If so, any difference between a member of T_j and a member of T_i must be in the interval $[4(j - i) - 2, 4(j - i) + 2]$. Any number in this range is always either (a) a member of the subset T_{j-i}; or (b) a member of U_{r+1}.

Case (a) is the only case we need concern ourselves with. In that case, the partition $q(r; m)$ would not be sum-free if the subset containing $(j - i)$ were the same as that containing both i and j. Therefore, every subset in the new partition which was formed by taking the union of the T_i is also strongly sum-free. Clearly, there are r such subsets in the new partition. Therefore, including U_{r+1}, we have $r + 1$ subsets in the new partition.

Lastly we observe that U_{r+1} is weakly sum-free, that the order of the new partition is $4m + 2$, and that it is a complete partition of $[1, 4m + 2]$.

The second construction used in the theorem partitions the set $[1, 13m + 8]$. It follows very similar lines to the above, but starts from the following weak Schur 3-partition of $[1, 21]$:

$S_1 = \{1, 2, 4, 8, 21\}$, $S_2 = \{3, 5, 6, 7, 18, 19, 20\}$, $S_3 = [9, 17]$.

We begin by defining

$U_{r+1} = \{1, 2, 4, 8\} \cup \{13i + 8 \mid i \in [1, m]\}$ and

$U_{r+2} = \{3, 5, 6, 7\} \cup \{13i + j \mid i \in [1, m], j \in \{5, 6, 7\}\}$.

It is again necessary to verify that, excluding the weak pairs $(1, 2), (2, 4)$ and $(4, 8)$, the difference between any two members of U_{r+1} is equivalent to either 0, 3, 6, or 7 mod 13, and so cannot be a member of the same subset. Excluding the weak pair $(3, 6)$, the difference between any two members of U_{r+2} is equivalent to either 0, 1, 2, 4, 11 or 12 mod 13, and so again cannot be a member of the same subset. The sets T_i are derived in this case by translation of S_3, such that

$T_i = [13i - 4, 13i + 4]$, for $i \in [1, m]$.

In this case, if two distinct subsets T_i, T_j (with $i < j$) are included in the same subset of the new weak Schur partition, then any difference between a member of T_j and a member of T_i must be in the interval $[13(j - i) - 8, 13(j - i) + 8]$. Any number in this range is always either (a) a member of the subset T_{j-i}; or (b) a member of $U_{r+1} \cup U_{r+2}$.

Once again, $U_k = \bigcup_{i \in Q_k} T_i$ for $k = 1, 2, \ldots, r$.

The proof follows immediately as before, noting that in this second case, each partition contains exactly four weak pairs. □

It is now simple to deduce that if there is an infinite sequence of strong partitions into r colors, whose orders increase with an ultimate growth rate of (say) γ as r increases, then there is a corresponding sequence of weak partitions with the same ultimate growth rate.

The orders now available for some 'small' partitions are shown in Table 1 below. The history and derivation of the smaller weak Schur partitions is well covered in [1], and orders of weak partitions shown for $1 \leq r \leq 5$ are from that source. All are believed to be the largest currently available, and the first four have been shown to be maximal.

The orders of weak partitions for $6 \leq r \leq 10$ are produced by the construction above and believed to exceed the highest values previously published. The orders of the strong partitions, on which the constructions are based, derive from [3].

r =	1	2	3	4	5	6	7	8	9	10
Weak Schur Partition	2	8	23	66	196	642	2146	6976	21848	70778
Strong Schur Partition	1	4	13	44	160	536	1680	5286	17694	60320
Ratio of Orders	2.00	2.00	1.77	1.50	1.23	1.20	1.28	1.32	1.23	1.17

Table 1 - Orders of largest available weak and strong Schur partitions

3. Conclusions

The construction described here is simple and effective, although it has some limitations.

All the partitions demonstrated above have very few weak pairs and so may be said to be only 'trivially weak', with order broadly equal to a fixed multiple of a known strong partition. As a result, we have not shown that the limiting growth rate in the weak case exceeds that in the strong case. Although there is room for a lot more work, this author believes that the limiting growth rates may well be finite and equal, and further are quite likely to be bounded by a number well below 4.

Nor does this paper provide a sequence in which every partition certainly exceeds the maximum possible strong partition: although it might do so if the ultimate growth rate in the strong case can later be shown to be less than 4. For the moment, though, many of the 'small' partitions represent significant improvements over previously demonstrated lower bounds on $WS(r)$.

Despite its limitations, the construction demonstrated in this paper sets a new baseline for constructing infinite sequences of weak Schur partitions in a way that consistently exceeds what is possible in the strong case.

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