EXPLICIT EXPRESSIONS FOR A CERTAIN SUBSET OF APPELL POLYNOMIALS: A PROBABILISTIC PERSPECTIVE

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Abstract

We consider the subset $\mathcal{E}$ of Appell polynomials whose generating function is given in terms of a real power of the moment generating function of a certain random variable Y. This subset contains different kinds of generalizations of the Bernoulli, Apostol-Euler, Cauchy-type, Hermite, and Miller-Lee polynomials, among others. We give a unified approach to obtain explicit expressions for those Appell sequences in $\mathcal{E}$. The main tool is a suitable probabilistic generalization of the Stirling numbers of the second kind.

1. Introduction and Main Result

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Unless otherwise specified, we assume from now on that $n \in \mathbb{N}_0$, $x \in \mathbb{R}$, and $z \in \mathbb{C}$ with $|z| \leq r$, where $r > 0$ may change from line to line. Denote by $\mathcal{G}$ the set of real sequences (u_n) such that $u_0 \neq 0$ and

$$\sum_{n=0}^{\infty} |u_n| \frac{r^n}{n!} < \infty,$$

for some radius $r > 0$. Let $(A_n(x))$ be a sequence of polynomials such that $(A_n(0)) \in \mathcal{G}$. Recall that $(A_n(x))$ is called an Appell sequence if one of the following equivalent conditions is satisfied

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} A_k(0) x^{n-k}, \qquad (1)$$

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or, in terms of its generating function,

$$\sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!} = e^{xz}\sum_{n=0}^{\infty} A_n(0)\frac{z^n}{n!}. \tag{2}$$

An interesting problem, widely considered in the literature, is to find explicit expressions for the different examples of Appell polynomials (see the references throughout Section 2). Usually, this problem has been faced in each particular case, taking into account the form of the Appell sequence under consideration. In this paper, we give a unified approach to obtain explicit expressions within a certain subset $\mathcal{E}$ of Appell polynomials which contains different kinds of generalizations of the classical Bernoulli, Apostol-Euler, and Cauchy-type polynomials. Although not considered here, $\mathcal{E}$ also contains the Hermite and the Miller-Lee polynomials (cf. Dattoli et al. [6] and [3]), among others.

To be more precise, let Y be a random variable such that

$$\mathbb{E}e^{r|Y|} < \infty, \tag{3}$$

for some $r > 0$, where $\mathbb{E}$ stands for mathematical expectation. We consider the subset $\mathcal{E}$ of Appell sequences $(A_n^{(\alpha)}(x))$ whose generating function is given by

$$\sum_{n=0}^{\infty} A_n^{(\alpha)}(x)\frac{z^n}{n!} = \frac{e^{xz}}{(\mathbb{E}e^{zY})^{\alpha}}, \tag{4}$$

for some real α. We point out that the particular case $\alpha = 1$ was first considered by Ta [19].

The main tool to obtain explicit expressions for Appell sequences in $\mathcal{E}$ is the following probabilistic generalization of the Stirling numbers of the second kind recently introduced in [4]. Given a random variable Y satisfying (3), we define the numbers $S_Y(n,m)$, $m = 0, 1, \ldots, n$, by means of their generating function

$$\frac{(\mathbb{E}e^{zY} - 1)^m}{m!} = \sum_{n=m}^{\infty} \frac{S_Y(n,m)}{n!} z^n, \quad m \in \mathbb{N}_0, \tag{5}$$

or, equivalently, as (cf. [4, Theorem 3.3])

$$S_Y(n,m) = \frac{1}{m!}\sum_{k=0}^{m} \binom{m}{k}(-1)^{m-k}\mathbb{E}S_k^n, \quad m = 0, 1, \ldots, n, \tag{6}$$

where

$$S_k = Y_1 + \cdots + Y_k, \quad k \in \mathbb{N} \qquad (S_0 = 0), \tag{7}$$

and $(Y_j)_{j\geq 1}$ is a sequence of independent copies of Y. As follows from (5) or (6), we have for $Y = 1$

$$S_1(n,m) = S(n,m),$$

$S(n,m)$ being the classical Stirling numbers of the second kind. Other generalizations of these numbers can be found in Mező [16], El-Desouky et al. [7] and the references therein.

With the preceding ingredients, we state the main result of this paper.

Theorem 1. *Let $\alpha \in \mathbb{R}$. For any Appell sequence $(A_n^{(\alpha)}(x)) \in \mathcal{E}$, we have*

$$A_n^{(\alpha)}(0) = \sum_{m=0}^{n} \binom{-\alpha}{m} m! S_Y(n,m) = \sum_{k=0}^{n} \binom{-\alpha}{k} \binom{n+\alpha}{n-k} \mathbb{E} S_k^n. \tag{8}$$

Proof. By dominated convergence, we can assume without loss of generality that $|\mathbb{E}e^{zY} - 1| < 1$, $|z| < r$. Applying the binomial expansion in (4), we have from (5)

$$\sum_{n=0}^{\infty} A_n^{(\alpha)}(0) \frac{z^n}{n!} = (\mathbb{E}e^{zY})^{-\alpha} = \left(1 + \mathbb{E}(e^{zY} - 1)\right)^{-\alpha} = \sum_{m=0}^{\infty} \binom{-\alpha}{m} \left(\mathbb{E}e^{zY} - 1\right)^m$$
$$= \sum_{m=0}^{\infty} \binom{-\alpha}{m} m! \sum_{n=m}^{\infty} \frac{S_Y(n,m)}{n!} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{n} \binom{-\alpha}{m} m! S_Y(n,m),$$

which shows the first equality in (8). On the other hand, we have from (6)

$$\sum_{m=0}^{n} \binom{-\alpha}{m} m! S_Y(n,m) = \sum_{m=0}^{n} \binom{-\alpha}{m} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \mathbb{E} S_k^n$$
$$= \sum_{k=0}^{n} \mathbb{E} S_k^n \sum_{m=k}^{n} \binom{-\alpha}{m} \binom{m}{k} (-1)^{m-k} = \sum_{k=0}^{n} \binom{-\alpha}{k} \binom{n+\alpha}{n-k} \mathbb{E} S_k^n,$$

where the last equality comes from the elementary combinatorial identity

$$\sum_{i=0}^{p} \binom{s+i}{i} = \binom{s+1+p}{p}, \quad p \in \mathbb{N}_0, \quad s \in \mathbb{R}.$$

This shows the second equality in (8) and completes the proof. □

In view of (1), Theorem 1 gives us two explicit expressions for any Appell sequence $(A_n^{(\alpha)}(x)) \in \mathcal{E}$. The first one in terms of the numbers $S_Y(n,m)$. In this respect, closed form expressions for such numbers for various choices of the random variable Y can be found in [4]. The second one in terms of the moments of the random variable Y. In fact, denote by

$$\mu_j = \mathbb{E}Y^j, \quad j \in \mathbb{N}_0, \tag{9}$$

the jth moment of Y. Observe that we have from (7)

$$\mathbb{E} S_k^n = \sum_{j_1+\cdots+j_k=n} \frac{n!}{j_1! \cdots j_k!} \mu_{j_1} \cdots \mu_{j_k}, \qquad k \in \mathbb{N}_0. \tag{10}$$

For $\alpha = 1$, Theorem 1 takes on the form

$$A_n^{(1)}(0) = \sum_{m=0}^{n} (-1)^m m! S_Y(n,m) = \sum_{k=0}^{n} \binom{n+1}{k+1} (-1)^k \mathbb{E} S_k^n,$$

the first three terms being

$$A_0^{(1)}(0) = 1, \quad A_1^{(1)}(0) = -\mu_1, \quad A_2^{(1)}(0) = 2\mu_1^2 - \mu_2.$$

2. Examples

In this section, we show that various kinds of generalizations of classical Appell sequences belong to $\mathcal{E}$, and obtain explicit expressions for them by applying Theorem 1.

Example 2.1 ***Generalizations of Bernoulli polynomials***. We consider the polynomials $B_n^{(\alpha)}(x;m)$ defined by means of the generating function

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;m) \frac{z^n}{n!} = \left(\frac{z^m/m!}{e^z - \sum_{k=0}^{m-1} z^k/k!} \right)^{\alpha} e^{xz}, \qquad \alpha \in \mathbb{R}, \quad m \in \mathbb{N}. \tag{11}$$

For $\alpha = 1$, such polynomials have been considered in Bretti et al. [5] and Khan and Riyasat [10], among others. Important special cases are the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α and the classical Bernoulli polynomials $B_n(x)$ respectively given by

$$B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x;1), \qquad B_n(x) = B_n^{(1)}(x). \tag{12}$$

Let β_m be a random variable having the beta density

$$\rho_m(\theta) = m(1-\theta)^{m-1}, \quad 0 \le \theta \le 1. \tag{13}$$

Corollary 1. *Let $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$. Then, $(B_n^{(\alpha)}(x;m) \in \mathcal{E}$ with*

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;m) \frac{z^n}{n!} = \left(\mathbb{E} e^{z\beta_m} \right)^{-\alpha} e^{xz}. \tag{14}$$

Moreover

$$B_n^{(\alpha)}(0;m) = n! \sum_{k=0}^{n} \binom{-\alpha}{k} \binom{n+\alpha}{n-k} (m!)^k \sum_{j_1+\cdots+j_k=n} \frac{1}{(m+j_1)! \cdots (m+j_k)!}. \tag{15}$$

In particular,

$$B_n^{(\alpha)}(0) = \sum_{k=0}^{n} \binom{-\alpha}{k} \binom{n+\alpha}{n-k} \frac{S(n+k,k)}{\binom{n+k}{n}}. \tag{16}$$

Proof. Using the finite Taylor expansion with remainder in integral form (see also [1, Lemma 7]), we have

$$e^z - \sum_{k=0}^{m-1} \frac{z^k}{k!} = \frac{z^m}{m!} \mathbb{E} e^{z\beta_m}.$$

This, together with (11), shows (14). By (9) and (13), the jth moment of the random variable $Y = \beta_m$ is given by

$$\mu_j = \mathbb{E}\left(\beta_m\right)^j = m \int_0^1 \theta^j (1-\theta)^{m-1}\, d\theta = m\beta(j+1,m) = \frac{1}{\binom{m+j}{m}}, \tag{17}$$

for any $j \in \mathbb{N}_0$. As follows from (10) and (17), we have in the case at hand

$$\mathbb{E} S_k^n = n!(m!)^k \sum_{j_1+\cdots+j_k=n} \frac{1}{(m+j_1)! \cdots (m+j_k)!}.$$

Hence, (15) follows from the second equality in (8). Finally, assume that $m = 1$. As follows from (13), the random variable $Y = \beta_1$ has the uniform distribution on $[0, 1]$. Sun[18] (see also [8, formula (38)]) showed the following probabilistic representation for the Stirling numbers of the second kind

$$S(n,k) = \binom{n}{k} \mathbb{E} S_k^{n-k}, \quad k = 0, 1, \ldots, n, \tag{18}$$

where S_k is the sum defined in (7) when $Y = \beta_1$. Therefore, equality (16) follows from (18) and the second equality in Theorem 1. The proof is complete. □

Formula (16) was already obtained by Todorov [21, Eq. (3)] using different techniques (see also Srivastava and Todorov [17]). A different representation for the Bernoulli numbers in terms of the Stirling numbers of the second kind was provided by Guo and Qi [9, Theorem 3.1]. For $\alpha \in \mathbb{N}$, a similar formula to that in (16) was obtained by Kim et al. [11] using umbral calculus techniques. For generalized Apostol-Bernoulli polynomials of integer order, we refer the reader to Luo and Srivastava [14].

Example 2.2 ***Generalizations of Apostol-Euler polynomials***. Let $\alpha \in \mathbb{R}$ and $0 \le \beta \le 1$. We consider the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\beta)$ of real order α defined via their generating function as

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\beta) \frac{z^n}{n!} = \frac{e^{xz}}{(1+\beta(e^z-1))^\alpha}. \tag{19}$$

The Apostol-Euler polynomials and the classical Euler polynomials are defined, respectively, as

$$E_n(x;\beta) = E_n^{(1)}(x;\beta), \qquad E_n(x) = E_n(x;1/2).$$

Up to a constant factor, the polynomials $E_n^{(\alpha)}(x;\beta)$ were introduced by Luo [13] (see also Wang et al. [23]).

Let $X(\beta)$ be a random variable having the Bernoulli law

$$P(X(\beta) = 1) = 1 - P(X(\beta) = 0) = \beta. \tag{20}$$

Corollary 2. *Let $\alpha \in \mathbb{R}$ and $0 \le \beta \le 1$. Then, $(E_n^{(\alpha)}(x;\beta) \in \mathcal{E}$ with*

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\beta)\frac{z^n}{n!} = \left(\mathbb{E}e^{zX(\beta)}\right)^{-\alpha} e^{xz}. \tag{21}$$

In addition,

$$E_n^{(\alpha)}(0;\beta) = \sum_{m=0}^{n} \binom{-\alpha}{m}\beta^m m! S(n,m). \tag{22}$$

Proof. Formula (21) is an immediate consequence of (19) and (20). On the other hand, since

$$\mathbb{E}e^{zX(\beta)} - 1 = \beta(e^z - 1),$$

formula (5) immediately yields

$$S_{X(\beta)}(n,m) = \beta^m S(n,m), \quad n,m \in \mathbb{N}_0, \quad m \le n.$$

Thus, (22) follows from the first equality in (8). □

When α is a positive integer, formula (22) was also shown by Kim et al. [11] using umbral calculus. A similar, but more involved formula than that in (22), was obtained by Luo [13].

Example 2.3 ***Cauchy-type polynomials***. For any $\alpha \in \mathbb{R}$, we consider the Cauchy-type polynomials $C_n^{(\alpha)}(x)$ defined via their generating function as

$$\sum_{n=0}^{\infty} C_n^{(\alpha)}(x)\frac{z^n}{n!} = \left(\frac{z}{\log(1+z)}\right)^{\alpha} e^{xz}. \tag{23}$$

For related polynomials and numbers, we refer the reader to Merlini et al. [15], Komatsu and Ramírez [12], and the references therein.

Let U and T be two independent random variables such that U is uniformly distributed on $[0,1]$ and T has the exponential density $\rho(\theta) = e^{-\theta}$, $\theta \ge 0$. On the other hand, denote by $s(n,k)$, $k = 0,1,\ldots,n$, the classical Stirling numbers of the first kind.

Corollary 3. *Let* $\alpha \in \mathbb{R}$. *Then,* $(C_n^{(\alpha)}(x)) \in \mathcal{E}$ *with*

$$\sum_{n=0}^{\infty} C_n^{(\alpha)}(x) \frac{z^n}{n!} = \left(\mathbb{E} e^{-zUT}\right)^{-\alpha} e^{xz}. \tag{24}$$

Moreover,

$$C_n^{(\alpha)}(0) = \sum_{k=0}^{n} \binom{-\alpha}{k} \binom{n+\alpha}{n-k} \frac{s(n+k,k)}{\binom{n+k}{n}}. \tag{25}$$

Proof. In [2], we have shown that

$$\mathbb{E} e^{-zUT} = \frac{\log(1+z)}{z},$$

which, in conjunction with (23), shows (24). Again in [2], the following probabilistic representation of $s(n,k)$ was established

$$s(n,k) = \binom{n}{k} \mathbb{E} S_k^{n-k}, \quad k = 0, 1, \ldots, n, \tag{26}$$

where S_k is the sum in (7) when $Y = -UT$. Thus, equality (25) follows from (26) and Theorem 1. This concludes the proof. □

It is worthy to note that identities (16) and (25) only differ in the kind of the Stirling numbers considered. In fact, these two identities are special cases of those in Wang [22, Theorem 6.3 and Example 7.1].

Example 2.4 ***Further Appell polynomials***. For any $\alpha \in \mathbb{R}$, denote by $D_n^{(\alpha)}(x)$ the polynomials defined as

$$\sum_{n=0}^{\infty} D_n^{(\alpha)}(x) \frac{z^n}{n!} = \left(\frac{z}{\sinh(z)}\right)^{\alpha} e^{xz}. \tag{27}$$

For $\alpha = 1$, such polynomials were introduced by Tempesta [20]. On the other hand, let V be a random variable uniformly distributed on $[-1,1]$.

Corollary 4. *Let* $\alpha \in \mathbb{R}$. *Then,* $(D_n^{(\alpha)}(x)) \in \mathcal{E}$ *with*

$$\sum_{n=0}^{\infty} D_n^{(\alpha)}(x) \frac{z^n}{n!} = \left(\mathbb{E} e^{zV}\right)^{-\alpha} e^{xz}. \tag{28}$$

In addition, $D_{2n+1}^{(\alpha)}(0) = 0$ *and*

$$D_{2n}^{(\alpha)}(0) = \sum_{k=0}^{2n} \binom{-\alpha}{k} \binom{2n+\alpha}{2n-k} \sum_{j=0}^{2n} \binom{2n}{j} (-k)^{2n-j}\, 2^j\, \frac{S(k+j,k)}{\binom{k+j}{k}}. \tag{29}$$

Proof. Formula (28) readily follows from (27) and the fact that V is uniformly distributed on $[-1,1]$. Since the function $z/\sinh(z)$ is symmetric, we see from (27) that $D_{2n+1}^{(\alpha)}(0)=0$. Finally, let U be a random variable uniformly distributed on $[0,1]$, and let $(U_j)_{j\geq 1}$ and $(V_j)_{j\geq 1}$ be two sequences of independent copies of U and V, respectively. Since the random variables $2U_j-1$ and V_j, $j\in\mathbb{N}$, have the same law, we have

$$\begin{aligned}\mathbb{E}(V_1+\cdots+V_k)^{2n} &= \mathbb{E}\left(2(U_1+\cdots+U_k)-k\right)^{2n}\\ &=\sum_{j=0}^{2n}\binom{2n}{j}(-k)^{2n-j}\,2^j\,\mathbb{E}(U_1+\cdots+U_k)^j\\ &=\sum_{j=0}^{2n}\binom{2n}{j}(-k)^{2n-j}\,2^j\,\frac{S(k+j,k)}{\binom{k+j}{k}},\end{aligned}$$

where we have used (18) in the last equality. Hence, (29) follows from Theorem 1. □

We point out that formula (29) in Corollary 4 has a close relation with that in Wang [22, Table 1, Entry D1].

Let $\alpha\in\mathbb{R}$ and $0\leq\beta\leq 1$. Finally, we consider the polynomials $\mathcal{B}_n^{(\alpha)}(x;\beta)$ defined by

$$\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\beta)\frac{z^n}{n!}=\left(\frac{z}{\beta(e^z-1)+(1-\beta)z}\right)^{\alpha}e^{xz}. \tag{30}$$

These polynomials are new generalizations of the Bernoulli polynomials, since we have $\mathcal{B}_n^{(\alpha)}(x;1)=B_n^{(\alpha)}(x)$, as defined in (12). Assume that U is a random variable uniformly distributed on $[0,1]$ and independent of $X(\beta)$, as defined in (20).

Corollary 5. *Let $\alpha\in\mathbb{R}$ and $0\leq\beta\leq 1$. Then, $(\mathcal{B}_n^{(\alpha)}(x;\beta))\in\mathcal{E}$ with*

$$\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\beta)\frac{z^n}{n!}=\left(\mathbb{E}e^{zX(\beta)U}\right)^{-\alpha}e^{xz}. \tag{31}$$

Moreover,

$$\mathcal{B}_n^{(\alpha)}(0;\beta)=\sum_{m=0}^{n}\binom{-\alpha}{m}\beta^m\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{S(n+k,k)}{\binom{n+k}{n}}. \tag{32}$$

Proof. Using (20) and the independence between $X(\beta)$ and U, we see that

$$\mathbb{E}e^{zX(\beta)U}=\beta\mathbb{E}e^{zU}+1-\beta=\beta\left(\mathbb{E}e^{zU}-1\right)+1. \tag{33}$$

This readily implies (31). On the other hand, formulas (5) and (33) imply that

$$S_{X(\beta)U}(n,m)=\beta^m S_U(n,m). \tag{34}$$

Finally, we have from (6) and (18)

$$S_U(n,m) = \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{S(n+k)}{\binom{n+k}{n}}.$$

This, together with Theorem 1 and (34), shows (32) and concludes the proof. □

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