#A62 INTEGERS 21 (2021)

FINITE SUMS OF ARITHMETIC PROGRESSIONS

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Received: 11/8/19, Revised: 8/27/20, Accepted: 4/28/21, Published: 6/3/21

Abstract

We give a purely combinatorial proof of a special case of the Deuber-Hindman theorem which is a two-fold generalization of Schur's extension of van der Waerden's theorem and Hindman's theorem. We also give a tower bound for a finite version of it.

1. Introduction

In this paper, we will consider the pointwise finite sums of l-term arithmetic progressions for a fixed $l \geq 3$. In terms of this type of sum, we have the following pleasant two-fold generalization of Hindman's theorem ([4], Section 3.5) and Schur's extension of van der Waerden's theorem ([6], Section 2.4), which is in fact a special case of the Deuber-Hindman theorem [1], and also can be deduced from Furstenberg's theorem ([5], Proposition 8.2.1).

For any positive integers c and $l \geq 3$, if $\mathbb{N}$ is c-colored, then there exist a color γ, and infinitely many l-term arithmetic progressions Q_i, $i \in \mathbb{N}$, such that all of their finite sums (with no repetition) are monochromatic with the color γ, and all the common differences of the above finite sums have also the color γ.

In Theorems 5, and 6, of this paper, we give a purely combinatorial proof of the above statement, avoiding topological dynamics as well as the theory of ultrafilters. It is interesting to see whether the method of the proof can be generalized to give a combinatorial proof of the Deuber-Hindman theorem. We are also interested in a finite version of the above theorem. It is well known that through a compactness argument, we can have a finite version. For instance, we have the following special case of Rodo's theorem ([4], Section 3.3), which is a two-fold generalization of van

[1]The research of the author was in part supported by a grant from IPM (No. 99030403).

der Waerden's theorem and a finite version of Hindman's theorem.

For positive integers c, n, and $l \geq 3$, there is a positive integer m such that whenever $\{1, 2, \ldots, m\}$ is c-colored, there exist l-term arithmetic progressions $P_1, P_2, \ldots, P_n \subset \{1, 2, \ldots, m\}$ such that all finite sums of P_i (with no repetition) are monochromatic with the same color.

If we denote the least such m by $f(l, n, c)$, then the proof given through the compactness argument does not give us an upper bound for $f(l, n, c)$. But it is not hard to see that the proof given for Theorem 5, can be made finitary (which may be regarded as an advantage of the proof over its counterparts using dynamical system or ultrafilters) to give us a primitive recursive upper bound for $f(l, n, c)$. To do so, we use the finitary Hindman numbers $\mathrm{Hind}(n, c)$, which is a tower function [2]. However, due to its iterated use of the function $\mathrm{Hind}(n, c)$, it gives us an upper bound belonging to the class of WOW functions [4]. In Theorem 7, we do a better job by giving a different proof which uses the function $\mathrm{Hind}(n, c)$ just one time, and thus obtaining a tower bound for $f(l, n, c)$. Also note that according to the Gowers elementary bound for the van der Waerden theorem, we will not worry about the van der Waerden part of the proof.

As the referee has pointed out, Theorem 7, follows from Rodo's theorem ([4], Chapter 3) and therefore, we can extract a primitive recursive upper bound for $f(l, n, c)$ according to the standard proof given there [4]. This is in fact based on an iteration of Theorem 2, in Section 3.1 of [4], which is itself a combination of Schur's theorem and van der Waerden's theorem. It is easily seen that the proof of Theorem 2, gives a tower bound which consequently, it would imply a WOW upper bound for $f(l, n, c)$.

2. Preliminaries

Let $l \geq 3$ be a positive integer and let $P = \{a_1, \ldots, a_l\}$ be an l-term arithmetic progression with $a_1 < \cdots < a_l$. We denote the sth term of P by $P[s] = a_s$. Now letting P and Q be two l-term arithmetic progressions, we define their pointwise sum (or briefly their sum) $P \boxplus Q$, as the l-term arithmetic progression with $(P \boxplus Q)[s] = P[s] + Q[s]$ for $1 \leq s \leq l$. Hence for the l-term arithmetic progressions $P_1, P_2, \ldots, P_m$, their finite sum $P_1 \boxplus P_2 \boxplus \cdots \boxplus P_m$ has unambiguous meaning. We also denote the common difference of an arithmetic progression P, by $\operatorname{add} P$. We also use the following notation for finite sums of arithmetic progressions

$$\boxplus_{i \in B} P_i = P_1 \boxplus P_2 \boxplus \cdots \boxplus P_m,$$

where $B = \{1, 2, \dots, m\}$. Obviously we have

$$(\boxplus_{i\in B} P_i)[s] = \sum_{i\in B} P_i[s].$$

We define a partial ordering between l-term arithmetic progressions by putting $P \prec Q$, whenever $P[s] < Q[s]$ for all $1 \le s \le l$.

We also fix some notation. For n a positive integer, put $[n] = \{1, 2, \dots, n\}$. Letting S be an infinite set, we denote the collection of finite nonempty subsets of S by $\mathcal{P}_f(S)$. For a finite set A, $\mathcal{P}^+(A)$ denotes the collection of nonempty subsets of A. Also $FS(S)$ will denote the set of all finite sums of elements of S with no repetition. Letting $A, B \in \mathcal{P}_f(\mathbb{N})$, by $A < B$ we mean that $\max A < \min B$.

Now let us state van der Waerden's theorem and Schur's extension of van der Waerden's theorem.

Theorem 1 ([4])**.** For positive integers c and $l \ge 3$, there is a positive integer n such that whenever $[n]$ is c-colored, there is a monochromatic l-term arithmetic progression $P \subseteq [n]$. We denote the least such n by $W(l, c)$.

Theorem 2 ([6])**.** For positive integers c and $l \ge 3$, there is a positive integer n such that whenever $\mathbf{c}$ is a c-coloring of $[n]$, there are $d, a, a+d, \dots, a+(l-1)d$ in $\{1, 2, \dots, n\}$ such that

$$\mathbf{c}(d) = \mathbf{c}(a) = \mathbf{c}(a+d) = \cdots = \mathbf{c}(a+(l-1)d).$$

We denote the least such n by $SB(l, c)$ (See [6] for historical remarks).

We will use the following strong version of Hindman's theorem.

Theorem 3 ([7])**.** Let $a_1 < a_2 < \cdots < a_m < \dots$ be an infinite strictly increasing sequence of positive integers. Let c be a positive integer and $FS(\{a_1, a_2, \dots\})$ be c-colored. Then there are $B_1 < B_2 < B_3 < \dots$ in $\mathcal{P}_f(\mathbb{N})$ such that, if

$$b_1 = \sum_{i\in B_1} a_i, b_2 = \sum_{i\in B_2} a_i, \dots, b_m = \sum_{i\in B_m} a_i, \dots,$$

then $FS(\{b_1, b_2, \dots\})$ is monochromatic.

We say that the two positive integers a, b are *power-disjoint* if the powers occurring in the expansions of a, b in base 2, are disjoint sets. More precisely, if we write $a = 2^{k_1} + \cdots + 2^{k_m}$ and $b = 2^{l_1} + \cdots + 2^{l_n}$, then the two sets $\{k_1, \dots, k_m\}$ and $\{l_1, \dots, l_n\}$ are disjoint. We denote the set $\{k_1, \dots, k_m\}$ by $\mathrm{pow}_2(a)$. We will use the following finitary version of Hindman's theorem [2], which strengthens the Disjoint Unions Theorem. First we introduce a notation. If T is a collection of pairwise disjoint sets, then $NU(T)$ will denote the set of non-empty unions of elements of T.

Theorem 4 ([2]). For positive integers n, c, there is a positive integer m such that for any m-element set $A = \{a_1, \dots, a_m\}$ of pairwise power-disjoint positive integers, whenever $\mathbf{c}$ is a c-coloring of $FS(A)$, there exist $\gamma \in [c]$ and $B_1, \dots, B_n$ in $\mathcal{P}^+([m])$ such that $B_1 < \cdots < B_n$, and for all $C \in NU\{B_1, \dots, B_n\}$, we have

$$\mathbf{c}\big(\sum_{i \in C} a_i\big) = \gamma.$$

Moreover, if $Hind(n, c)$ denotes the least such m, then $Hind(n, c)$ is a tower function.

3. Purely Combinatorial Proofs

In the following theorem, we give a purely combinatorial proof of the two-fold generalization of van der Waerden's theorem and Hindman's theorem, mentioned in the introduction. We will start with a suitable sequence of n-term arithmetic progressions $P_1^0, P_2^0, P_3^0, \dots$ such that n is large enough to use the van der Waerden theorem later. Then by using a strong version of Hindman's theorem, we construct a sequence of n-term arithmetic progressions $P_1^1, P_2^1, P_3^1, \dots$ such that each P_i^1 is a finite sum of P_i^0's, and the positive integers $P_1^1[1], P_2^1[1], P_3^1[1], \dots$ have the same color. We iterate this process n times until we get a sequence of n-term arithmetic progressions $P_1^n, P_2^n, P_3^n, \dots$ such that each P_i^n is a finite sum of P_i^{n-1}, and also for each $s \in \{1, 2, \dots, n\}$, the positive integers $P_1^n[s], P_2^n[s], P_3^n[s], \dots$ have the same color. Now by uniformity of the construction, it will suffice to choose one arithmetic progression from $\{P_i^n; i \in \mathbb{N}\}$, say P_1^n, and apply van der Waerden's theorem inside P_1^n to obtain a monochromatic l-term arithmetic progression $Q_1 \subset P_1^n$. In fact all other Q_i will occupy the same places in P_i^n as Q_1 does in P_1^n.

Theorem 5. Let c and $l \geq 3$ be positive integers. Let $\mathbf{c}$ be a c-coloring of $\mathbb{N}$, then there are l-term arithmetic progressions $Q_1, Q_2, Q_3, \dots$ such that

(i) $Q_1 \prec Q_2 \prec Q_3 \prec \cdots$,

(ii) there is $\gamma \in [c]$ such that for all $C \in \mathcal{P}_f(\mathbb{N})$, and all $s \in \{1, \dots, l\}$, we have

$$\mathbf{c}\big((\boxplus_{i \in C} Q_i)[s]\big) = \gamma.$$

Proof. Let $n = W(l, c)$, and let $a_1 < a_2 < \cdots < a_m < \cdots$ be a strictly increasing sequence of positive integers with $a_{m+1} > a_1 + \cdots + a_m + mn$. For $i \in \mathbb{N}$ we put

$$P_i^0 = \{a_i, a_i + 1, \dots, a_i + (n-1)\}.$$

Obviously P_i^0 is an n-term arithmetic progression and we have

$$P_1^0 \prec P_2^0 \prec P_3^0 \prec \cdots .$$

In fact it is easily seen that for any $C_1 < C_2$ in $\mathcal{P}_f(\mathbb{N})$, we have

$$\boxplus_{i\in C_1} P_i^0 \prec \boxplus_{i\in C_2} P_i^0. \tag{1}$$

Now for $1 \leq k \leq n$, we inductively define the n-term arithmetic progressions $P_1^k, P_2^k, P_3^k, \dots$ so that there are $\alpha_1^k, \alpha_2^k, \dots, \alpha_k^k \in [c]$, such that the following two conditions are satisfied

(a) for all $C \in \mathcal{P}_f(\mathbb{N})$ and all $s \in \{1, \dots, k\}$, we have

$$\mathbf{c}\big((\boxplus_{i\in C} P_i^k)[s]\big) = \alpha_s^k,$$

(b) for all $C_1 < C_2$ in $\mathcal{P}_f(\mathbb{N})$, we have

$$\boxplus_{i\in C_1} P_i^k \prec \boxplus_{i\in C_2} P_i^k.$$

Suppose we have defined $P_1^k, P_2^k, P_3^k, \dots$ with the above properties. We do the job for $k+1$. The second condition implies that

$$P_1^k[k+1] < P_2^k[k+1] < \cdots < P_m^k[k+1] < \cdots.$$

Now by Hindman's theorem, there are $B_1 < B_2 < \cdots < B_m < \cdots$ in $\mathcal{P}_f(\mathbb{N})$ such that, if we put

$$b_1 = \sum_{i\in B_1} P_i^k[k+1], b_2 = \sum_{i\in B_2} P_i^k[k+1], \dots, b_m = \sum_{i\in B_m} P_i^k[k+1], \dots,$$

then $\mathbf{c}$ has a constant value on $FS(\{b_1, b_2, \dots\})$ which we denote it by α. Now we set

$$P_1^{k+1} = \boxplus_{i\in B_1} P_i^k, P_2^{k+1} = \boxplus_{i\in B_2} P_i^k, \dots, P_m^{k+1} = \boxplus_{i\in B_m} P_i^k, \dots,$$

as well as we set

$$\alpha_1^{k+1} = \alpha_1^k, \alpha_2^{k+1} = \alpha_2^k, \dots, \alpha_k^{k+1} = \alpha_k^k, \alpha_{k+1}^{k+1} = \alpha.$$

We check the conditions (a) and (b) for $k+1$. Let $C \in \mathcal{P}_f(\mathbb{N})$ and $1 \leq s \leq k+1$, hence we have

$$(\boxplus_{i\in C} P_i^{k+1})[s] = (\boxplus_{i\in C} \boxplus_{j\in B_i} P_j^k)[s] = (\boxplus_{i\in D} P_i^k)[s],$$

where $D = \bigcup_{i\in C} B_i$. Suppose $1 \leq s \leq k$; then from the induction hypothesis, it follows that

$$\mathbf{c}\big((\boxplus_{i\in D} P_i^k)[s]\big) = \alpha_s^k = \alpha_s^{k+1}. \tag{2}$$

Also for $s = k+1$, we have

$$(\boxplus_{i\in C} \boxplus_{j\in B_i} P_j^k)[k+1] = \sum_{i\in C} \sum_{j\in B_i} P_j^k[k+1] = \sum_{i\in C} b_i \in FS(\{b_1, b_2, \dots\}),$$

which implies that

$$\mathbf{c}\big((\boxplus_{i\in C}\boxplus_{j\in B_i} P_j^k)[k+1]\big) = \mathbf{c}\big(\sum_{i\in C} b_i\big) = \alpha = \alpha_{k+1}^{k+1}. \tag{3}$$

Now putting the equations (2) and (3) together, we deduce

$$\mathbf{c}\big((\boxplus_{i\in C} P_i^{k+1})[s]\big) = \alpha_s^{k+1}$$

for $1 \le s \le k+1$. This finishes the proof of the condition (a). Now we turn to checking (b). Let $C_1 < C_2$ be in $\mathcal{P}_f(\mathbb{N})$. We must show that

$$\boxplus_{i\in C_1} P_i^{k+1} \prec \boxplus_{i\in C_2} P_i^{k+1},$$

which is equivalent to

$$\boxplus_{i\in C_1}\boxplus_{j\in B_i} P_j^k \prec \boxplus_{i\in C_2}\boxplus_{j\in B_i} P_j^k. \tag{4}$$

Letting $D_1 = \bigcup_{i\in C_1} B_i$, $D_2 = \bigcup_{i\in C_2} B_i$, we get $D_1 < D_2$, and the relation (4) becomes

$$\boxplus_{i\in D_1} P_i^k \prec \boxplus_{i\in D_2} P_i^k,$$

which is exactly our induction hypothesis. This proves the condition (b).

Now consider $P_1^n[1], P_1^n[2], \ldots, P_1^n[n]$ and recall that $n = W(l,c)$. By construction, we have

$$\mathbf{c}(P_1^n[1]) = \alpha_1^n, \ldots, \mathbf{c}(P_1^n[n]) = \alpha_n^n.$$

Through induced coloring, it follows from van der Waerden's theorem that there exist $\gamma \in [c]$ and positive integers a, d such that

$$\alpha_a^n = \alpha_{a+d}^n = \cdots = \alpha_{a+(l-1)d}^n = \gamma.$$

We define the desire arithmetic progressions $Q_i, i \in \mathbb{N}$ as follows

$$Q_i = \big\{P_i^n[a], P_i^n[a+d], \ldots, P_i^n[a+(l-1)d]\big\}.$$

It is easily seen by condition (b) that $Q_1 \prec Q_2 \prec Q_3 \prec \cdots$. Also for all $C \in \mathcal{P}_f(\mathbb{N})$ and all $1 \le s \le l$, we have

$$\mathbf{c}\big((\boxplus_{i\in C} Q_i)[s]\big) = \mathbf{c}\big(\sum_{i\in C} Q_i[s]\big) = \mathbf{c}\big(\sum_{i\in C} P_i^n[a+(s-1)d]\big) = \alpha_{a+(s-1)d}^n = \gamma.$$

This finishes the proof of Theorem 5. □

Now we turn to the two-fold generalization of Schur's extension of van der Waerden's theorem and Hindman's theorem. To simplify the description of the proof, we make a convention. We say that an arithmetic progression P is *homogenous* if the common difference of P equals $P[1]$. The proof of Theorem 6 proceeds similar to the proof of Theorem 5 with the exception that for the starting step, we arrange the sequence of n-term arithmetic progressions $P_1^0, P_2^0, P_3^0, \ldots$ in such a way that they become homogenous. This will imply that all the sequences $P_i^k, 1 \leq k \leq n, i \in \mathbb{N}$, constructed in the next steps, and also their finite sums, become homogenous. Then as in the proof of Theorem 5, we choose P_1^n but this time we use Schur's extension of van der Waerden's theorem inside P_1^n, and then we show that the proof works. It is worth mentioning that what essentially makes the proof working, is the simple fact that, if Q is an arithmetic progression which is a subset of a homogenous P, then the common difference of Q is a term of P. We use this fact in a uniform way.

Theorem 6. Let c and $l \geq 3$ be positive integers. Let $\mathbf{c}$ be a c-coloring of $\mathbb{N}$, then there are l-term arithmetic progressions $Q_1, Q_2, Q_3, \ldots$ such that

(i) $Q_1 \prec Q_2 \prec Q_3 \prec \cdots$,

(ii) there is $\gamma \in [c]$ such that for all $C \in \mathcal{P}_f(\mathbb{N})$ and all $s \in \{1, \ldots, l\}$, we have

$$\mathbf{c}\big((\boxplus_{i\in C} Q_i)[s]\big) = \mathbf{c}\big(\operatorname{add}(\boxplus_{i\in C} Q_i)\big) = \gamma.$$

Proof. We start with $n = SB(l, c)$, and a strictly increasing sequence of positive integers $a_1 < a_2 < \cdots < a_m < \cdots$ with $a_{m+1} > n(a_1 + \cdots + a_m)$. For $i \in \mathbb{N}$, we put $P_i^0 = \{a_i, a_i + a_i, \ldots, a_i + (n-1)a_i\}$. In this case for all $1 \leq k \leq n$ and all $C \in \mathcal{P}_f(\mathbb{N})$, we will have

$$\operatorname{add}(\boxplus_{i\in C} P_i^k) = (\boxplus_{i\in C} P_i^k)[1]. \tag{5}$$

We prove the equation (5) by induction on k. First observe that

$$\begin{aligned} \operatorname{add}(\boxplus_{i\in C} P_i^0) = (\boxplus_{i\in C} P_i^0)[2] - (\boxplus_{i\in C} P_i^0)[1] &= \sum_{i\in C} P_i^0[2] - \sum_{i\in C} P_i^0[1] \\ &= \sum_{i\in C} (a_i + a_i) - \sum_{i\in C} a_i \\ &= \sum_{i\in C} a_i = \sum_{i\in C} P_i^0[1] \\ &= (\boxplus_{i\in C} P_i^0)[1]. \end{aligned}$$

Also for $k+1$, recall the subsets B_i in definition of the arithmetic progressions P_i^{k+1}, so we have

$$\begin{aligned} \operatorname{add}(\boxplus_{i\in C} P_i^{k+1}) &= \operatorname{add}(\boxplus_{i\in C} \boxplus_{j\in B_i} P_j^k) = \operatorname{add}(\boxplus_{i\in D} P_i^k) \\ &= (\boxplus_{i\in D} P_i^k)[1] = (\boxplus_{i\in C} P_i^{k+1})[1], \end{aligned}$$

where $D = \bigcup_{i\in C} B_i$. This proves the equation (5). The proof now proceeds as in the proof of Theorem 5, in particular, the relation (1) can be proved easily for these new P_i^0. Now recall $P_1^n[1], P_1^n[2], \ldots, P_1^n[n]$ so that for $s \in \{1, \ldots, n\}$ and $C \in \mathcal{P}_f(\mathbb{N})$, we have

$$\mathbf{c}\big((\boxplus_{i\in C} P_i^n)[s]\big) = \alpha_s^n.$$

Through induced coloring, and this time using $= SB(l, c)$, we obtain $\gamma \in [c]$ and positive integers a, d such that

$$\alpha_d^n = \alpha_a^n = \alpha_{a+d}^n = \cdots = \alpha_{a+(l-1)d}^n = \gamma.$$

Again, define the desire arithmetic progressions $Q_i, i \in \mathbb{N}$ by

$$Q_i = \big\{P_i^n[a], P_i^n[a+d], \ldots, P_i^n[a+(l-1)d]\big\}.$$

Thus for all $C \in \mathcal{P}_f(\mathbb{N})$, we have

$$\begin{aligned}
\operatorname{add}(\boxplus_{i\in C} Q_i) &= (\boxplus_{i\in C} Q_i)[2] - (\boxplus_{i\in C} Q_i)[1] = \sum_{i\in C} Q_i[2] - \sum_{i\in C} Q_i[1] \\
&= \sum_{i\in C} P_i^n[a+d] - \sum_{i\in C} P_i^n[a] = \sum_{i\in C}\big(P_i[a+d] - P_i[a]\big) \\
&= \sum_{i\in C}\sum_{t=1}^{d}\big(P_i^n[a+t] - P_i^n[a+(t-1)]\big) = \sum_{i\in C}\sum_{t=1}^{d} \operatorname{add} P_i^n \\
&= \sum_{i\in C} d.\operatorname{add} P_i^n = d\sum_{i\in C}\operatorname{add} P_i^n = d.\operatorname{add}(\boxplus_{i\in C} P_i^n) \\
&= (\boxplus_{i\in C} P_i^n)[1] + (d-1)\operatorname{add}(\boxplus_{i\in C} P_i^n) = (\boxplus_{i\in C} P_i^n)[d].
\end{aligned}$$

Note that in the second and third equations from the end, we have respectively used the equation (5), and the easily checked fact $\sum_{i\in C} \operatorname{add} P_i^n = \operatorname{add}(\boxplus_{i\in C} P_i^n)$. So we conclude that

$$\mathbf{c}\big(\operatorname{add}(\boxplus_{i\in C} Q_i)\big) = \mathbf{c}\big((\boxplus_{i\in C} P_i^n)[d]\big) = \alpha_d^n = \gamma,$$

and the rest of the proof is the same as the proof of Theorem 5. □

4. Tower Bound for the Finite Case

In this section, we prove

Theorem 7. For positive integers n, c and $l \geq 3$, let $f(n, l, c)$ be the least positive integer p such that whenever $\mathbf{c}$ is a c-coloring of $[p]$, there are l-term arithmetic progressions $Q_1, Q_2, \ldots, Q_n$ such that

(i) $Q_1 \prec \cdots \prec Q_n$,

(ii) $\max(Q_1 \boxplus \cdots \boxplus Q_n) \le p$,

(iii) there is $\gamma \in [c]$ such that for all $C \in \mathcal{P}^+([n])$ and all $s \in \{1, \dots, l\}$, we have

$$\mathbf{c}\big((\boxplus_{i \in C} Q_i)[s]\big) = \gamma.$$

Then $f(n, l, c)$ is a tower function.

Proof. Let $q = W(l, c^{2^{\mathrm{Hind}(n,c)}})$. We will show that $f(n, l, c) \le 2^{q^3}$. So from Gower's elementary bounds for the van der Waerden numbers [3] and Theorem 4, it follows that $f(n, l, c)$ is a tower function. Suppose that $p \ge 2^{q^3}$ and $\mathbf{c}$ is a c-coloring of $[p]$. We show that p satisfies the requirements of the theorem. Put $m = \mathrm{Hind}(n, c)$. Let $h_i, 1 \le i \le m$ be positive integers defined by $h_i = (m+i)+(i-1)q$. For $1 \le i \le m$, we define the q-term arithmetic progressions P_i as follows

$$P_i = \{2^i, 2^i + 2^{h_i}, 2^i + 2.2^{h_i}, \dots, 2^i + (q-1)2^{h_i}\}.$$

Clearly $P_1 \prec P_2 \prec \cdots \prec P_m$. We claim that for each $1 \le s \le q$, the positive integers $P_1[s], P_2[s], \dots, P_m[s]$ are pairwise power-disjoint. Let $1 \le s \le q$, $2^u \le q-1 < 2^{u+1}$ and $s - 1 = 2^{u_1} + \cdots + 2^{u_k}$ with $u_1 < u_2 < \cdots < u_k$, so $u_k \le u \le q-1$. Since $i \le m < h_1 \le h_i$, and

$$P_i[s] = 2^i + (s-1)2^{h_i} = 2^i + 2^{u_1+h_i} + \cdots + 2^{u_k+h_i},$$

we have that

$$\mathrm{pow}_2(P_i[s]) \subseteq \{i, h_i, h_i+1, \dots, h_i + (q-1)\} =: A_i$$

for $1 \le i \le m$. Now to prove the claim, it would be enough to show that $A_1, \dots, A_m$ are pairwise disjoint. In fact we show that

$$\{1, 2, \dots, m\} < A_1 - \{1\} < A_2 - \{2\} < \cdots < A_m - \{m\},$$

which easily implies the disjointness of $A_1, \dots, A_m$. First observe that

$$\min(A_1 - \{1\}) = h_1 = m + 1 > m.$$

Also for $1 \le i \le m-1$, we have

$$\begin{aligned} \min(A_{i+1} - \{i+1\}) = h_{i+1} &= (m+i+1) + iq \\ &> (m+i) + (i-1)q + (q-1) \\ &= h_i + (q-1) \\ &= \max(A_i - \{i\}), \end{aligned}$$

thus the claim is proved. Also we have

$$\begin{aligned}
\max(\boxplus_{i\in[m]}P_i) = (\boxplus_{i\in[m]}P_i)[q] = \sum_{i\in[m]} P_i[q] &\leq m2^m + m(q-1)2^{h_m} \\
&\leq q.2^q + q^2.2^{2m+(m-1)q} \\
&\leq 2^{2q} + q^2.2^{2q+q^2} \\
&\leq 2^{2q} + 2^q.2^{2q^2} \\
&\leq 2^{q+1}.2^{2q^2} \leq 2^{q^3} \leq p.
\end{aligned}$$

Now we define a coloring $\mathbf{c}^*$ on $[q]$ as follows. For $u, v \in [q]$, we put $\mathbf{c}^*(u) = \mathbf{c}^*(v)$ if for all $B \in \mathcal{P}^+([m])$ we have

$$\mathbf{c}\big((\boxplus_{i\in B}P_i)[u]\big) = \mathbf{c}\big((\boxplus_{i\in B}P_i)[v]\big).$$

Obviously the number of colors is c^{2^m-1}, so from $q = W(l, c^{2^m})$ it follows that there are $a, a+d, \ldots, a+(l-1)d$ in $\{1, 2, \ldots, q\}$ such that

$$\mathbf{c}^*(a) = \mathbf{c}^*(a+d) = \cdots = \mathbf{c}^*(a+(l-1)d),$$

which means that for all $B \in \mathcal{P}^+([m])$ and all $k_1, k_2 \in \{0, \ldots, l-1\}$, we have

$$\mathbf{c}\big((\boxplus_{i\in B}P_i)[a+k_1d]\big) = \mathbf{c}\big((\boxplus_{i\in B}P_i)[a+k_2d]\big).$$

We denote the above color by $\pi(B)$. So we have the well-defined function

$$\pi \colon \mathcal{P}^+([m]) \longrightarrow [c].$$

Now consider the following m-element set of power-disjoint (due to the claim) positive integers

$$\big\{P_1[a], P_2[a], \ldots, P_m[a]\big\}.$$

From $m = \mathrm{Hind}(n, c)$, we infer that there exist $B_1 < B_2 < \cdots < B_n$ in $\mathcal{P}^+([m])$ and $\gamma \in [c]$ so that for all $C \in NU\{B_1, \ldots, B_n\}$, we have

$$\pi(C) = \mathbf{c}\big(\sum_{i\in C} P_i[a]\big) = \gamma.$$

The desired arithmetic progressions $Q_1, \ldots, Q_n$ are defined as follows. For $1 \leq i \leq n$, we set

$$Q_i = \big\{(\boxplus_{j\in B_i}P_j)[a], (\boxplus_{j\in B_i}P_j)[a+d], \ldots, (\boxplus_{j\in B_i}P_j)[a+(l-1)d]\big\}.$$

Obviously $Q_1 \prec Q_2 \prec \cdots \prec Q_n$ and from $B_1 < B_2 < \cdots < B_n$, it is easily seen that

$$\max(Q_1 \boxplus \cdots \boxplus Q_n) \leq \max(P_1 \boxplus \cdots \boxplus P_m) \leq p.$$

Now for $C \in \mathcal{P}^+([n])$ and $1 \le s \le l$, we have

$$\begin{aligned}\mathbf{c}\big((\boxplus_{i\in C} Q_i)[s]\big) = \mathbf{c}\big(\sum_{i\in C} Q_i[s]\big) &= \mathbf{c}\big(\sum_{i\in C}(\boxplus_{j\in B_i} P_j)[a+(s-1)d]\big)\\ &= \mathbf{c}\big(\sum_{i\in C}\sum_{j\in B_i} P_j[a+(s-1)d]\big)\\ &= \mathbf{c}\big(\sum_{i\in D} P_i[a+(s-1)d]\big) = \pi(D) = \gamma,\end{aligned}$$

where $D = \bigcup_{i\in C} B_i \in NU\{B_1,\ldots,B_n\}$. This finishes the proof of the theorem. □

Acknowledgment. We would like to thank the referee for the encouraging comments and useful suggestions. We would also like to thank Professor Landman for his comments which improved the presentation of the paper.

References

[1] W. Deuber and N. Hindman, Partitions and sums of (m, p, c)-sets, *J. Combin. Theory Ser. A* **45** (1987), 300–302.

[2] P. Dodos and V. Kanellopoulos, *Ramsey Theory for Product Spaces*, American Mathematical Society, Providence, RI, 2016.

[3] W. T. Gowers, A new proof of Szemerédi's theorem, *Geom. Funct. Anal.* **11** (2001), 465–588.

[4] R. L. Graham and B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, John Wiley and Sons, Hoboken, NJ, 1990.

[5] H. Furstenberg, *Recurrence in Ergodic theory and Combinatorial Number Theory*, Princeton University Press, Princetone, NJ, 1981.

[6] H. J. Prömel, *Ramsey Theory for Discrete Structures*, Springer, Cham, 2013.

[7] A. D. Taylor, A canonical partition relation for finite subsets of ω, *J. Combinatorial Theory Ser. A* **21** (1976), 137–146.