PROPER DIVISOR GRAPH OF A POSITIVE INTEGER

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Received: 7/10/20, Accepted: 5/31/21, Published: 6/14/21

Abstract

The proper divisor graph Υ_n of a positive integer n is the simple graph whose vertices are the proper divisors of n, and in which two distinct vertices u, v are adjacent if and only if n divides uv. The graph Υ_n plays an important role in the study of the zero divisor graph of the ring $\mathbb{Z}_n$. In this paper, we study some graph theoretical properties of Υ_n and determine the graph parameters such as clique number, chromatic number, chromatic index, independence number, matching number, domination number, vertex and edge covering numbers of Υ_n. We also determine the automorphism group of Υ_n.

1. Introduction

All graphs considered in this paper are finite and simple. We refer to the book [10] for unexplained graph terminology and the basics on graph theory. Let G be a graph with vertex set $V(G)$. If $V(G) = \Phi$, then G is called the *empty graph.* If $V(G) \neq \Phi$ but G has no edge, then G is called a *null graph.* We write $u \sim v$ for

[1]The author is supported by Scholarship for Higher Education (SHE) under Innovation in Science Pursuit for Inspired Research (INSPIRE) programme of the Department of Science and Technology, Government of India.

[2]The author is partially supported by Project No. MTR/2017/000372 of the Science and Engineering Research Board, Department of Science and Technology, Government of India.

two distinct vertices $u, v \in V(G)$ if they are adjacent in G. The *degree* of a vertex $v \in V(G)$ is denoted by $\deg(v)$. A vertex of degree one is called a *pendant vertex* of G. The automorphism group of G is denoted by $\mathrm{Aut}(G)$. A vertex $v \in V(G)$ is called a *cut vertex* if the induced subgraph $G[V(G) \setminus \{v\}]$ of G has more components than G.

A *clique* in G is a subset K of $V(G)$ such that the subgraph induced by K is complete. The maximum size of a clique in G, denoted by $\omega(G)$, is called the *clique number* of G. A *vertex coloring* of G is an assignment of colors to the vertices of G such that two adjacent vertices receive different colors. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors required for a vertex coloring of G. If $\chi(H) = \omega(H)$ for every induced subgraph H of G, then G is called a *perfect graph*. An *edge coloring* of G is an assignment of colors to the edges of G such that two adjacent edges receive different colors. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number of colors required for an edge coloring of G.

An *independent set* in G is a set of vertices such that no two of them are adjacent. The maximum size of an independent set in G, denoted by $\alpha(G)$, is called the *independence number* of G. A *matching* in G is a set of edges such that no two of them are adjacent. The maximum size of a matching in G, denoted by $\alpha'(G)$, is called the *matching number* of G. A matching M in G is called a *perfect matching* if each vertex of G is incident with some edge contained in M. A *vertex cover* in G is a set of vertices that contains at least one endpoint of every edge. The minimum size of a vertex cover in G, denoted by $\beta(G)$, is called the *vertex covering number* of G. An *edge cover* in G is a set of edges such that every vertex of G is incident with some edge contained in it. The minimum size of an edge cover in G, denoted by $\beta'(G)$, is called the *edge covering number* of G. A *dominating set* in G is set X of vertices such that every vertex of $V(G) \setminus X$ is adjacent with some vertex in X. The minimum size of a dominating set in G, denoted by $\gamma(G)$, is called the *domination number* of G.

1.1. The Proper Divisor Graph Υ_n

Let n be a positive integer. An integer d is called a *proper divisor* of n if $1 < d < n$ and d divides n. The *proper divisor graph* of n, denoted by Υ_n, is the graph with vertices the proper divisors of n, and two distinct vertices u and v are adjacent if and only if n divides the product uv.

The graph Υ_n was recently introduced in the paper [6]. Note that Υ_n is the empty graph if and only if $n = 1$ or n is a prime. If n is composite, then Υ_n is a connected graph by [6, Lemma 2.6].

1.1.1. Use of the graph Υ_n

Let G be a graph on m vertices with $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $H_1, H_2, \ldots, H_m$ be m pairwise vertex disjoint graphs. The *G-generalized join graph* of $H_1, H_2 \ldots, H_m$ is the graph obtained from G by replacing each vertex v_i with the graph H_i and then adding new edges from each vertex of H_i to every vertex of H_j, $1 \leq i \neq j \leq m$, whenever v_i and v_j are adjacent in G (such graphs are called generalized composition graphs in [8]). Note that if $m = 2$ and $G = K_2$, then the G-generalized join graph of H_1 and H_2 coincides with the usual join graph $H_1 \vee H_2$ of H_1 and H_2.

The notion of zero divisor graph of a commutative ring was first introduced by I. Beck in [4] by taking all elements of the ring as vertices of the graph. It was later modified by Anderson and Livingston in [2] as the following. The *zero divisor graph* $\Gamma(R)$ of a commutative ring R with unity is the graph with vertex set consisting of the zero divisors of R, and two distinct vertices a and b are adjacent if and only if $ab = 0$ in R. Note that $\Gamma(R)$ is the empty graph if R is an integral domain.

Let n be composite. Since every proper divisor of n is a zero divisor of the ring $\mathbb{Z}_n$ of integers modulo n, Υ_n is an induced subgraph of the zero divisor graph $\Gamma(\mathbb{Z}_n)$. The graph Υ_n plays an important role in [6] while studying the spectrum of the Laplacian matrix of $\Gamma(\mathbb{Z}_n)$. By [6, Lemma 2.7], $\Gamma(\mathbb{Z}_n)$ is the Υ_n-generalized join graph of certain complete graphs and null graphs corresponding to the proper divisors of n. It was proved in [6, Proposition 4.1] that $\Gamma(\mathbb{Z}_n)$ is Laplacian integral if and only if all the eigenvalues of the $m \times m$ vertex weighted Laplacian matrix $\mathbb{L}(\Upsilon_n)$ (defined in [6, p.275]) of Υ_n are integers, where m is the number of proper divisors of n. Further, by [6, Theorem 5.8], the algebraic connectivity of $\Gamma(\mathbb{Z}_n)$ coincides with the second smallest eigenvalue of $\mathbb{L}(\Upsilon_n)$ if n is not a prime power nor a product of two distinct primes, and the Laplacian spectral radius of $\Gamma(\mathbb{Z}_n)$ coincides with the largest eigenvalue of $\mathbb{L}(\Upsilon_n)$ if n is not a prime power. It is also known that $\Gamma(\mathbb{Z}_n)$ is perfect if and only if Υ_n is prefect (see Section 5.2).

One can refer to the book [3] for different kinds of matrices associated with graphs and the papers [1, 2, 7, 12] for more on the zero divisor graph of $\mathbb{Z}_n$.

1.2. Aim of This Paper

It is clear from the discussion in Section 1.1 that the structure of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ is completely dependent on that of the proper divisor graph Υ_n. In this paper, we shall mainly be interested in studying the automorphism group and different parameters of Υ_n.

In Section 2, we discuss some basic properties of Υ_n such as vertex degrees, pendant vertices, diameter and cut vertices. Similarity of two positive integers is defined in Section 3. For two composites m and n, we prove that the proper divisor graphs Υ_m and Υ_n are isomorphic if and only if m and n are similar (except for distinct $m, n \in \{p_1^3, q_1q_2\}$, where p_1, q_1, q_2 are primes with $q_1 \neq q_2$). We then determine the automorphism group of Υ_n in Section 4. In Section 5, the graph parameters clique number, chromatic number, chromatic index, independence number, matching number, domination number, vertex and edge covering numbers of Υ_n are determined. We also provide algorithms for coloring the edges of Υ_n when n is a prime power or a product of distinct primes.

2. Basic Properties of Υ_n

Let $n > 1$ be an integer. The number of proper divisors of n is denoted by $\pi(n)$. Let k denote the number of distinct prime divisors of n and

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

be the prime power factorization of n, where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. We have

$$\pi(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) - 2, \tag{1}$$

which follows from the paragraph before Lemma 2.6 in [6].

Lemma 1. *Let a, b be two positive integers. If a is a proper divisor of b with $\pi(a)+1 = \pi(b)$, then $b = p^t$ and $a = p^{t-1}$ for some prime p and some integer $t \geq 2$.*

Proof. If b has at least two distinct prime divisors, then formula (1) implies that $\pi(b) - \pi(a) \geq 2$, which is not possible. So $b = p^t$ for some prime p and some positive integer t. Since b has a proper divisor a, we must have $t \geq 2$. Then, using (1), $\pi(a) + 1 = \pi(b)$ implies that $a = p^{t-1}$. □

In the rest of the paper, for obvious reason, we shall consider proper divisor graphs Υ_n only when n is composite. If $n = p_1^2$, then $\Upsilon_n \cong K_1$. To avoid this triviality, if $n = p_1^{\alpha_1}$, then we shall also assume that $\alpha_1 \geq 3$. We thus have $|V(\Upsilon_n)| = \pi(n) \geq 2$. Then connectedness of Υ_n implies that the degree of every vertex is at least one. For a positive integer m, we denote by $[m] := \{1, 2, \ldots, m\}$.

2.1. Vertex Degrees

In the following proposition, we determine the degree of a vertex of Υ_n based on a divisibility condition involving n and the square of that vertex.

Proposition 1. *Let u be a vertex of Υ_n. Then the degree of u is given by:*

$$\deg(u) = \begin{cases} \pi(u) & \text{if } n | u^2; \\ \pi(u) + 1 & \text{if } n \nmid u^2. \end{cases}$$

Proof. If $v \sim u$ in Υ_n, then $uv = rn$ for some integer $r \geq 1$ with $r \neq u$ and so $v = r\frac{n}{u}$. Since v is a proper divisor of n, we get that r divides u. Conversely, if r is a divisor of u with $r \neq u$, then $w = r\frac{n}{u}$ is a proper divisor of n and so is a vertex of Υ_n. Also, $w \sim u$ if $w \neq u$.

Thus, the number of neighbours of u is equal to the number of positive divisors r of u satisfying $r \neq u$ and $u \neq r\frac{n}{u}$. It follows that $\deg(u) = \pi(u)$ if $u = r\frac{n}{u}$ for some positive divisor r of u with $r \neq u$, otherwise $\deg(u) = \pi(u) + 1$. Then the fact that $u = r\frac{n}{u}$ if and only if n divides u^2 completes the proof. □

Corollary 1. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. If $\alpha_i = \alpha_j$ for some $i, j \in [k]$, then* $\deg\left(\frac{n}{p_i}\right) = \deg\left(\frac{n}{p_j}\right)$.

Proof. Write $u_i = \frac{n}{p_i}$ and $u_j = \frac{n}{p_j}$. Since $\alpha_i = \alpha_j$, we have that n divides u_i^2 if and only if n divides u_j^2, and it follows from (1) that $\pi(u_i) = \pi(u_j)$. Then the corollary follows from Proposition 1. □

As an application of Proposition 1, all vertices of degree one and two in Υ_n are determined in the following.

Proposition 2. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then the following hold:*

(i) If $k = 1$ and $\alpha_1 \in \{3, 4\}$, then p_1 and p_1^2 are the pendant vertices of Υ_n.

(ii) If $k = 1$ and $\alpha_1 \geq 5$, then p_1 is the only pendant vertex of Υ_n.

(iii) If $k \geq 2$, then $p_1, p_2, \ldots, p_k$ are precisely the pendant vertices of Υ_n.

Proof. The vertex p_i, $i \in [k]$, has exactly one neighbour in Υ_n, namely $\frac{n}{p_i}$. So each of $p_1, p_2, \ldots, p_k$ is a pendant vertex of Υ_n.

Now let u be a vertex of Υ_n with $\deg(u) = 1$. By Proposition 1, we have $\pi(u) = 0$ or 1, and $n|u^2$ if the latter holds. If $\pi(u) = 0$, then u must be a prime. If $\pi(u) = 1$, then u must be the square of a prime. In that case, $n|u^2$ implies $n \in \{p_1^3, p_1^4\}$. It can be seen that p_1^2 is also a pendant vertex of Υ_n for $n \in \{p_1^3, p_1^4\}$. □

Corollary 2. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, and let l be the number of pendant vertices of Υ_n. Then*

$$l = \begin{cases} 1 & \text{if } n = p_1^{\alpha_1} \text{ with } \alpha_1 \geq 5; \\ 2 & \text{if } n \in \{p_1^3, p_1^4\}; \\ k & \text{if } k \geq 2. \end{cases}$$

Proposition 3. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$, and let $V_2(n)$ be the set of all degree two vertices of Υ_n. Then the following hold:*

(i) $V_2(p_1^3) = \Phi$; $V_2(p_1^4) = \{p_1^3\}$; $V_2(n) = \{p_1^2, p_1^3\}$ if $n \in \{p_1^5, p_1^6\}$;

(ii) $V_2(p_1p_2) = \Phi$*;* $V_2(p_1^2p_2) = \{p_1^2, p_1p_2\}$*;* $V_2(p_1^2p_2^2) = \{p_1^2, p_2^2, p_1p_2\}$*;*

(iii) $V_2(n) = \{p_i^2 : 1 \le i \le k, \alpha_i \ge 2\}$ *if* $n \notin \{p_1^3, p_1^4, p_1^5, p_1^6, p_1p_2, p_1^2p_2, p_1^2p_2^2\}$.

Proof. We shall prove (iii) only as the statements of (i) and (ii) can easily be verified. Assume that $n \notin \{p_1^3, p_1^4, p_1^5, p_1^6, p_1p_2, p_1^2p_2, p_1^2p_2^2\}$. If $\alpha_i \ge 2$ for some $i \in [k]$, then p_i^2 is adjacent with the two vertices $\frac{n}{p_i}$, $\frac{n}{p_i^2}$ only, and so $\deg(p_i^2) = 2$.

Conversely, let u be a vertex of Υ_n with $\deg(u) = 2$. If n divides u^2, then Proposition 1 gives that $\pi(u) = \deg(u) = 2$, which is possible if and only if u is the cube of a prime or the product of two distinct primes. This forces $n \in \{p_1^4, p_1^5, p_1^6, p_1^2p_2, p_1^2p_2^2\}$ (as u is a proper divisor of n), a contradiction. So $n \nmid u^2$. Then $\pi(u) = 1$ by Proposition 1 again, which is possible if and only if u is the square of a prime. Therefore, $V_2(n) = \{p_i^2 : 1 \le i \le k, \alpha_i \ge 2\}$. □

Proposition 4. *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ *and* u, v *be two distinct vertices of* Υ_n*. If* u *divides* v*, then* $\deg(u) \le \deg(v)$*, and the equality holds if and only if* $k = 1$*,* $u = p_1^{\lceil \alpha_1/2 \rceil - 1}$ *and* $v = p_1^{\lceil \alpha_1/2 \rceil}$*.*

Proof. Since u is a proper divisor of v, we have $\pi(u) + 1 \le \pi(v)$. Then, using Proposition 1, we get

$$\deg(u) \le \pi(u) + 1 \le \pi(v) \le \deg(v). \tag{2}$$

If $k = 1$, $u = p_1^{\lceil \alpha_1/2 \rceil - 1}$ and $v = p_1^{\lceil \alpha_1/2 \rceil}$, then $n \nmid u^2$ and $n|v^2$. Applying Proposition 1, we get $\deg(u) = \pi(u) + 1 = \lceil \alpha_1/2 \rceil - 1 = \pi(v) = \deg(v)$. Conversely, suppose that $\deg(u) = \deg(v)$. Then (2) gives

$$\deg(u) = \pi(u) + 1 = \pi(v) = \deg(v).$$

It follows that both u and v are powers of a prime by Lemma 1, and that $n \nmid u^2$ and $n|v^2$ by Proposition 1. Thus we have the following:

(i) n itself is a prime power and so $n = p_1^{\alpha_1}$.

(ii) $u = p_1^{t-1}$ and $v = p_1^t$ for some integer t with $2 \le t \le \alpha_1 - 1$ (Lemma 1).

(iii) $p_1^{\lceil \alpha_1/2 \rceil} \nmid u$ as $n \nmid u^2$, and $p_1^{\lceil \alpha_1/2 \rceil} | v$ as $n \mid v^2$.

It follows from (ii) and (iii) that $t - 1 \le \lceil \alpha_1/2 \rceil - 1$ and $t \ge \lceil \alpha_1/2 \rceil$. This gives $t = \lceil \alpha_1/2 \rceil$, thus completing the proof. □

Remark 1. Recall that a graph is called nearly irregular if it contains exactly one pair of vertices with equal degree. Let G be a connected nearly irregular graph on $m \ge 2$ vertices. If u and v are the two vertices of G with equal degree, then $\deg(u) = \deg(v) = \lceil \frac{m+1}{2} \rceil - 1$. This follows from Proposition 4 with $n = p_1^{m+1}$ and the fact that there is precisely one connected nearly irregular graph on m vertices up to isomorphism ([5, Theorem 1.12]).

2.2. Diameter

The following proposition determines the diameter of Υ_n which is denoted by $\mathrm{diam}(\Upsilon_n)$.

Proposition 5. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then*

$$\mathrm{diam}(\Upsilon_n) = \begin{cases} 1 & \text{if } n \in \{p_1^3, p_1 p_2\}; \\ 2 & \text{if } n = p_1^{\alpha_1} \text{ with } \alpha_1 \geq 4; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. It follows from the proof of connectedness of Υ_n in [6, Lemma 2.6] that $\mathrm{diam}(\Upsilon_n) \leq 3$. If $n \in \{p_1^3, p_1 p_2\}$, then $\Upsilon_n \cong K_2$ and so $\mathrm{diam}(\Upsilon_n) = 1$. Suppose that $n = p_1^{\alpha_1}$ with $\alpha_1 \geq 4$. The vertex $p_1^{\alpha_1 - 1}$ is adjacent with all other vertices of Υ_n. Further, $\alpha_1 \geq 4$ implies that $p_1 \sim p_1^{\alpha_1 - 1} \sim p_1^2$ is the shortest path between p_1 and p_1^2. Hence $\mathrm{diam}(\Upsilon_n) = 2$.

Now consider $k \geq 2$ with $n \neq p_1 p_2$. By Proposition 2(iii), p_1 and p_2 are pendant vertices of Υ_n. We have $p_1 \sim \frac{n}{p_1}$ and $p_2 \sim \frac{n}{p_2}$. The vertices $p_1, p_2, \frac{n}{p_1}, \frac{n}{p_2}$ are pairwise distinct and $p_1 \sim \frac{n}{p_1} \sim \frac{n}{p_2} \sim p_2$ is the shortest path between p_1 and p_2. Therefore, $\mathrm{diam}(\Upsilon_n) = 3$. □

2.3. Minimum and Maximum Degrees

By Corollary 2, Υ_n has at least one pendant vertex and hence the minimum degree of Υ_n is one. The vertices of maximum degree in Υ_n are determined in Proposition 6 below. We need the following lemma.

Lemma 2. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $k \geq 2$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Set $u_i := \frac{n}{p_i}$ for $i \in [k]$. Then $\deg(u_1) \geq \deg(u_j)$ for $2 \leq j \leq k$, and equality holds if and only if $\alpha_1 = \alpha_j$ or $n = p_1^2 p_2$.*

Proof. We have $\alpha_1 \geq \alpha_j$. If $\alpha_1 = \alpha_j$, then $\deg(u_1) = \deg(u_j)$ by Corollary 1. Assume that $\alpha_1 > \alpha_j$. Then using Proposition 1 and the inequality $(\alpha_1 + 1)\alpha_j < \alpha_1(\alpha_j + 1)$, we get that

$$\begin{aligned} \deg(u_j) \leq \pi(u_j) + 1 &= (\alpha_1 + 1) \cdots (\alpha_{j-1} + 1)\alpha_j(\alpha_{j+1} + 1) \cdots (\alpha_k + 1) - 1 \\ &\leq \alpha_1(\alpha_2 + 1) \cdots (\alpha_j + 1) \cdots (\alpha_k + 1) - 2 \qquad (3) \\ &= \pi(u_1) \leq \deg(u_1). \end{aligned}$$

If $k \geq 3$, then the inequality (3) is strict and this gives $\deg(u_1) > \deg(u_j)$. Suppose that $\deg(u_1) = \deg(u_j)$ with $\alpha_1 > \alpha_j$. Then $k = 2$, $j = 2$ and we must have

$$\deg(u_2) = \pi(u_2) + 1 = (\alpha_1 + 1)\alpha_2 - 1 = \alpha_1(\alpha_2 + 1) - 2 = \pi(u_1) = \deg(u_1).$$

It follows that $n \nmid u_2^2$ by Proposition 1 and that $\alpha_1 = \alpha_2 + 1$. The former implies that $\alpha_2 = 1$ and so $\alpha_1 = 2$. This given $n = p_1^2 p_2$. If $n = p_1^2 p_2$, then it can be seen that $\deg(u_1) = 2 = \deg(u_2)$. □

Proposition 6. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Then the following hold:*

(i) If $n = p_1^3$, then both the vertices p_1 and p_1^2 of Υ_n are of maximum degree.

(ii) If $n = p_1^2 p_2$, then $\frac{n}{p_1} = p_1 p_2$ and $\frac{n}{p_2} = p_1^2$ are the vertices of Υ_n of maximum degree.

(iii) If $n \notin \{p_1^3, p_1^2 p_2\}$, then $\frac{n}{p_1}, \ldots, \frac{n}{p_t}$ are precisely the vertices of Υ_n of maximum degree, where $t \in [k]$ is the largest integer such that $\alpha_t = \alpha_1$.

Proof. The statements made in (i) and (ii) can easily be verified. We shall prove (iii). If $n = p_1^{\alpha_1}$ with $\alpha_1 \geq 4$, then $\frac{n}{p_1}$ is the only vertex that is adjacent with all other vertices of Υ_n and so it is the only vertex of Υ_n of maximum degree.

Now consider $k \geq 2$. Let $u \in V(\Upsilon_n)$ be of maximum degree. Set $u_i := \frac{n}{p_i}$ for $i \in [k]$. Since u is a divisor of u_j for some $j \in [k]$, we have $\deg(u) \leq \deg(u_j)$ by Proposition 4. The maximality of $\deg(u)$ then gives that $\deg(u) = \deg(u_j)$. Since $k \geq 2$, Proposition 4 again implies that $u = u_j$. Thus, the vertices of Υ_n of maximum degree are contained in $\{u_1, u_2, \ldots, u_k\}$. By the definition of t, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_t$ and $\alpha_1 > \alpha_j$ for $t+1 \leq j \leq k$. Since $n \neq p_1^2 p_2$, Lemma 2 gives that $u_1, u_2, \ldots, u_t$ are precisely the vertices of maximum degree in Υ_n. □

The number of vertices of Υ_n with minimum degree one (that is, pendant vertices) is given in Corollary 2. As a consequence of Proposition 6, we have the following result on the number of vertices of Υ_n having maximum degree.

Corollary 3. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$ and L be the number of vertices of Υ_n with maximum degree. If $t \in [k]$ is the largest integer with $\alpha_t = \alpha_1$, then*

$$L = \begin{cases} 2 & \text{if } n \in \{p_1^3, p_1^2 p_2\}; \\ t & \text{otherwise.} \end{cases}$$

Corollary 4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Then the maximum degree $\Delta(\Upsilon_n)$ of Υ_n is given by:*

$$\Delta(\Upsilon_n) = \deg\left(\frac{n}{p_1}\right) = \begin{cases} \pi\left(\frac{n}{p_1}\right) & \text{if } k = 1, \text{ or } k \geq 2 \text{ with } \alpha_1 \geq 2; \\ \pi\left(\frac{n}{p_1}\right) + 1 & \text{otherwise.} \end{cases}$$

Proof. This follows from Propositions 1 and 6. □

2.4. Cut Vertices

Let v be a vertex of Υ_n. For two disjoint nonempty subsets A and B of $V(\Upsilon_n)\setminus\{v\}$, we say that (A, B) is a *separation* of $\Upsilon_n[V(\Upsilon_n) \setminus \{v\}]$ if $V(\Upsilon_n) \setminus \{v\} = A \cup B$ and there is no edge of $\Upsilon_n[V(\Upsilon_n) \setminus \{v\}]$ with one endpoint in A and the other in B. Since Υ_n is connected, we have that v is a cut vertex of Υ_n if and only if there exists a separation of $\Upsilon_n[V(\Upsilon_n) \setminus \{v\}]$.

If $n \in \{p_1^3, p_1p_2\}$, then $\Upsilon_n \cong K_2$ has no cut vertex. We shall find the cut vertices of Υ_n for the remaining values of n.

Proposition 7. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ with $n \notin \{p_1^3, p_1p_2\}$. Then $\frac{n}{p_1}, \frac{n}{p_2}, \ldots, \frac{n}{p_k}$ are precisely the cut vertices of Υ_n.*

Proof. Each p_i, $i \in [k]$, is a pendant vertex of Υ_n by Proposition 2 (with $p_i \sim \frac{n}{p_i}$). Since $n \notin \{p_1^3, p_1p_2\}$, we have $\pi(n) = |V(\Upsilon_n)| \geq 3$ and then the connectedness of Υ_n implies that each $\frac{n}{p_i}$ is a cut vertex. We claim that any cut vertex of Υ_n is one of $\frac{n}{p_1}, \frac{n}{p_2}, \ldots, \frac{n}{p_k}$.

If $k = 1$, then $\frac{n}{p_1}$ is adjacent with all other vertices of Υ_n and so it must be the only cut vertex of Υ_n. Now consider $k \geq 2$. Suppose that there exists a cut vertex v of Υ_n different from $\frac{n}{p_1}, \frac{n}{p_2}, \ldots, \frac{n}{p_k}$. Let (A, B) be a separation of $\Upsilon_n[V(\Upsilon_n) \setminus \{v\}]$. Since $\frac{n}{p_i} \sim \frac{n}{p_j}$ for distinct $i, j \in [k]$, all the vertices $\frac{n}{p_i}$, $i \in [k]$, are either in A or in B. Without loss of generality, we may assume that $\frac{n}{p_i} \in A$ for all $i \in [k]$. Since B is nonempty, there is a vertex $u \in B$. Since u is divisible by p_j for some $j \in [k]$, we must have $u \sim \frac{n}{p_j}$, contradicting that (A, B) is a separation of $\Upsilon_n[V(\Upsilon_n)\setminus\{v\}]$. □

3. Similarity and Isomorphisms

Let n and m be two positive integers with their prime power factorisations:

$$n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k} \quad \text{and} \quad m = q_1^{\beta_1}q_2^{\beta_2}\cdots q_l^{\beta_l},$$

where p_i, q_j are primes and α_i, β_j are positive integers with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_l$. We say that n and m are *similar* if $k = l$ and $\alpha_i = \beta_i$ for every $i \in [k]$.

If two composites n and m are similar, then the proper divisor graphs Υ_n and Υ_m are isomorphic. This can be seen from the fact that the construction of the graph Υ_n does not depend on the actual primes involved as divisors of n. What about the converse statement? If $n = p_1^3$ and $m = q_1q_2$, then $\Upsilon_n \cong K_2 \cong \Upsilon_m$, but n and m are not similar. We prove that the converse statement is also true with this particular example as the only exception. More precisely, we have the following:

Theorem 1. *Let m and n be composite integers. Then Υ_m and Υ_n are isomorphic if and only if m and n are similar, except for $m, n \in \{p_1^3, q_1q_2\}$ with $m \neq n$, where p_1, q_1, q_2 are primes with $q_1 \neq q_2$.*

The proof of Theorem 1 follows from Propositions 8 and 9 below.

Proposition 8. *Let $n = p_1^{\alpha_1}$ with $\alpha_1 \geq 4$. If $\Upsilon_n \cong \Upsilon_m$ for some composite m, then n and m are similar.*

Proof. We have $|V(\Upsilon_n)| = |V(\Upsilon_m)|$ as $\Upsilon_n \cong \Upsilon_m$. If $m = q_1^{\beta_1}$ for some prime q_1 and positive integer β_1, then $\alpha_1 - 1 = \pi(n) = |V(\Upsilon_n)| = |V(\Upsilon_m)| = \pi(m) = \beta_1 - 1$ gives $\beta_1 = \alpha_1$.

Therefore, it is enough to prove that m is a prime power. This is true if $\alpha_1 = 4$, as there is no integer m with at least two distinct prime divisors for which $\pi(m) = |V(\Upsilon_m)| = |V(\Upsilon_n)| = \alpha_1 - 1 = 3$. If $\alpha_1 \geq 5$, then the claim follows from Proposition 2 and the fact that Υ_n and Υ_m must have the same number of pendant vertices. □

Proposition 9. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\beta_1} q_2^{\beta_2} \cdots q_l^{\beta_l}$ with $k \geq 2$, $l \geq 2$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_l$. If $\Upsilon_n \cong \Upsilon_m$, then n and m are similar.*

Proof. By Proposition 2(iii), $p_1, p_2, \ldots, p_k$ are the pendant vertices of Υ_n and $q_1, q_2, \ldots, q_l$ are that of Υ_m. Since $\Upsilon_n \cong \Upsilon_m$, they have the same number of pendant vertices and hence $k = l$. The fact that $|V(\Upsilon_n)| = |V(\Upsilon_m)|$ gives

$$(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) = (\beta_1 + 1)(\beta_2 + 1) \cdots (\beta_k + 1) \tag{4}$$

Using (4), it can be seen that $n = p_1^2 p_2$ if and only if $m = q_1^2 q_2$, and so n, m are similar in this case. Therefore, we shall assume that $n \neq p_1^2 p_2$ (and hence $m \neq q_1^2 q_2$).

Let $\phi : \Upsilon_n \to \Upsilon_m$ be a graph isomorphism. Since ϕ maps pendant vertices of Υ_n to that of Υ_m, it induces a bijection from $\{p_1, p_2, \ldots, p_k\}$ to $\{q_1, q_2, \ldots, q_k\}$. In order to prove the proposition, it is enough to show that $\alpha_i = \beta_s$ if $\phi(p_i) = q_s$ for $i, s \in [k]$.

The only neighbour of p_i in Υ_n is $\frac{n}{p_i}$ and that of q_s in Υ_m is $\frac{m}{q_s}$. Since $\phi(p_i) = q_s$, we must have $\phi\left(\frac{n}{p_i}\right) = \frac{m}{q_s}$ and then

$$\deg\left(\frac{n}{p_i}\right) = \deg\left(\frac{m}{q_s}\right). \tag{5}$$

Note that n divides $\left(\frac{n}{p_i}\right)^2$ if and only if $\alpha_i \geq 2$, and m divides $\left(\frac{m}{q_s}\right)^2$ if and only if $\beta_s \geq 2$. So we have the following by Proposition 1:

$$\alpha_i = 1: \qquad \deg\left(\frac{n}{p_i}\right) = (\alpha_1 + 1) \cdots (\alpha_{i-1} + 1)\alpha_i(\alpha_{i+1} + 1) \cdots (\alpha_k + 1) - 1; \tag{6}$$

$$\alpha_i \geq 2: \qquad \deg\left(\frac{n}{p_i}\right) = (\alpha_1+1)\cdots(\alpha_{i-1}+1)\alpha_i(\alpha_{i+1}+1)\cdots(\alpha_k+1)-2; \quad (7)$$

$$\beta_s = 1: \qquad \deg\left(\frac{m}{q_s}\right) = (\beta_1+1)\cdots(\beta_{s-1}+1)\beta_s(\beta_{s+1}+1)\cdots(\beta_k+1)-1; \quad (8)$$

$$\beta_s \geq 2: \qquad \deg\left(\frac{m}{q_s}\right) = (\beta_1+1)\cdots(\beta_{s-1}+1)\beta_s(\beta_{s+1}+1)\cdots(\beta_k+1)-2. \quad (9)$$

If $\alpha_i \geq 2$ and $\beta_s \geq 2$, then equations (4), (5), (7) and (9) give $\frac{\alpha_i+1}{\alpha_i} = \frac{\beta_s+1}{\beta_s}$, that is, $\alpha_i = \beta_s$.

Now suppose that $\alpha_i = 1$. If $\beta_s = 1$, then we are done. Suppose that $\beta_s \geq 2$. We shall get a contradiction by showing that $m = q_1^2 q_2$. Putting $\alpha_i = 1$ in (4), we get

$$(\alpha_1+1)\cdots(\alpha_{i-1}+1)(\alpha_{i+1}+1)\cdots(\alpha_k+1) = \frac{1}{2}(\beta_1+1)\cdots(\beta_s+1)\cdots(\beta_k+1). \quad (10)$$

From (5), (6) and (9), we get

$$(\alpha_1+1)\cdots(\alpha_{i-1}+1)(\alpha_{i+1}+1)\cdots(\alpha_k+1) = (\beta_1+1)\cdots(\beta_{s-1}+1)\beta_s (\beta_{s+1}+1)\cdots(\beta_k+1)-1. \quad (11)$$

An easy calculation using the equations (10) and (11) gives that

$$(\beta_s - 1)\prod_{\substack{t=1 \\ t\neq s}}^{k}(\beta_t+1) = 2. \quad (12)$$

Since $k \geq 2$, $\beta_s \geq 2$ and $\beta_t \geq 1$ for $t \neq s$, it follows from (12) that $k = 2$, $\beta_s = 2$ and $\beta_t = 1$, where $\{s,t\} = \{1,2\}$. Since $\beta_s > \beta_t$, we must have $s = 1, t = 2$ and hence $m = q_1^2 q_2$.

Similarly, if $\alpha_i \geq 2$ and $\beta_s = 1$, then we shall get a contradiction by showing that $n = p_1^2 p_2$ (using the equations (4), (5), (7) and (8)). □

From the proof of Proposition 9, we have the following result which is useful in determining the automorphism group of Υ_n in the next section.

Corollary 5. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $k \geq 2$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. If $n \neq p_1^2 p_2$ and $\phi : \Upsilon_n \to \Upsilon_n$ is a graph automorphism, then ϕ permutes the pendant vertices $p_1, p_2, \ldots, p_k$ of Υ_n such that $\phi(p_i) = p_j$ implies $\alpha_i = \alpha_j$ for $1 \leq i, j \leq k$.*

4. The Automorphism Group $\mathrm{Aut}(\Upsilon_n)$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. We study the automorphism group $\mathrm{Aut}(\Upsilon_n)$ of Υ_n in this section. We have the following result when $k = 1$.

Proposition 10. *If $n = p_1^{\alpha_1}$, then $|\mathrm{Aut}(\Upsilon_n)| = 2$.*

Proof. Recall that $\alpha_1 \geq 3$ by our assumption. By Proposition 4, the degrees of the vertices of Υ_n are pairwise distinct, with the exception of two vertices $u := p_1^{\lceil \frac{\alpha_1}{2} \rceil - 1}$ and $v := p_1^{\lceil \frac{\alpha_1}{2} \rceil}$ for which the degrees are the same. Therefore, every automorphism of Υ_n must fix each of the vertices contained in $V(\Upsilon_n) \setminus \{u, v\}$.

The map $\phi : \Upsilon_n \to \Upsilon_n$ with $\phi(u) = v$, $\phi(v) = u$ and which is identity on $V(\Upsilon_n) \setminus \{u, v\}$ is an automorphism of Υ_n. This follows from the fact that a vertex of $V(\Upsilon_n) \setminus \{u, v\}$ is adjacent with either both u and v, or none of them (note that $u \sim v$ if and only if α_1 is odd). Therefore, $|\mathrm{Aut}(\Upsilon_n)| = 2$. $\square$

If $k \geq 2$, then $p_1, p_2, \ldots, p_k$ are precisely the pendant vertices of Υ_n (Proposition 2(iii)). Therefore, for $\phi \in \mathrm{Aut}(\Upsilon_n)$, $\phi(p_i)$ is also a prime for every $i \in [k]$.

Lemma 3. *Let $k \geq 2$ and ϕ be an automorphism of Υ_n. Then the following hold:*

(i) $\phi\left(\frac{n}{p_i}\right) = \frac{n}{\phi(p_i)}$ *for* $1 \leq i \leq k$.

(ii) If $n \neq p_1^2 p_2$ *and* $w = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \in V(\Upsilon_n)$, *then* $\phi(w) = \phi(p_1)^{t_1} \phi(p_2)^{t_2} \cdots \phi(p_k)^{t_k}$, *where* $t_i \geq 1$ *if and only* $s_i \geq 1$ *for* $1 \leq i \leq k$.

Proof. (i) Since p_i and $\phi(p_i)$ both are pendant vertices with $p_i \sim \frac{n}{p_i}$ and $\phi(p_i) \sim \frac{n}{\phi(p_i)}$, we must have $\phi\left(\frac{n}{p_i}\right) = \frac{n}{\phi(p_i)}$ for $1 \leq i \leq k$.

(ii) Let u, v be vertices of Υ_n such that $\phi(u) = v$. We claim that if p_i divides u, then $p_j = \phi(p_i)$ divides v. Assume that $p_i | u$. If $u \neq \frac{n}{p_i}$, then $u \sim \frac{n}{p_i}$ implies that $v = \phi(u) \sim \phi\left(\frac{n}{p_i}\right) = \frac{n}{\phi(p_i)} = \frac{n}{p_j}$ and hence p_j divides v. If $u = \frac{n}{p_i}$, then p_i^2 divides n and so $\alpha_i \geq 2$. Since $n \neq p_1^2 p_2$, we have $\alpha_j = \alpha_i \geq 2$ by Corollary 5. Then $v = \phi\left(\frac{n}{p_i}\right) = \frac{n}{\phi(p_i)} = \frac{n}{p_j}$ is divisible by p_j. Applying similar argument to the automorphism ϕ^{-1} of Υ_n and using the fact that $\phi^{-1}(v) = u$, we get that if p_l divides v, then the prime $\phi^{-1}(p_l)$ divides u.

Now taking $u = w$, it follows from the above that $\phi(w) = \phi(p_1)^{t_1} \phi(p_2)^{t_2} \cdots \phi(p_k)^{t_k}$, where $t_i \geq 1$ if and only $s_i \geq 1$ for $1 \leq i \leq k$. $\square$

Lemma 4. *Let $k \geq 2$ with $n \neq p_1^2 p_2$ and ϕ be an automorphism of Υ_n. Then* $\phi\left(p_i^{s_i}\right) = \phi(p_i)^{s_i}$ *and* $\phi\left(\frac{n}{p_i^{s_i}}\right) = \frac{n}{\phi(p_i)^{s_i}}$ *for* $1 \leq i \leq k$ *and* $1 \leq s_i \leq \alpha_i$.

Proof. Let $u = p_i^{s_i}$. By Lemma 3(ii), we have $\phi(u) = \phi(p_i)^{t_i}$ for some positive integer t_i. Since $k \geq 2$, Proposition 1 gives that $\deg(u) = \pi(u) + 1 = s_i$ and $\deg(\phi(u)) = \pi(\phi(u)) + 1 = t_i$. Then $\deg(u) = \deg(\phi(u))$ gives that $s_i = t_i$ and hence $\phi(u) = \phi(p_i)^{s_i}$. The second part that $\phi\left(\frac{n}{p_i^{s_i}}\right) = \frac{n}{\phi(p_i)^{s_i}}$ can be obtained from the following.

Let $p_j = \phi(p_i)$. Then $\alpha_i = \alpha_j$ by Corollary 5. We claim that $\phi\left(\frac{n}{p_i^l}\right) = \frac{n}{p_j^l}$ for $1 \leq l \leq \alpha_i$. The proof is by induction on l. If $l = 1$, then the claim follows from Lemma 3(i). Assume that $\phi\left(\frac{n}{p_i^l}\right) = \frac{n}{p_j^l}$ for $1 \leq l \leq m < \alpha_i$. The neighbours of p_i^{m+1} are precisely $\frac{n}{p_i}, \frac{n}{p_i^2}, \ldots, \frac{n}{p_i^{m+1}}$ and that of $\phi\left(p_i^{m+1}\right) = \phi(p_i)^{m+1} = p_j^{m+1}$ are $\frac{n}{p_j}, \frac{n}{p_j^2}, \ldots, \frac{n}{p_j^{m+1}}$. Since ϕ is one-one and the neighbours of p_i^{m+1} are mapped to the neighbours of $\phi\left(p_i^{m+1}\right)$, the induction hypothesis then implies that $\phi\left(\frac{n}{p_i^{m+1}}\right) = \frac{n}{p_j^{m+1}}$. This proves the claim. □

Proposition 11. *Let $k \geq 2$ with $n \neq p_1^2 p_2$ and ϕ be an automorphism of Υ_n. Then for any vertex $p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ of Υ_n, we have*

$$\phi\left(p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}\right) = \phi(p_1)^{s_1} \phi(p_2)^{s_2} \cdots \phi(p_k)^{s_k}.$$

Proof. Let $u = p_1^{s_1} \cdots p_k^{s_k}$. By Lemma 3(ii), we have $\phi(u) = \phi(p_1)^{t_1} \cdots \phi(p_k)^{t_k}$, where $t_i \geq 1$ if and only $s_i \geq 1$ for $1 \leq i \leq k$. Suppose that $s_i \neq t_i$ for some i. Then $s_i \geq 1$ and $t_i \geq 1$. Let $\phi(p_i) = p_j$. Then $\alpha_i = \alpha_j$ by Corollary 5.

Claim 1: $u \neq \frac{n}{p_i^{s_i}}$. If possible, suppose that $u = \frac{n}{p_i^{s_i}}$. Then $\alpha_i = 2s_i$. By Lemma 4, we have $\phi(u) = \phi\left(\frac{n}{p_i^{s_i}}\right) = \frac{n}{p_j^{s_i}}$. This gives $s_i + t_i = \alpha_j$. Then $\alpha_j = \alpha_i = 2s_i$ implies that $s_i = t_i$, contradicting our assumption.

Claim 2: $\phi(u) \neq \frac{n}{\phi(p_i)^{t_i}}$. If possible, suppose that $\phi(u) = \frac{n}{\phi(p_i)^{t_i}}$. Then $\alpha_j = 2t_i$. Since $\phi\left(\frac{n}{p_i^{t_i}}\right) = \frac{n}{\phi(p_i)^{t_i}}$ by Lemma 4, injectivity of ϕ gives $u = \frac{n}{p_i^{t_i}}$. This implies $s_i + t_i = \alpha_i$. Then $\alpha_i = \alpha_j$ gives that $s_i = t_i$, contradicting our assumption.

Since $u \sim \frac{n}{p_i^{s_i}}$, we have $\phi(u) \sim \phi\left(\frac{n}{p_i^{s_i}}\right) = \frac{n}{\phi(p_i)^{s_i}}$ (Lemma 4). This implies $s_i < t_i$ (as $s_i \neq t_i$). Since $\phi(u) \sim \frac{n}{\phi(p_i)^{t_i}} = \phi\left(\frac{n}{p_i^{t_i}}\right)$, we get $u \sim \frac{n}{p_i^{t_i}}$. But this is not possible as $s_i < t_i$. Therefore, $s_i = t_i$ for all $i \in [k]$ and hence $\phi(u) = \phi(p_1)^{s_1} \phi(p_2)^{s_2} \cdots \phi(p_k)^{s_k}$. □

Corollary 6. *If $k \geq 2$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_k \geq 1$, then $|\mathrm{Aut}(\Upsilon_n)| = 2$ or 1 according as $n = p_1^2 p_2$ or not.*

Proof. If $n = p_1^2 p_2$, then Υ_n is a path of length three and hence $|\mathrm{Aut}(\Upsilon_n)| = 2$. Assume that $n \neq p_1^2 p_2$. Let ϕ be an automorphism of Υ_n. Since $\alpha_1 > \alpha_2 > \cdots > \alpha_k$, ϕ fixes each of the pendant vertices $p_1, p_2, \ldots, p_k$ by Corollary 5. Then Proposition 11 implies that each vertex of Υ_n is fixed by ϕ and so ϕ is the identity map. □

Corollary 7. *If $k \geq 2$ and two automorphisms of Υ_n agree on the pendant vertices, then they are equal.*

Lemma 5. *Let $k \geq 2$ and $\alpha_{i_1} = \alpha_{i_2} = \cdots = \alpha_{i_a}$ for some subset $A := \{i_1, i_2, \ldots, i_a\}$ of $[k]$. Given a permutation τ of $\{p_{i_1}, p_{i_2}, \ldots, p_{i_a}\}$, define the map $\overline{\tau} : \Upsilon_n \to \Upsilon_n$ by*

$$\overline{\tau}\left(p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}\right) = \left(\prod_{i_j \in A} \tau(p_{i_j})^{s_{i_j}}\right) \left(\prod_{l \in [k] \setminus A} p_l^{s_l}\right)$$

for $p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \in V(\Upsilon_n)$. Then $\overline{\tau}$ is an automorphism of Υ_n.

Proof. If $\tau(p_{i_j}) = p_{i_r}$ for $i_j, i_r \in A$, then $s_{i_j} \leq \alpha_{i_j} = \alpha_{i_r}$ and so $\overline{\tau}$ is well-defined.

Let $u = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ and $v = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$ be two vertices of Υ_n. If $\overline{\tau}(u) = \overline{\tau}(v)$, then comparing the prime powers on both sides we get $s_{i_j} = t_{i_j}$ for $i_j \in A$ and $s_l = t_l$ for $l \in [k] \setminus A$. So $s_r = t_r$ for $r \in [k]$ and hence $\overline{\tau}$ is injective. Write $v = \left(\prod_{i_j \in A} p_{i_j}^{t_{i_j}}\right)\left(\prod_{l \in [k] \setminus A} p_l^{t_l}\right)$ and define

$$w := \left(\prod_{i_j \in A} \tau^{-1}(p_{i_j})^{t_{i_j}}\right) \left(\prod_{l \in [k] \setminus A} p_l^{t_l}\right).$$

If $\tau^{-1}(p_{i_j}) = p_{i_r}$ for some $i_r \in A$, then $t_{i_j} \leq \alpha_{i_j} = \alpha_{i_r}$ and so w is a vertex of Υ_n. Now it is clear that $\overline{\tau}(w) = v$, implying $\overline{\tau}$ is surjective.

Since $\alpha_{i_1} = \cdots = \alpha_{i_a}$, it can be observed that $\overline{\tau}(u) \sim \overline{\tau}(v)$ if and only if $s_r + t_r \geq \alpha_r$ for all $r \in [k]$. The later holds if and only if $u \sim v$. Hence $u \sim v$ if and only if $\overline{\tau}(u) \sim \overline{\tau}(v)$. Thus $\overline{\tau}$ is an automorphism of Υ_n. □

In the following proposition, we determine the full automorphism group of Υ_n when $k \geq 2$ with $n \neq p_1^2 p_2$. Let $\alpha_{r_1}, \alpha_{r_2}, \ldots, \alpha_{r_b}$ be the distinct integers in the list $\alpha_1, \alpha_2, \ldots, \alpha_k$. For $1 \leq i \leq b$, define

$$A_{r_i} := \{j \in [k] : \alpha_j = \alpha_{r_i}\}$$

and set $|A_{r_i}| = k_i$. Then $A_{r_1} \cup A_{r_2} \cup \cdots \cup A_{r_b}$ is a partition of $[k]$ and so $k_1 + k_2 + \cdots + k_b = k$. For a given positive integer m, S_m denotes the symmetric group defined on the set $[m]$.

Proposition 12. *Let $k \geq 2$ with $n \neq p_1^2 p_2$. Then* $\mathrm{Aut}(\Upsilon_n) \cong S_{k_1} \times S_{k_2} \times \cdots \times S_{k_b}$, *where the integers $k_1, k_2, \ldots, k_b$ are as defined above.*

Proof. For $1 \leq i \leq b$, consider the sets A_{r_i} as defined above and let $X_{r_i} := \{p_j : j \in A_{r_i}\}$. Then $X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_b}$ is a partition of the set $X = \{p_1, p_2, \ldots, p_k\}$. Given $\phi \in \mathrm{Aut}(\Upsilon_n)$, let ϕ_{r_i} denote the restriction of ϕ to X_{r_i}. Then ϕ_{r_i} is a permutation of

X_{r_i} by Corollary 5. This gives $(\phi_{r_1}, \ldots, \phi_{r_b}) \in Sym(X_{r_1}) \times \cdots \times Sym(X_{r_b})$, where $Sym(X_{r_i})$ is the symmetric group defined on X_{r_i}. Thus the map $f : \mathrm{Aut}(\Upsilon_n) \to Sym(X_{r_1}) \times \cdots \times Sym(X_{r_b})$ taking ϕ to $(\phi_{r_1}, \phi_{r_2}, \ldots, \phi_{r_b})$ is well defined. We prove that f is a group isomorphism.

Let $\phi, \psi \in \mathrm{Aut}(\Upsilon_n)$. We claim that $f(\phi\psi) = f(\phi)f(\psi)$. It is enough to show that $(\phi\psi)_{r_i} = \phi_{r_i}\psi_{r_i}$ for $1 \le i \le b$. Indeed, for $p_j \in X_{r_i}$, we have

$$(\phi\psi)_{r_i}(p_j) = (\phi\psi)(p_j) = \phi(\psi(p_j)) = \phi(\psi_{r_i}(p_j)) = \phi_{r_i}(\psi_{r_i}(p_j)) = (\phi_{r_i}\psi_{r_i})(p_j).$$

Thus f is a group homomorphism. Corollary 7 implies that f is injective. Consider $(\tau_1, \ldots, \tau_b) \in Sym(X_{r_1}) \times \cdots \times Sym(X_{r_b})$. For $1 \le i \le b$, let $\overline{\tau}_i$ be the automorphism of Υ_n as obtained in Lemma 5. Define $\overline{\tau} := \overline{\tau}_1\overline{\tau}_2 \cdots \overline{\tau}_b$, the composition of $\overline{\tau}_1, \overline{\tau}_2, \ldots, \overline{\tau}_b$. Then $\overline{\tau} \in \mathrm{Aut}(\Upsilon_n)$ and observe that $\overline{\tau}_{r_i} = \tau_i$ for each $i \in [b]$. This gives $f(\overline{\tau}) = (\tau_1, \tau_2, \ldots, \tau_b)$ and hence f is surjective.

Thus f is a group isomorphism and so $\mathrm{Aut}(\Upsilon_n) \cong Sym(X_{r_1}) \times \cdots \times Sym(X_{r_b})$. Since $|X_{r_i}| = |A_{r_i}| = k_i$, we have $Sym(X_{r_i}) \cong S_{k_i}$ for every $i \in [b]$ and hence the result follows. □

5. Graph Parameters of Υ_n

In this section, we shall determine the graph parameters clique number, chromatic number, chromatic index, domination number, independence number, matching number, vertex and edge covering numbers of Υ_n.

5.1. Clique Number $\omega(\Upsilon_n)$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime power factorization of n such that the integers $\alpha_1, \ldots, \alpha_l$ are odd and $\alpha_{l+1}, \ldots, \alpha_k$ are even for some $l \in \{0, 1, 2, \ldots, k\}$. Consider the subsets A and B of $V(\Upsilon_n)$ as defined below:

$$A := \{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} : \lceil \alpha_i/2 \rceil \le r_i \le \alpha_i, 1 \le i \le k\} \setminus \{n\},$$
$$B := \left\{ \frac{n}{p_j^{\lceil \alpha_j/2 \rceil}} : 1 \le j \le l \right\},$$

where $B = \Phi$ if $l = 0$. Let $\mathcal{K}$ denote the induced subgraph of Υ_n with vertex set $A \cup B$. Observe that A and B are disjoint, and that any two distinct vertices in $A \cup B$ are adjacent. Thus $\mathcal{K}$ is a clique in Υ_n with

$$|V(\mathcal{K})| = |A| + |B| = \left\lceil \frac{\alpha_1}{2} \right\rceil \left\lceil \frac{\alpha_2}{2} \right\rceil \cdots \left\lceil \frac{\alpha_l}{2} \right\rceil \left(\frac{\alpha_{l+1}}{2} + 1 \right) \cdots \left(\frac{\alpha_k}{2} + 1 \right) + l - 1. \quad (13)$$

Proposition 13. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime power factorization of n such that the integers $\alpha_1, \ldots, \alpha_l$ are odd and $\alpha_{l+1}, \ldots, \alpha_k$ are even for some $l \in \{0, 1, 2, \ldots, k\}$. Then*

$$\omega(\Upsilon_n) = \left\lceil \frac{\alpha_1}{2} \right\rceil \left\lceil \frac{\alpha_2}{2} \right\rceil \cdots \left\lceil \frac{\alpha_l}{2} \right\rceil \left(\frac{\alpha_{l+1}}{2} + 1 \right) \cdots \left(\frac{\alpha_k}{2} + 1 \right) + l - 1.$$

Proof. Consider the clique $\mathcal{K}$ in Υ_n defined above with vertex set $A \cup B$. Let H be an arbitrary clique in Υ_n. We prove that $|V(H)| \leq |V(\mathcal{K})|$ and then (13) would complete the proof. It is enough to show that there exists an injective map $\phi : V(H) \to V(\mathcal{K})$.

Let $y = p_1^{s_1} \cdots p_l^{s_l} p_{l+1}^{s_{l+1}} \cdots p_k^{s_k}$ be a vertex of H. If $\lceil \alpha_i/2 \rceil \leq s_i \leq \alpha_i$ for every $i \in [k]$, then y is a vertex of $\mathcal{K}$ that is contained in A. In this case, we define $\phi(y) := y \in A$.

Suppose that $s_i < \lceil \frac{\alpha_i}{2} \rceil$ for some $i \in [k]$. Let $j \in [k]$ be the smallest integer such that $s_j < \lceil \frac{\alpha_j}{2} \rceil$. For every vertex $z = p_1^{t_1} \cdots p_l^{t_l} p_{l+1}^{t_{l+1}} \cdots p_k^{t_k} \in V(H) \setminus \{y\}$, the fact that $y \sim z$ gives

$$t_j \geq \begin{cases} \lceil \frac{\alpha_j}{2} \rceil & \text{if } j \leq l; \\ \lceil \frac{\alpha_j}{2} \rceil + 1 = \frac{\alpha_j}{2} + 1 & \text{if } j \geq l + 1. \end{cases}$$

Thus, y is the only vertex of H with $s_j < \lceil \frac{\alpha_j}{2} \rceil$. If $j \leq l$, we define

$$\phi(y) := \frac{n}{p_j^{\lceil \alpha_j/2 \rceil}} \in B.$$

If $j \geq l + 1$, then there is no such vertex z of H with $t_j = \lceil \frac{\alpha_j}{2} \rceil = \frac{\alpha_j}{2}$. In this case, we define

$$\phi(y) := p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}} p_j^{\alpha_j/2} p_{j+1}^{\alpha_{j+1}} \cdots p_k^{\alpha_k} \in A.$$

It follows from the construction of the map $\phi : V(H) \to V(\mathcal{K})$ that ϕ is well-defined and it is one-one. □

As a consequence of Proposition 13, we have the following:

Corollary 8. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then $\omega(\Upsilon_n) \geq k$.*

Corollary 8 can also be seen directly as follows. Since $\frac{n}{p_i} \sim \frac{n}{p_j}$ for $1 \leq i \neq j \leq k$, the induced subgraph of Υ_n with vertex set $\left\{ \frac{n}{p_i} : 1 \leq i \leq k \right\}$ is a clique in Υ_n.

5.2. Chromatic Number $\chi(\Upsilon_n)$

In the following proposition, we prove that the chromatic number and the clique number of Υ_n are equal.

Proposition 14. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime power factorization of n such that the integers $\alpha_1, \ldots, \alpha_l$ are odd and $\alpha_{l+1}, \ldots, \alpha_k$ are even for some $l \in \{0, 1, 2, \ldots, k\}$. Then*

$$\chi(\Upsilon_n) = \omega(\Upsilon_n) = \left\lceil \frac{\alpha_1}{2} \right\rceil \left\lceil \frac{\alpha_2}{2} \right\rceil \cdots \left\lceil \frac{\alpha_l}{2} \right\rceil \left(\frac{\alpha_{l+1}}{2} + 1 \right) \cdots \left(\frac{\alpha_k}{2} + 1 \right) + l - 1.$$

Proof. Consider the clique $\mathcal{K}$ in Υ_n (defined in Section 5.1 with $V(\mathcal{K}) = A \cup B$). We have $\omega(\Upsilon_n) = |V(\mathcal{K})|$ by (13) and Proposition 13. Assign $\omega(\Upsilon_n)$ distinct colors to the vertices of $\mathcal{K}$. Out of the $\omega(\Upsilon_n)$ colors used so far, we shall choose k of them (possible as $\omega(\Upsilon_n) \geq k$ by Corollary 8) and assign these k colors suitably to the remaining vertices of Υ_n.

For $1 \leq i \leq k$, set $\gamma_i := \lceil \frac{\alpha_i}{2} \rceil$ and let c_i be the color assigned to the vertex $w_i \in V(\mathcal{K})$, where

$$w_i = \begin{cases} p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_i^{\gamma_i - 1} p_{i+1}^{\alpha_{i+1}} \cdots p_l^{\alpha_l} \cdots p_k^{\alpha_k} & \text{if } 1 \leq i \leq l; \\ p_1^{\alpha_1} \cdots p_l^{\alpha_l} \cdots p_{i-1}^{\alpha_{i-1}} p_i^{\gamma_i} p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k} & \text{if } l+1 \leq i \leq k. \end{cases}$$

Let u be a vertex of Υ_n outside $\mathcal{K}$. Then $p_i^{\gamma_i}$ does not divide u for some $i \in [k]$. We assign the color c_t to the vertex u, where $t \in [k]$ is the smallest integer such that $p_t^{\gamma_t} \nmid u$. Thus u and w_t receive the same color. In this way, we color all the vertices of Υ_n. Note that if x, y are two distinct vertices of Υ_n with the same color c_t, then $p_t^{\alpha_t} \nmid xy$ implies $x \nsim y$. $\square$

Since $\chi(\Upsilon_n) = \omega(\Upsilon_n)$, it is natural to ask whether Υ_n is perfect. In [9], the *zero-divisor type graph* $\Gamma^T(\mathbb{Z}_n)$ of $\mathbb{Z}_n$ is defined and it is proved in Theorem 4.1 that $\Gamma^T(\mathbb{Z}_n)$ is perfect if and only if the zero divisor graph $\Gamma(\mathbb{Z}_n)$ is perfect. Further, using the Strong Perfect Graph Theorem, the author proved that the graph $\Gamma^T(\mathbb{Z}_n)$ is perfect if and only if $n \in \{p_1^{\alpha_1}, p_1^{\alpha_1} p_2^{\alpha_2}, p_1^{\alpha_1} p_2 p_3, p_1 p_2 p_3 p_4\}$. From the construct of $\Gamma^T(\mathbb{Z}_n)$, it can be seen that the proper divisor graph Υ_n is isomorphic to $\Gamma^T(\mathbb{Z}_n)$. As a consequence, it follows that Υ_n is prefect if and only if $n \in \{p_1^{\alpha_1}, p_1^{\alpha_1} p_2^{\alpha_2}, p_1^{\alpha_1} p_2 p_3, p_1 p_2 p_3 p_4\}$.

5.3. Matching Number $\alpha'(\Upsilon_n)$

Proposition 15. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $\mathcal{M}$ be the collection of all edges of Υ_n of the form $\{x, \frac{n}{x}\}$ with $x < \sqrt{n}$. Then $\mathcal{M}$ is a matching in Υ_n of maximum size and*

$$\alpha'(\Upsilon_n) = |\mathcal{M}| = \lfloor \pi(n)/2 \rfloor.$$

Proof. Since $|V(\Upsilon_n)| = \pi(n)$, we have $\alpha'(\Upsilon_n) \leq \lfloor \frac{\pi(n)}{2} \rfloor$. Clearly, two distinct edges contained in $\mathcal{M}$ do not share any common vertex. So $\mathcal{M}$ is a matching in Υ_n. Every vertex of Υ_n is an end point of some edge contained in $\mathcal{M}$, with the exception of the vertex $\sqrt{n}$ when n is a perfect square (in which case $|V(\Upsilon_n)|$ is odd). This gives $|\mathcal{M}| = \lfloor \frac{\pi(n)}{2} \rfloor$ and it follows that the matching $\mathcal{M}$ is of maximum size. $\square$

Corollary 9. *Υ_n has a perfect matching if and only if n is not a perfect square.*

Corollary 10. *Let $Z = \{x \in V(\Upsilon_n) : x < \sqrt{n}\}$. Then $|Z| = \lfloor \frac{\pi(n)}{2} \rfloor$.*

Proof. This follows from Proposition 15 using the fact that Z is in bijective correspondence with the set $\mathcal{M} := \big\{\{x, \frac{n}{x}\} : x \in V(\Upsilon_n),\ x < \sqrt{n}\big\}$. □

Corollary 11. *The edge covering number $\beta'(\Upsilon_n)$ of Υ_n is given by: $\beta'(\Upsilon_n) = \lceil \pi(n)/2 \rceil$.*

Proof. Since Υ_n has no isolated vertices, we have $\alpha'(\Upsilon_n)+\beta'(\Upsilon_n) = |V(\Upsilon_n)| = \pi(n)$ by [10, Theorem 3.1.22]. Then Proposition 15 gives that $\beta'(\Upsilon_n) = \pi(n)-\lfloor \pi(n)/2 \rfloor = \lceil \pi(n)/2 \rceil$. □

5.4. Independence Number $\alpha(\Upsilon_n)$

If $n = p_1p_2$, then $\Upsilon_n \cong K_2$ and so $\alpha(\Upsilon_n) = 1$. If $n \neq p_1p_2$, then no two vertices among $p_1, p_2, \dots, p_k$ are adjacent and hence $\alpha(\Upsilon_n) \geq k$. We prove the following:

Proposition 16. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ and $\mathcal{I} := \{x \in V(\Upsilon_n) : x \leq \sqrt{n}\}$. Then $\mathcal{I}$ is an independent set in Υ_n of maximum size and*

$$\alpha(\Upsilon_n) = |\mathcal{I}| = \lceil \pi(n)/2 \rceil.$$

Proof. If $x, y \in \mathcal{I}$ with $x \neq y$, then at least one of them is less than $\sqrt{n}$ and so $xy < n$. This implies that x and y are not adjacent. Thus $\mathcal{I}$ is an independent set in Υ_n.

Let $\mathcal{J}$ be an independent set in Υ_n of maximum size. Then $|\mathcal{I}| \leq |\mathcal{J}|$. We claim that $|\mathcal{I}| \geq |\mathcal{J}|$. This follows if $\mathcal{J} \setminus \mathcal{I} = \Phi$. Assume that $\mathcal{J} \setminus \mathcal{I} \neq \Phi$. Let $y \in \mathcal{J} \setminus \mathcal{I}$. Then $y > \sqrt{n}$. This implies $\frac{n}{y} < \sqrt{n}$ and so $\frac{n}{y} \in \mathcal{I}$. Since $y \sim \frac{n}{y}$ (as $n \neq y^2$) and $\mathcal{J}$ is an independent set, it follows that $\frac{n}{y} \notin \mathcal{J}$ and hence $\frac{n}{y} \in \mathcal{I} \setminus \mathcal{J}$. Thus $y \mapsto \frac{n}{y}$ defines an injective map from $\mathcal{J} \setminus \mathcal{I}$ to $\mathcal{I} \setminus \mathcal{J}$. Then $|\mathcal{J}| = |\mathcal{I} \cap \mathcal{J}| + |\mathcal{J} \setminus \mathcal{I}| \leq |\mathcal{I} \cap \mathcal{J}| + |\mathcal{I} \setminus \mathcal{J}| = |\mathcal{I}|$. Thus $|\mathcal{I}| = |\mathcal{J}|$ and hence the independent set $\mathcal{I}$ is of maximum size.

We have $\sqrt{n} \in \mathcal{I}$ if and only if n is a perfect square if and only if $\pi(n) = |V(\Upsilon_n)|$ is odd. Consider the set Z defined in Corollary 10. When n is a perfect square, we have $Z = \mathcal{I} \setminus \{\sqrt{n}\}$ and this gives $|\mathcal{I}| = |Z| + 1 = \lfloor \frac{\pi(n)}{2} \rfloor + 1 = \lceil \frac{\pi(n)}{2} \rceil$, otherwise $Z = \mathcal{I}$ and we get $|\mathcal{I}| = |Z| = \lfloor \frac{\pi(n)}{2} \rfloor = \lceil \frac{\pi(n)}{2} \rceil$. □

Corollary 12. *The vertex covering number $\beta(\Upsilon_n)$ of Υ_n is given by: $\beta(\Upsilon_n) = \lfloor \pi(n)/2 \rfloor$.*

Proof. We have $\alpha(\Upsilon_n) + \beta(\Upsilon_n) = |V(\Upsilon_n)| = \pi(n)$ by [10, Lemma 3.1.21]. Then Proposition 16 gives that $\beta(\Upsilon_n) = \pi(n) - \lceil \pi(n)/2 \rceil = \lfloor \pi(n)/2 \rfloor$. □

5.5. Domination Number $\gamma(\Upsilon_n)$

Proposition 17. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. If $n \neq p_1p_2$, then $Y := \left\{\frac{n}{p_1}, \frac{n}{p_2} \ldots, \frac{n}{p_k}\right\}$ is a dominating set of minimum size and so*

$$\gamma(\Upsilon_n) = \begin{cases} 1 & \text{if } n = p_1p_2; \\ k & \text{otherwise.} \end{cases}$$

Proof. Every vertex of Υ_n is adjacent or equal to at least one of the vertices in Y. This implies that Y is a dominating set in Υ_n. Assume that $n \neq p_1p_2$. Then the set consisting of the k distinct edges $\left\{p_i, \frac{n}{p_i}\right\}$, $1 \leq i \leq k$, is a matching in Υ_n. The fact that $p_1, p_2, \ldots, p_k$ are pendant vertices of Υ_n then implies that any dominating set in Υ_n must contain p_i or $\frac{n}{p_i}$ for every $i \in [k]$. Thus every dominating set must contain at least k vertices and hence the dominating set Y is of minimum size. The rest is clear. □

5.6. Chromatic Index $\chi'(\Upsilon_n)$

Clearly, $\chi'(\Upsilon_n) \geq \Delta(\Upsilon_n)$. We shall prove that $\chi'(\Upsilon_n) = \Delta(\Upsilon_n)$. The following result proved in [1, Remark 1] is helpful for us, which was obtained as an application of Vizing's Adjacency Lemma [11, Corollary 3.6(iii)].

Lemma 6 ([1])**.** *Let G be a simple graph and $D := \{u \in V(G) : \deg(u) = \Delta(G)\}$. Suppose that, for every vertex $u \in D$, there is an edge $\{u, w\}$ of G such that $\Delta(G) - \deg(w) + 2 > |D|$. Then $\chi'(G) = \Delta(G)$.*

Proposition 18. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Then $\chi'(\Upsilon_n) = \Delta(\Upsilon_n)$.*

Proof. If $n = p_1^3$ or p_1p_2, then $\Upsilon_n \cong K_2$ and so $\chi'(\Upsilon_n) = 1 = \Delta(\Upsilon_n)$. Assume that $n \notin \{p_1^3, p_1p_2\}$. Define $u_i := \frac{n}{p_i}$ for $1 \leq i \leq k$ and $D := \{v \in V(\Upsilon_n) : \deg(v) = \Delta(\Upsilon_n)\}$. By Proposition 6, we have $D \subseteq \{u_1, u_2, \ldots, u_k\}$ and so $1 \leq |D| \leq k$.

Let $v \in D$. Then $v = u_i$ for some $i \in [k]$. We have $u_i \sim p_i$ and $u_i \sim u_j$ for $j \in [k] \setminus \{i\}$. Since $n \neq p_1p_2$, the vertex p_i is different from each such u_j. Thus $\deg(v) = \deg(u_i) \geq k$. For the edge $\{v, p_i\} = \{u_i, p_i\}$ of Υ_n, we have $\Delta(\Upsilon_n) - \deg(p_i) + 2 = \deg(v) - 1 + 2 \geq k + 1 > |D|$. Hence $\chi'(\Upsilon_n) = \Delta(\Upsilon_n)$ by Lemma 6. □

5.6.1. Coloring the Edges of Υ_n

When n is a prime power or a product of distinct primes, we are able to provide algorithms in the following for a proper edge coloring of Υ_n using $\Delta(\Upsilon_n)$ distinct colors. It would be interesting to develop such algorithms for other values of n as well.

(i) $n = p_1^{\alpha_1}$**:**

We have $\chi'(\Upsilon_n) = \Delta(\Upsilon_n) = \pi\left(\frac{n}{p_1}\right) = \alpha_1 - 2$ by Proposition 18 and Corollary 4. Consider $\alpha_1 - 2$ distinct colours, say $c_0, c_1, \ldots, c_{\alpha_1-3}$. For a given edge $\{u, v\}$ of Υ_n, there exists a unique $j \in \{0, 1, \ldots, \alpha_1 - 3\}$ such that $\frac{uv}{n} = p_1^j$. Then assign the color c_j to the edge $\{u, v\}$. If two distinct adjacent edges $\{u, v\}$ and $\{v, w\}$ of Υ_n receive the same color c_t, then $\frac{uv}{n} = p_1^t = \frac{vw}{n}$ gives $u = w$, a contradiction. Thus, two distinct adjacent edges receive different colors, implying a proper edge coloring of Υ_n.

(ii) $n = p_1 p_2 \cdots p_k$**,** $k \geq 2$**:**

Here $\chi'(\Upsilon_n) = \Delta(\Upsilon_n) = \pi\left(\frac{n}{p_1}\right) + 1 = 2^{k-1} - 1$ by Proposition 18 and Corollary 4. Consider the colors c_j for $j \in I$, where I is the index set defined by

$$I := \{j : j|n \text{ and } 1 < j < \sqrt{n}\}.$$

We thus have a total of $|I| = \pi(n)/2 = 2^{k-1} - 1$ colors (see Corollary 10). We assign colors to the edges of Υ_n in the following way. Let $\{u, v\}$ be an edge of Υ_n. Set $l = \frac{uv}{n}$. Then $1 \leq l < n$ and $l|n$. We have $l \neq \sqrt{n}$ as n is not a perfect square. If $l \neq 1$, then $l \in I$ if $l < \sqrt{n}$, and $\frac{n}{l} \in I$ if $l > \sqrt{n}$.

- If $1 < l < \sqrt{n}$, then we call $\{u, v\}$ an edge of Type-I and assign the color c_l to it.
- If $\sqrt{n} < l < n$, then we call $\{u, v\}$ an edge of Type-II and assign the color $c_{\frac{n}{l}}$ to it.
- If $l = 1$, then we call $\{u, v\}$ an edge of Type-III.

We first prove that there is no conflict of interest in the above assignment of colors to Type-I and Type II edges, and then we shall color Type-III edges suitably.

Let $\{u, v\}$ and $\{v, w\}$ be distinct adjacent edges of Υ_n of Type-I/Type-II. Suppose that they have received the same color. If both $\{u, v\}$ and $\{v, w\}$ are of the same type, then it follows that $\frac{uv}{n} = \frac{vw}{n}$ which gives $u = w$, a contradiction. So assume that they are of different types. Without loss of generality, we may assume that $\{u, v\}$ is of Type-I and $\{v, w\}$ is of Type-II. Then it follows that $\frac{uv}{n} = \frac{n^2}{vw}$ and this gives $uv^2w = n^3$. Since n is a product of distinct primes and $v \neq n$, we have that p_j does not divide v for some $j \in [k]$. Then uv^2w is not divisible by p_j^3. This implies that there are no such u, v, w with $uv^2w = n^3$. Thus, two distinct adjacent edges of Type-I/Type-II have received different colors.

Now consider an edge $\{x, y\}$ of Type-III. Then $xy = n$. Note that any other edge that is adjacent with $\{x, y\}$ must be of Type-I or Type-II. Let $a = \deg(x) - 1$ and $b = \deg(y) - 1$. If we prove that $a + b < 2^{k-1} - 1$, then we can color the edge $\{x, y\}$

with any of the $2^{k-1} - 1 - (a + b)$ colors which are not used for the Type-I/Type-II edges adjacent with $\{x, y\}$.

If one of x or y, say x, is a pendant vertex, then $\deg(y) = \Delta(\Upsilon_n)$ by Propositions 2(iii) and 6(iii). This gives $a = 0$ and $b = \Delta(\Upsilon_n) - 1 = 2^{k-1} - 2$ and so the claim follows. Assume that none of x, y is a pendant vertex. Then none of x, y is a prime. Since $n = xy$ and n is a product of distinct primes, there exist distinct primes p_i, p_j each dividing x but not y, and there exist distinct primes p_s, p_t each dividing y but not x. Since $\deg(x) = \pi(x) + 1 \leq 2^{k-2} - 1$, we have $a \leq 2^{k-2} - 2$. Similarly, $b \leq 2^{k-2} - 2$. It follows that $a + b \leq 2^{k-1} - 4 < 2^{k-1} - 1$.

Acknowledgement. The authors wish to sincerely thank the referee for his/her comments and suggestions which helped us to improve the submitted version of the article.

References

[1] S. Akbari and A. Mohammadian, On the zero-divisor graph of a commutative ring, *J. Algebra* **274** (2004), no. 2, 847–855.

[2] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), no. 2, 434–447.

[3] R. B. Bapat, *Graphs and Matrices*, Second edition, Universitext, Springer, London; Hindustan Book Agency, New Delhi, 2014.

[4] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), no. 1, 208–226.

[5] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2009.

[6] S. Chattopadhyay, K. L. Patra and B. K. Saho, Laplacian eigenvalues of the zero divisor graph of the ring $\mathbb{Z}_n$, *Linear Algebra Appl.* **584** (2020), 267–286.

[7] T. Ju and M. Wu, On iteration digraph and zero-divisor graph of the ring $\mathbb{Z}_n$, *Czechoslovak Math. J.* **64(139)** (2014), no. 3, 611—628.

[8] A. J. Schwenk, Computing the characteristic polynomial of a graph, Graphs and Combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973), pp. 153–172, Lecture Notes in Math., Vol. 406, Springer, Berlin, 1974.

[9] B. Smith, Perfect zero-divisor graphs of $\mathbb{Z}_n$, *Rose-Hulman Undergrad. Math. J.* **17** (2016), no. 2, Art. 6, 113–132.

[10] D. B. West, *Introduction to Graph Theory*, Second edition, Prentice Hall of India Private Limited, New Delhi, 2003.

[11] H. P. Yap, *Some Topics in Graph Theory*, London Mathematical Society Lecture Note Series, 108, Cambridge University Press, Cambridge, 1986.

[12] M. Young, Adjacency matrices of zero-divisor graphs of integers modulo n, *Involve* **8** (2015), no. 5, 753–761.