PROPER DIVISOR GRAPH OF A POSITIVE INTEGER

Hitesh Kumar
National Institute of Science Education and Research, Bhubaneswar, India
Homi Bhabha National Institute, Mumbai, India
hitess.kumar@niser.ac.in

Kamal Lochan Patra
National Institute of Science Education and Research, Bhubaneswar, India
Homi Bhabha National Institute, Mumbai, India
klpatra@niser.ac.in

Binod Kumar Sahoo
National Institute of Science Education and Research, Bhubaneswar, India
Homi Bhabha National Institute, Mumbai, India
bksahoo@niser.ac.in

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Abstract
The proper divisor graph $\Upsilon_n$ of a positive integer $n$ is the simple graph whose vertices are the proper divisors of $n$, and in which two distinct vertices $u, v$ are adjacent if and only if $n$ divides $uv$. The graph $\Upsilon_n$ plays an important role in the study of the zero divisor graph of the ring $\mathbb{Z}_n$. In this paper, we study some graph theoretical properties of $\Upsilon_n$ and determine the graph parameters such as clique number, chromatic number, chromatic index, independence number, matching number, domination number, vertex and edge covering numbers of $\Upsilon_n$. We also determine the automorphism group of $\Upsilon_n$.

1. Introduction
All graphs considered in this paper are finite and simple. We refer to the book [10] for unexplained graph terminology and the basics on graph theory. Let $G$ be a graph with vertex set $V(G)$. If $V(G) = \emptyset$, then $G$ is called the empty graph. If $V(G) \neq \emptyset$ but $G$ has no edge, then $G$ is called a null graph. We write $u \sim v$ for

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two distinct vertices \(u, v \in V(G)\) if they are adjacent in \(G\). The \textit{degree} of a vertex \(v \in V(G)\) is denoted by \(\deg(v)\). A vertex of degree one is called a \textit{pendant vertex} of \(G\). The automorphism group of \(G\) is denoted by \(\text{Aut}(G)\). A vertex \(v \in V(G)\) is called a \textit{cut vertex} if the induced subgraph \(G[V(G)\setminus\{v\}]\) of \(G\) has more components than \(G\).

A \textit{clique} in \(G\) is a subset \(K\) of \(V(G)\) such that the subgraph induced by \(K\) is complete. The maximum size of a clique in \(G\), denoted by \(\omega(G)\), is called the \textit{clique number} of \(G\). A \textit{vertex coloring} of \(G\) is an assignment of colors to the vertices of \(G\) such that two adjacent vertices receive different colors. The \textit{chromatic number} of \(G\), denoted by \(\chi(G)\), is the minimum number of colors required for a vertex coloring of \(G\). If \(\chi(H) = \omega(H)\) for every induced subgraph \(H\) of \(G\), then \(G\) is called a \textit{perfect graph}. An \textit{edge coloring} of \(G\) is an assignment of colors to the edges of \(G\) such that two adjacent edges receive different colors. The \textit{chromatic index} of \(G\), denoted by \(\chi'(G)\), is the minimum number of colors required for an edge coloring of \(G\).

An \textit{independent set} in \(G\) is a set of vertices such that no two of them are adjacent. The maximum size of an independent set in \(G\), denoted by \(\alpha(G)\), is called the \textit{independence number} of \(G\). A \textit{matching} in \(G\) is a set of edges such that no two of them are adjacent. The maximum size of a matching in \(G\), denoted by \(\alpha'(G)\), is called the \textit{matching number} of \(G\). A matching \(M\) in \(G\) is called a \textit{perfect matching} if each vertex of \(G\) is incident with some edge contained in \(M\). A \textit{vertex cover} in \(G\) is a set of vertices that contains at least one endpoint of every edge. The minimum size of a vertex cover in \(G\), denoted by \(\beta(G)\), is called the \textit{vertex covering number} of \(G\). An \textit{edge cover} in \(G\) is a set of edges such that every vertex of \(G\) is incident with some edge contained in it. The minimum size of an edge cover in \(G\), denoted by \(\beta'(G)\), is called the \textit{edge covering number} of \(G\). A \textit{dominating set} in \(G\) is set \(X\) of vertices such that every vertex of \(V(G)\setminus X\) is adjacent with some vertex in \(X\). The minimum size of a dominating set in \(G\), denoted by \(\gamma(G)\), is called the \textit{domination number} of \(G\).

1.1. The Proper Divisor Graph \(\Upsilon_n\)

Let \(n\) be a positive integer. An integer \(d\) is called a \textit{proper divisor} of \(n\) if \(1 < d < n\) and \(d\) divides \(n\). The \textit{proper divisor graph} of \(n\), denoted by \(\Upsilon_n\), is the graph with vertices the proper divisors of \(n\), and two distinct vertices \(u\) and \(v\) are adjacent if and only if \(n\) divides the product \(uv\).

The graph \(\Upsilon_n\) was recently introduced in the paper [6]. Note that \(\Upsilon_n\) is the empty graph if and only if \(n = 1\) or \(n\) is a prime. If \(n\) is composite, then \(\Upsilon_n\) is a connected graph by [6, Lemma 2.6].
1.1.1. Use of the graph $\Upsilon_n$

Let $G$ be a graph on $m$ vertices with $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $H_1, H_2, \ldots, H_m$ be $m$ pairwise vertex disjoint graphs. The $G$-generalized join graph of $H_1, H_2, \ldots, H_m$ is the graph obtained from $G$ by replacing each vertex $v_i$ with the graph $H_i$ and then adding new edges from each vertex of $H_i$ to every vertex of $H_j$, $1 \leq i \neq j \leq m$, whenever $v_i$ and $v_j$ are adjacent in $G$ (such graphs are called generalized composition graphs in [8]). Note that if $m = 2$ and $G = K_2$, then the $G$-generalized join graph of $H_1$ and $H_2$ coincides with the usual join graph $H_1 \vee H_2$ of $H_1$ and $H_2$.

The notion of zero divisor graph of a commutative ring was first introduced by I. Beck in [4] by taking all elements of the ring as vertices of the graph. It was later modified by Anderson and Livingston in [2] as the following. The zero divisor graph $\Gamma(R)$ of a commutative ring $R$ with unity is the graph with vertex set consisting of the zero divisors of $R$, and two distinct vertices $a$ and $b$ are adjacent if and only if $ab = 0$ in $R$. Note that $\Gamma(R)$ is the empty graph if $R$ is an integral domain.

Let $n$ be composite. Since every proper divisor of $n$ is a zero divisor of the ring $\mathbb{Z}_n$ of integers modulo $n$, $\Upsilon_n$ is an induced subgraph of the zero divisor graph $\Gamma(\mathbb{Z}_n)$. The graph $\Upsilon_n$ plays an important role in [6] while studying the spectrum of the Laplacian matrix of $\Gamma(\mathbb{Z}_n)$. By [6, Lemma 2.7], $\Gamma(\mathbb{Z}_n)$ is the $\Upsilon_n$-generalized join graph of certain complete graphs and null graphs corresponding to the proper divisors of $n$. It was proved in [6, Proposition 4.1] that $\Gamma(\mathbb{Z}_n)$ is Laplacian integral if and only if all the eigenvalues of the $m \times m$ vertex weighted Laplacian matrix $L(\Upsilon_n)$ (defined in [6, p.275]) of $\Upsilon_n$ are integers, where $m$ is the number of proper divisors of $n$. Further, by [6, Theorem 5.8], the algebraic connectivity of $\Gamma(\mathbb{Z}_n)$ coincides with the second smallest eigenvalue of $L(\Upsilon_n)$ if $n$ is not a prime power nor a product of two distinct primes, and the Laplacian spectral radius of $\Gamma(\mathbb{Z}_n)$ coincides with the largest eigenvalue of $L(\Upsilon_n)$ if $n$ is not a prime power. It is also known that $\Gamma(\mathbb{Z}_n)$ is perfect if and only if $\Upsilon_n$ is prefect (see Section 5.2).

One can refer to the book [3] for different kinds of matrices associated with graphs and the papers [1, 2, 7, 12] for more on the zero divisor graph of $\mathbb{Z}_n$.

1.2. Aim of This Paper

It is clear from the discussion in Section 1.1 that the structure of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ is completely dependent on that of the proper divisor graph $\Upsilon_n$. In this paper, we shall mainly be interested in studying the automorphism group and different parameters of $\Upsilon_n$.

In Section 2, we discuss some basic properties of $\Upsilon_n$ such as vertex degrees, pendant vertices, diameter and cut vertices. Similarity of two positive integers is defined in Section 3. For two composites $m$ and $n$, we prove that the proper divisor graphs $\Upsilon_m$ and $\Upsilon_n$ are isomorphic if and only if $m$ and $n$ are similar (except for distinct $m, n \in \{p_1^1, q_1 q_2\}$, where $p_1, q_1, q_2$ are primes with $q_1 \neq q_2$). We then
determine the automorphism group of $\Upsilon_n$ in Section 4. In Section 5, the graph parameters clique number, chromatic number, chromatic index, independence number, matching number, domination number, vertex and edge covering numbers of $\Upsilon_n$ are determined. We also provide algorithms for coloring the edges of $\Upsilon_n$ when $n$ is a prime power or a product of distinct primes.

2. Basic Properties of $\Upsilon_n$

Let $n > 1$ be an integer. The number of proper divisors of $n$ is denoted by $\pi(n)$. Let $k$ denote the number of distinct prime divisors of $n$ and $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the prime power factorization of $n$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. We have

$$\pi(n) = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1) - 2,$$

which follows from the paragraph before Lemma 2.6 in [6].

Lemma 1. Let $a, b$ be two positive integers. If $a$ is a proper divisor of $b$ with $\pi(a) + 1 = \pi(b)$, then $b = p^t$ and $a = p^{t-1}$ for some prime $p$ and some integer $t \geq 2$.

Proof. If $b$ has at least two distinct prime divisors, then formula (1) implies that $\pi(b) - \pi(a) \geq 2$, which is not possible. So $b = p^t$ for some prime $p$ and some positive integer $t$. Since $b$ has a proper divisor $a$, we must have $t \geq 2$. Then, using (1), $\pi(a) + 1 = \pi(b)$ implies that $a = p^{t-1}$.

In the rest of the paper, for obvious reason, we shall consider proper divisor graphs $\Upsilon_n$ only when $n$ is composite. If $n = p_1^{\alpha_1}$, then $\Upsilon_n \cong K_1$. To avoid this triviality, if $n = p_1^{\alpha_1}$, then we shall also assume that $\alpha_1 \geq 3$. We thus have $|V(\Upsilon_n)| = \pi(n) \geq 2$. Then connectedness of $\Upsilon_n$ implies that the degree of every vertex is at least one. For a positive integer $m$, we denote by $[m] := \{1, 2, \ldots, m\}$.

2.1. Vertex Degrees

In the following proposition, we determine the degree of a vertex of $\Upsilon_n$ based on a divisibility condition involving $n$ and the square of that vertex.

Proposition 1. Let $u$ be a vertex of $\Upsilon_n$. Then the degree of $u$ is given by:

$$\deg(u) = \begin{cases} \pi(u) & \text{if } n | u^2; \\ \pi(u) + 1 & \text{if } n \nmid u^2. \end{cases}$$
Proposition 2. Let \( d \) be determined in the following.

Thus, the number of neighbours of \( u \) is equal to the number of positive divisors \( r \) of \( u \) satisfying \( r \neq u \) and \( u \neq r \frac{u}{r} \). It follows that \( \text{deg}(u) = \pi(u) \) if \( u = r \frac{u}{r} \) for some positive divisor \( r \) of \( u \) with \( r \neq u \), otherwise \( \text{deg}(u) = \pi(u) + 1 \). Then the fact that \( u = r \frac{u}{r} \) if and only if \( n \) divides \( u^2 \) completes the proof.

\[ \text{deg} \left( \frac{u}{r_i} \right) = \text{deg} \left( \frac{u}{r} \right) \]

Corollary 1. Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \). If \( \alpha_i = \alpha_j \) for some \( i, j \in [k] \), then \( \text{deg} \left( \frac{n}{p_i} \right) = \text{deg} \left( \frac{n}{p} \right) \).

Proof. Write \( u_i = \frac{n}{p_i} \) and \( u_j = \frac{n}{p_j} \). Since \( \alpha_i = \alpha_j \), we have that \( n \) divides \( u_i^2 \) if and only if \( n \) divides \( u_j^2 \), and it follows from (1) that \( \pi(u_i) = \pi(u_j) \). Then the corollary follows from Proposition 1.

As an application of Proposition 1, all vertices of degree one and two in \( \Upsilon_n \) are determined in the following.

Proposition 2. Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \). Then the following hold:

(i) If \( k = 1 \) and \( \alpha_1 \in \{3,4\} \), then \( p_1 \) and \( p_1^2 \) are the pendant vertices of \( \Upsilon_n \).

(ii) If \( k = 1 \) and \( \alpha_1 \geq 5 \), then \( p_1 \) is the only pendant vertex of \( \Upsilon_n \).

(iii) If \( k \geq 2 \), then \( p_1, p_2, \ldots, p_k \) are precisely the pendant vertices of \( \Upsilon_n \).

Proof. The vertex \( p_i, i \in [k] \), has exactly one neighbour in \( \Upsilon_n \), namely \( \frac{n}{p_i} \). So each of \( p_1, p_2, \ldots, p_k \) is a pendant vertex of \( \Upsilon_n \).

Now let \( u \) be a vertex of \( \Upsilon_n \) with \( \text{deg}(u) = 1 \). By Proposition 1, we have \( \pi(u) = 0 \) or \( 1 \), and \( n|u^2 \) if the latter holds. If \( \pi(u) = 0 \), then \( u \) must be a prime. If \( \pi(u) = 1 \), then \( u \) must be the square of a prime. In that case, \( n|u^2 \) implies \( n \in \{p_1^3, p_1^4\} \). It can be seen that \( p_1^2 \) is also a pendant vertex of \( \Upsilon_n \) for \( n \in \{p_1^3, p_1^4\} \).

Corollary 2. Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \), and let \( l \) be the number of pendant vertices of \( \Upsilon_n \). Then

\[ l = \begin{cases} 1 & \text{if } n = p_1^{\alpha_1} \text{ with } \alpha_1 \geq 5; \\ 2 & \text{if } n \in \{p_1^3, p_1^4\}; \\ k & \text{if } k \geq 2. \end{cases} \]

Proposition 3. Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) with \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \), and let \( V_2(n) \) be the set of all degree two vertices of \( \Upsilon_n \). Then the following hold:

(i) \( V_2(p_1^3) = \Phi; V_2(p_1^4) = \{p_1^3\}; V_2(n) = \{p_1^2, p_1^3\} \) if \( n \in \{p_1^3, p_1^4\} \);
(ii) $V_2(p_1p_2) = \Phi$; $V_2(p_1^2p_2) = \{p_1^2, p_1p_2\}$; $V_2(p_1^2p_2^2) = \{p_1^3, p_1^2p_2, p_1p_2^2\}$;
(iii) $V_2(n) = \{p_i^2 : 1 \leq i \leq k, \alpha_i \geq 2\} \text{ if } n \notin \{p_1^3, p_1^4, p_1^6, p_1p_2, p_1^2p_2, p_1^3p_2^2\}$.

Proof. We shall prove (iii) only as the statements of (i) and (ii) can easily be verified. Assume that $n \notin \{p_1^3, p_1^4, p_1^6, p_1p_2, p_1^2p_2, p_1^3p_2^2\}$. If $\alpha_i \geq 2$ for some $i \in [k]$, then $p_i^2$ is adjacent with the two vertices $\frac{n}{p_i}$, $\frac{n}{p_i^2}$ only, and so $\deg(p_i^2) = 2$.

Conversely, let $u$ be a vertex of $\Upsilon_n$ with $\deg(u) = 2$. If $n$ divides $u^2$, then Proposition 1 gives that $\pi(u) = \deg(u) = 2$, which is possible if and only if $u$ is the cube of a prime or the product of two distinct primes. This forces $n \in \{p_1^3, p_1^4, p_1^6, p_1p_2, p_1^2p_2, p_1^3p_2^2\}$ (as $u$ is a proper divisor of $n$), a contradiction. So $n \nmid u^2$.

Then $\pi(u) = 1$ by Proposition 1 again, which is possible if and only if $u$ is the square of a prime. Therefore, $V_2(n) = \{p_i^2 : 1 \leq i \leq k, \alpha_i \geq 2\}$. \hfill \qed

Proposition 4. Let $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $u, v$ be two distinct vertices of $\Upsilon_n$. If $u$ divides $v$, then $\deg(u) \leq \deg(v)$, and the equality holds if and only if $k = 1$, $u = p_1^{[\alpha_1/2] - 1}$ and $v = p_1^{[\alpha_1/2]}$.

Proof. Since $u$ is a proper divisor of $v$, we have $\pi(u) + 1 \leq \pi(v)$. Then, using Proposition 1, we get

$$\deg(u) \leq \pi(u) + 1 \leq \pi(v) \leq \deg(v). \quad (2)$$

If $k = 1$, $u = p_1^{[\alpha_1/2] - 1}$ and $v = p_1^{[\alpha_1/2]}$, then $n \nmid u^2$ and $n|v^2$. Applying Proposition 1, we get $\deg(u) = \pi(u) + 1 = [\alpha_1/2] - 1 = \pi(v) = \deg(v)$. Conversely, suppose that $\deg(u) = \deg(v)$. Then (2) gives

$$\deg(u) = \pi(u) + 1 = \pi(v) = \deg(v).$$

It follows that both $u$ and $v$ are powers of a prime by Lemma 1, and that $n \nmid u^2$ and $n|v^2$ by Proposition 1. Thus we have the following:

(i) $n$ itself is a prime power and so $n = p_1^{\alpha_1}$.
(ii) $u = p_1^{t-1}$ and $v = p_1^t$ for some integer $t$ with $2 \leq t \leq \alpha_1 - 1$ (Lemma 1).
(iii) $p_1^{[\alpha_1/2]} \nmid u$ as $n \nmid u^2$, and $p_1^{[\alpha_1/2]} | v$ as $n | v^2$.

It follows from (ii) and (iii) that $t - 1 \leq [\alpha_1/2] - 1$ and $t \geq [\alpha_1/2]$. This gives $t = [\alpha_1/2]$, thus completing the proof. \hfill \qed

Remark 1. Recall that a graph is called nearly irregular if it contains exactly one pair of vertices with equal degree. Let $G$ be a connected nearly irregular graph on $m \geq 2$ vertices. If $u$ and $v$ are the two vertices of $G$ with equal degree, then $\deg(u) = \deg(v) = [\frac{m+1}{2}] - 1$. This follows from Proposition 4 with $n = p_1^{m+1}$ and the fact that there is precisely one connected nearly irregular graph on $m$ vertices up to isomorphism ([5, Theorem 1.12]).
2.2. Diameter

The following proposition determines the diameter of $\Upsilon_n$ which is denoted by $\text{diam}(\Upsilon_n)$.

**Proposition 5.** Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. Then

$$\text{diam}(\Upsilon_n) = \begin{cases} 1 & \text{if } n \in \{p_1^3, p_1p_2\}; \\ 2 & \text{if } n = p_1^{\alpha_1} \text{ with } \alpha_1 \geq 4; \\ 3 & \text{otherwise.} \end{cases}$$

**Proof.** It follows from the proof of connectedness of $\Upsilon_n$ in [6, Lemma 2.6] that $\text{diam}(\Upsilon_n) \leq 3$. If $n \in \{p_1^3, p_1p_2\}$, then $\Upsilon_n \cong K_2$ and so $\text{diam}(\Upsilon_n) = 1$. Suppose that $n = p_1^{\alpha_1} \text{ with } \alpha_1 \geq 4$. The vertex $p_1^{\alpha_1-1}$ is adjacent with all other vertices of $\Upsilon_n$. Further, $\alpha_1 \geq 4$ implies that $p_1 \sim p_1^{\alpha_1-1} \sim p_1^2$ is the shortest path between $p_1$ and $p_1^2$. Hence $\text{diam}(\Upsilon_n) = 2$.

Now consider $k \geq 2$ with $n \neq p_1p_2$. By Proposition 2(iii), $p_1$ and $p_2$ are pendant vertices of $\Upsilon_n$. We have $p_1 \sim \frac{n}{p_1}$ and $p_2 \sim \frac{n}{p_2}$. The vertices $p_1, p_2, \frac{n}{p_1}, \frac{n}{p_2}$ are pairwise distinct and $p_1 \sim \frac{n}{p_1} \sim \frac{n}{p_2} \sim p_2$ is the shortest path between $p_1$ and $p_2$. Therefore, $\text{diam}(\Upsilon_n) = 3$. \hfill \Box

2.3. Minimum and Maximum Degrees

By Corollary 2, $\Upsilon_n$ has at least one pendant vertex and hence the minimum degree of $\Upsilon_n$ is one. The vertices of maximum degree in $\Upsilon_n$ are determined in Proposition 6 below. We need the following lemma.

**Lemma 2.** Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ with $k \geq 2$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Set $u_i := \frac{n}{p_i}$ for $i \in [k]$. Then $\deg(u_1) \geq \deg(u_j)$ for $2 \leq j \leq k$, and equality holds if and only if $\alpha_1 = \alpha_j$ or $n = p_1^{\alpha_1}p_2^{\alpha_2}$.

**Proof.** We have $\alpha_1 \geq \alpha_j$. If $\alpha_1 = \alpha_j$, then $\deg(u_1) = \deg(u_j)$ by Corollary 1. Assume that $\alpha_1 > \alpha_j$. Then using Proposition 1 and the inequality $(\alpha_1 + 1)\alpha_j < \alpha_1(\alpha_j + 1)$, we get that

$$\deg(u_j) \leq \pi(u_j) + 1 = (\alpha_1 + 1)(\alpha_j + 1)(\alpha_j + 2)(\alpha_k + 1) - 1 \leq \alpha_1(\alpha_2 + 1)(\alpha_j + 2)(\alpha_k + 1) - 2 = \pi(u_1) \leq \deg(u_1).$$

If $k \geq 3$, then the inequality (3) is strict and this gives $\deg(u_1) > \deg(u_j)$. Suppose that $\deg(u_1) = \deg(u_j)$ with $\alpha_1 > \alpha_j$. Then $k = 2, j = 2$ and we must have

$$\deg(u_2) = \pi(u_2) + 1 = (\alpha_1 + 1)(\alpha_2 - 1) = \alpha_1(\alpha_2 + 1) - 2 = \pi(u_1) = \deg(u_1).$$
It follows that $n \mid u_2^2$ by Proposition 1 and that $\alpha_1 = \alpha_2 + 1$. The former implies that $\alpha_2 = 1$ and so $\alpha_1 = 2$. This given $n = p_1^2 p_2$. If $n = p_1^2 p_2$, then it can be seen that $\deg(u_1) = 2 = \deg(u_2)$. 

**Proposition 6.** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Then the following hold:

(i) If $n = p_1^3$, then both the vertices $p_1$ and $p_1^2$ of $\Upsilon_n$ are of maximum degree.

(ii) If $n = p_1^2 p_2$, then $\frac{n}{p_1} = p_1 p_2$ and $\frac{n}{p_2} = p_1^2$ are the vertices of $\Upsilon_n$ of maximum degree.

(iii) If $n \notin \{p_1^3, p_1^2 p_2\}$, then $\frac{n}{p_1}, \ldots, \frac{n}{p_t}$ are precisely the vertices of $\Upsilon_n$ of maximum degree, where $t \in [k]$ is the largest integer such that $\alpha_t = \alpha_1$.

**Proof.** The statements made in (i) and (ii) can easily be verified. We shall prove (iii). If $n = p_1^{\alpha_1}$ with $\alpha_1 \geq 4$, then $\frac{n}{p_1}$ is the only vertex that is adjacent with all other vertices of $\Upsilon_n$ and so it is the only vertex of $\Upsilon_n$ of maximum degree.

Now consider $k \geq 2$. Let $u \in V(\Upsilon_n)$ be of maximum degree. Set $u_i := \frac{n}{p_i}$ for $i \in [k]$. Since $u$ is a divisor of $u_i$ for some $j \in [k]$, we have $\deg(u) \leq \deg(u_j)$ by Proposition 4. The maximality of $\deg(u)$ then gives that $\deg(u) = \deg(u_j)$. Since $k \geq 2$, Proposition 4 again implies that $u = u_j$. Thus, the vertices of $\Upsilon_n$ of maximum degree are contained in $\{u_1, u_2, \ldots, u_k\}$. By the definition of $t$, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_t$ and $\alpha_1 > \alpha_j$ for $t + 1 \leq j \leq k$. Since $n \neq p_1^2 p_2$, Lemma 2 gives that $u_1, u_2, \ldots, u_t$ are precisely the vertices of maximum degree in $\Upsilon_n$. 

The number of vertices of $\Upsilon_n$ with minimum degree one (that is, pendant vertices) is given in Corollary 2. As a consequence of Proposition 6, we have the following result on the number of vertices of $\Upsilon_n$ having maximum degree.

**Corollary 3.** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$ and $L$ be the number of vertices of $\Upsilon_n$ with maximum degree. If $t \in [k]$ is the largest integer with $\alpha_t = \alpha_1$, then

$$L = \begin{cases} 2 & \text{if } n \in \{p_1^3, p_1^2 p_2\}; \\ t & \text{otherwise.} \end{cases}$$

**Corollary 4.** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Then the maximum degree $\Delta(\Upsilon_n)$ of $\Upsilon_n$ is given by:

$$\Delta(\Upsilon_n) = \deg\left(\frac{n}{p_1}\right) = \begin{cases} \pi\left(\frac{n}{p_1}\right) & \text{if } k = 1, \text{ or } k \geq 2 \text{ with } \alpha_1 \geq 2; \\ \pi\left(\frac{n}{p_t}\right) + 1 & \text{otherwise.} \end{cases}$$

**Proof.** This follows from Propositions 1 and 6. 

2.4. Cut Vertices

Let \( v \) be a vertex of \( \Upsilon_n \). For two disjoint nonempty subsets \( A \) and \( B \) of \( V(\Upsilon_n) \setminus \{v\} \), we say that \((A, B)\) is a separation of \( \Upsilon_n[V(\Upsilon_n) \setminus \{v\}] \) if \( V(\Upsilon_n) \setminus \{v\} = A \cup B \) and there is no edge of \( \Upsilon_n[V(\Upsilon_n) \setminus \{v\}] \) with one endpoint in \( A \) and the other in \( B \). Since \( \Upsilon_n \) is connected, we have that \( v \) is a cut vertex of \( \Upsilon_n \) if and only if there exists a separation of \( \Upsilon_n[V(\Upsilon_n) \setminus \{v\}] \).

If \( n \in \{p_1^3, p_1 p_2\} \), then \( \Upsilon_n \cong K_2 \) has no cut vertex. We shall find the cut vertices of \( \Upsilon_n \) for the remaining values of \( n \).

**Proposition 7.** Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) with \( n \notin \{p_1^3, p_1 p_2\} \). Then \( \frac{n}{p_1}, \frac{n}{p_2}, \ldots, \frac{n}{p_k} \) are precisely the cut vertices of \( \Upsilon_n \).

**Proof.** Each \( p_i, i \in [k], \) is a pendant vertex of \( \Upsilon_n \) by Proposition 2 (with \( p_i \sim \frac{n}{p_i} \)). Since \( n \notin \{p_1^3, p_1 p_2\} \), we have \( \pi(n) = |V(\Upsilon_n)| \geq 3 \) and then the connectedness of \( \Upsilon_n \) implies that each \( \frac{n}{p_i} \) is a cut vertex. We claim that any cut vertex of \( \Upsilon_n \) is one of \( \frac{n}{p_1}, \frac{n}{p_2}, \ldots, \frac{n}{p_k} \).

If \( k = 1 \), then \( \frac{n}{p_1} \) is adjacent with all other vertices of \( \Upsilon_n \) and so it must be the only cut vertex of \( \Upsilon_n \). Now consider \( k \geq 2 \). Suppose that there exists a cut vertex \( v \) of \( \Upsilon_n \) different from \( \frac{n}{p_1}, \frac{n}{p_2}, \ldots, \frac{n}{p_k} \). Let \((A, B)\) be a separation of \( \Upsilon_n[V(\Upsilon_n) \setminus \{v\}] \).

Since \( \frac{n}{p_1} \sim \frac{n}{p_i} \) for distinct \( i, j \in [k] \), all the vertices \( \frac{n}{p_i}, i \in [k] \), are either in \( A \) or in \( B \). Without loss of generality, we may assume that \( \frac{n}{p_i} \in A \) for all \( i \in [k] \). Since \( B \) is nonempty, there is a vertex \( u \in B \). Since \( u \) is divisible by \( p_j \) for some \( j \in [k] \), we must have \( u \sim \frac{n}{p_j} \), contradicting that \((A, B)\) is a separation of \( \Upsilon_n[V(\Upsilon_n) \setminus \{v\}] \). \( \square \)

3. Similarity and Isomorphisms

Let \( n \) and \( m \) be two positive integers with their prime power factorisations:

\[
 n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \quad \text{and} \quad m = q_1^{\beta_1} q_2^{\beta_2} \cdots q_l^{\beta_l},
\]

where \( p_i, q_j \) are primes and \( \alpha_i, \beta_j \) are positive integers with \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \) and \( \beta_1 \geq \beta_2 \geq \cdots \geq \beta_l \). We say that \( n \) and \( m \) are similar if \( k = l \) and \( \alpha_i = \beta_i \) for every \( i \in [k] \).

If two composites \( n \) and \( m \) are similar, then the proper divisor graphs \( \Upsilon_n \) and \( \Upsilon_m \) are isomorphic. This can be seen from the fact that the construction of the graph \( \Upsilon_n \) does not depend on the actual primes involved as divisors of \( n \). What about the converse statement? If \( n = p_1^3 \) and \( m = q_1 q_2 \), then \( \Upsilon_n \cong K_2 \cong \Upsilon_m \), but \( n \) and \( m \) are not similar. We prove that the converse statement is also true with this particular example as the only exception. More precisely, we have the following:
**Theorem 1.** Let $m$ and $n$ be composite integers. Then $\Upsilon_m$ and $\Upsilon_n$ are isomorphic if and only if $m$ and $n$ are similar, except for $m, n \in \{p_1^2, q_1q_2\}$ with $m \neq n$, where $p_1, q_1, q_2$ are primes with $q_1 \neq q_2$.

The proof of Theorem 1 follows from Propositions 8 and 9 below.

**Proposition 8.** Let $n = p_k^{\alpha_1}$ with $\alpha_1 \geq 4$. If $\Upsilon_n \cong \Upsilon_m$ for some composite $m$, then $n$ and $m$ are similar.

*Proof.* We have $|V(\Upsilon_n)| = |V(\Upsilon_m)|$ as $\Upsilon_n \cong \Upsilon_m$. If $m = q_1^{\beta_1}$ for some prime $q_1$ and positive integer $\beta_1$, then $\alpha_1 - 1 = \pi(n) = |V(\Upsilon_n)| = |V(\Upsilon_m)| = \pi(m) = \beta_1 - 1$ gives $\beta_1 = \alpha_1$.

Therefore, it is enough to prove that $m$ is a prime power. This is true if $\alpha_1 = 4$, as there is no integer $m$ with at least two distinct prime divisors for which $\pi(m) = |V(\Upsilon_n)| = |V(\Upsilon_m)| = \alpha_1 - 1 = 3$. If $\alpha_1 \geq 5$, then the claim follows from Proposition 2 and the fact that $\Upsilon_n$ and $\Upsilon_m$ must have the same number of pendant vertices. \(\square\)

**Proposition 9.** Let $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\beta_1}q_2^{\beta_2} \cdots q_l^{\beta_l}$ with $k \geq 2$, $l \geq 2$, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_l$. If $\Upsilon_n \cong \Upsilon_m$, then $n$ and $m$ are similar.

*Proof.* By Proposition 2(iii), $p_1, p_2, \ldots, p_k$ are the pendant vertices of $\Upsilon_n$ and $q_1, q_2, \ldots, q_l$ are that of $\Upsilon_m$. Since $\Upsilon_n \cong \Upsilon_m$, they have the same number of pendant vertices and hence $k = l$. The fact that $|V(\Upsilon_n)| = |V(\Upsilon_m)|$ gives

\[(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) = (\beta_1 + 1)(\beta_2 + 1) \cdots (\beta_l + 1) \tag{4}\]

Using (4), it can be seen that $n = p_1^{\alpha_1}p_2^{\alpha_2}$ if and only if $m = q_1^{\beta_1}q_2^{\beta_2}$, and so $n, m$ are similar in this case. Therefore, we shall assume that $n \neq p_1^{\alpha_1}p_2^{\alpha_2}$ (and hence $m \neq q_1^{\beta_1}q_2^{\beta_2}$).

Let $\phi : \Upsilon_n \to \Upsilon_m$ be a graph isomorphism. Since $\phi$ maps pendant vertices of $\Upsilon_n$ to that of $\Upsilon_m$, it induces a bijection from $\{p_1, p_2, \ldots, p_k\}$ to $\{q_1, q_2, \ldots, q_l\}$. In order to prove the proposition, it is enough to show that $\alpha_i = \beta_s$ if $\phi(p_i) = q_s$ for $i, s \in [k]$.

The only neighbour of $p_i$ in $\Upsilon_n$ is $\frac{n}{p_i}$ and that of $q_s$ in $\Upsilon_m$ is $\frac{m}{q_s}$. Since $\phi(p_i) = q_s$, we must have $\phi \left( \frac{n}{p_i} \right) = \frac{m}{q_s}$ and then

\[\deg \left( \frac{n}{p_i} \right) = \deg \left( \frac{m}{q_s} \right). \tag{5}\]

Note that $n$ divides $\left( \frac{n}{p_i} \right)^2$ if and only if $\alpha_i \geq 2$, and $m$ divides $\left( \frac{m}{q_s} \right)^2$ if and only if $\beta_s \geq 2$. So we have the following by Proposition 1:

\[\alpha_i = 1 : \quad \deg \left( \frac{n}{p_i} \right) = (\alpha_1 + 1) \cdots (\alpha_{i-1} + 1)\alpha_i(\alpha_{i+1} + 1) \cdots (\alpha_k + 1) - 1; \tag{6}\]
\[ \alpha_i \geq 2 : \quad \deg \left( \frac{n}{p_i} \right) = (\alpha_1 + 1) \cdots (\alpha_{i-1} + 1)\alpha_i(\alpha_{i+1} + 1) \cdots (\alpha_k + 1) - 2; \quad (7) \]

\[ \beta_s = 1 : \quad \deg \left( \frac{m}{q_s} \right) = (\beta_1 + 1) \cdots (\beta_{s-1} + 1)\beta_s(\beta_{s+1} + 1) \cdots (\beta_k + 1) - 1; \quad (8) \]

\[ \beta_s \geq 2 : \quad \deg \left( \frac{m}{q_s} \right) = (\beta_1 + 1) \cdots (\beta_{s-1} + 1)\beta_s(\beta_{s+1} + 1) \cdots (\beta_k + 1) - 2. \quad (9) \]

If \( \alpha_i \geq 2 \) and \( \beta_s \geq 2 \), then equations (4), (5), (7) and (9) give \( \frac{\alpha_i + 1}{\alpha_s} = \frac{\beta_s + 1}{\beta_t} \), that is, \( \alpha_i = \beta_s \).

Now suppose that \( \alpha_i = 1 \). If \( \beta_s = 1 \), then we are done. Suppose that \( \beta_s \geq 2 \). We shall get a contradiction by showing that \( m = q_1^2 q_2 \). Putting \( \alpha_i = 1 \) in (4), we get

\[ (\alpha_1 + 1) \cdots (\alpha_{i-1} + 1)(\alpha_{i+1} + 1) \cdots (\alpha_k + 1) = \frac{1}{2}(\beta_1 + 1) \cdots (\beta_s + 1) \cdots (\beta_k + 1). \quad (10) \]

From (5), (6) and (9), we get

\[ (\alpha_1 + 1) \cdots (\alpha_{i-1} + 1)(\alpha_{i+1} + 1) \cdots (\alpha_k + 1) = (\beta_1 + 1) \cdots (\beta_{s-1} + 1)\beta_s \]
\[ (\beta_{s+1} + 1) \cdots (\beta_k + 1) - 1. \quad (11) \]

An easy calculation using the equations (10) and (11) gives that

\[ (\beta_s - 1) \prod_{t=1}^{k} (\beta_t + 1) = 2. \quad (12) \]

Since \( k \geq 2 \), \( \beta_s \geq 2 \) and \( \beta_t \geq 1 \) for \( t \neq s \), it follows from (12) that \( k = 2 \), \( \beta_s = 2 \) and \( \beta_t = 1 \), where \( \{s, t\} = \{1, 2\} \). Since \( \beta_s > \beta_t \), we must have \( s = 1, t = 2 \) and hence \( m = q_1^2 q_2 \).

Similarly, if \( \alpha_i \geq 2 \) and \( \beta_s = 1 \), then we shall get a contradiction by showing that \( n = p_1^\alpha p_2^\alpha \) (using the equations (4), (5), (7) and (8)).

From the proof of Proposition 9, we have the following result which is useful in determining the automorphism group of \( \Upsilon_n \) in the next section.

**Corollary 5.** Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) with \( k \geq 2 \) and \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \). If \( n \neq p_1^2 p_2 \) and \( \phi : \Upsilon_n \to \Upsilon_n \) is a graph automorphism, then \( \phi \) permutes the pendant vertices \( p_1, p_2, \ldots, p_k \) of \( \Upsilon_n \) such that \( \phi(p_i) = p_j \) implies \( \alpha_i = \alpha_j \) for \( 1 \leq i, j \leq k \).
4. The Automorphism Group $\text{Aut}(\Upsilon_n)$

Let $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. We study the automorphism group $\text{Aut}(\Upsilon_n)$ of $\Upsilon_n$ in this section. We have the following result when $k = 1$.

**Proposition 10.** If $n = p_1^{\alpha_1}$, then $|\text{Aut}(\Upsilon_n)| = 2$.

*Proof.* Recall that $\alpha_1 \geq 3$ by our assumption. By Proposition 4, the degrees of the vertices of $\Upsilon_n$ are pairwise distinct, with the exception of two vertices $u := p_1^{\alpha_1 - 1}$ and $v := p_1^{\alpha_1 - 1}$ for which the degrees are the same. Therefore, every automorphism of $\Upsilon_n$ must fix each of the vertices contained in $V(\Upsilon_n) \setminus \{u, v\}$.

The map $\phi : \Upsilon_n \to \Upsilon_n$ with $\phi(u) = v$, $\phi(v) = u$ and which is identity on $V(\Upsilon_n) \setminus \{u, v\}$ is an automorphism of $\Upsilon_n$. This follows from the fact that a vertex of $V(\Upsilon_n) \setminus \{u, v\}$ is adjacent with either both $u$ and $v$, or none of them (note that $u \sim v$ if and only if $\alpha_1$ is odd). Therefore, $|\text{Aut}(\Upsilon_n)| = 2$. $\square$

If $k \geq 2$, then $p_1, p_2, \ldots, p_k$ are precisely the pendant vertices of $\Upsilon_n$ (Proposition 2(iii)). Therefore, for $\phi \in \text{Aut}(\Upsilon_n)$, $\phi(p_i)$ is also a prime for every $i \in [k]$.

**Lemma 3.** Let $k \geq 2$ and $\phi$ be an automorphism of $\Upsilon_n$. Then the following hold:

(i) $\phi \left( \frac{n}{p_i} \right) = \frac{n}{\phi(p_i)}$ for $1 \leq i \leq k$.

(ii) If $n \neq p_1^2 p_2$ and $w = p_1^{t_1}p_2^{t_2} \cdots p_k^{t_k} \in V(\Upsilon_n)$, then $\phi(w) = \phi(p_1)^{t_1}\phi(p_2)^{t_2} \cdots \phi(p_k)^{t_k}$, where $t_i \geq 1$ if and only $s_i \geq 1$ for $1 \leq i \leq k$.

*Proof.* (i) Since $p_i$ and $\phi(p_i)$ both are pendant vertices with $p_i \sim \frac{n}{p_i}$ and $\phi(p_i) \sim \frac{n}{\phi(p_i)}$, we must have $\phi \left( \frac{n}{p_i} \right) = \frac{n}{\phi(p_i)}$ for $1 \leq i \leq k$.

(ii) Let $u, v$ be vertices of $\Upsilon_n$ such that $\phi(u) = v$. We claim that if $p_i$ divides $u$, then $p_j = \phi(p_i)$ divides $v$. Assume that $p_i | u$. If $u = \frac{n}{p_i}$, then $u \sim \frac{n}{p_i}$ implies that $v = \phi(u) \sim \phi \left( \frac{n}{p_i} \right) = \frac{n}{\phi(p_i)} = \frac{n}{p_j}$ and hence $p_j$ divides $v$. If $u = \frac{n}{p_i}$, then $p_i^2$ divides $n$ and so $\alpha_i \geq 2$. Since $n \neq p_1^2 p_2$, we have $\alpha_j = \alpha_i \geq 2$ by Corollary 5. Then $v = \phi \left( \frac{n}{p_i} \right) = \frac{n}{\phi(p_i)} = \frac{n}{p_j}$ is divisible by $p_j$. Applying similar argument to the automorphism $\phi^{-1}$ of $\Upsilon_n$, and using the fact that $\phi^{-1}(v) = u$, we get that if $p_i$ divides $v$, then the prime $\phi^{-1}(p_i)$ divides $u$.

Now taking $u = w$, it follows from the above that $\phi(w) = \phi(p_1)^{t_1}\phi(p_2)^{t_2} \cdots \phi(p_k)^{t_k}$, where $t_i \geq 1$ if and only $s_i \geq 1$ for $1 \leq i \leq k$. $\square$

**Lemma 4.** Let $k \geq 2$ with $n \neq p_1^2 p_2$ and $\phi$ be an automorphism of $\Upsilon_n$. Then $\phi \left( p_i^{s_i} \right) = \phi(p_i)^{s_i}$ and $\phi \left( \frac{n}{p_i^r} \right) = \frac{n}{\phi(p_i)^r}$ for $1 \leq i \leq k$ and $1 \leq s_i \leq \alpha_i$. 


Proof. Let $u = p_i^{s_i}$. By Lemma 3(ii), we have $\phi(u) = \phi(p_i)^{t_i}$ for some positive integer $t_i$. Since $k \geq 2$, Proposition 1 gives that $\deg(u) = \pi(u) + 1 = s_i$ and $\deg(\phi(u)) = \pi(\phi(u)) + 1 = t_i$. Then $\deg(u) = \deg(\phi(u))$ gives that $s_i = t_i$ and hence $\phi(u) = \phi(p_i)^{s_i}$. The second part that $\phi\left(\frac{n}{p_i}\right) = \frac{n}{\phi(p_i)}$ can be obtained from the following.

Let $p_j = \phi(p_i)$. Then $\alpha_i = \alpha_j$ by Corollary 5. We claim that $\phi\left(\frac{n}{p_j}\right) = \frac{n}{p_j}$ for $1 \leq l \leq \alpha_i$. The proof is by induction on $l$. If $l = 1$, then the claim follows from Lemma 3(i). Assume that $\phi\left(\frac{n}{p_j}\right) = \frac{n}{p_j}$ for $1 \leq l \leq m < \alpha_i$. The neighbours of $p_i^{m+1}$ are precisely $\frac{n}{p_i}, \frac{n}{p_i^2}, \ldots, \frac{n}{p_i^m}$ and that of $\phi\left(p_i^{m+1}\right) = \phi(p_i)^{m+1} = p_j^{m+1}$ are $\frac{n}{p_j}, \frac{n}{p_j^2}, \ldots, \frac{n}{p_j^m}$. Since $\phi$ is one-one and the neighbours of $p_i^{m+1}$, the induction hypothesis then implies that $\phi\left(\frac{n}{p_i^{m+1}}\right) = \frac{n}{p_j^{m+1}}$. This proves the claim.

Proposition 11. Let $k \geq 2$ with $n \neq p_2^2 p_2$ and $\phi$ be an automorphism of $\Upsilon_n$. Then for any vertex $p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ of $\Upsilon_n$, we have

$$\phi(p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}) = \phi(p_1)^{s_1} \phi(p_2)^{s_2} \cdots \phi(p_k)^{s_k}.$$ 

Proof. Let $u = p_1^{s_1} \cdots p_k^{s_k}$. By Lemma 3(ii), we have $\phi(u) = \phi(p_1)^{t_1} \cdots \phi(p_k)^{t_k}$, where $t_i \geq 1$ if and only $s_i \geq 1$ for $1 \leq i \leq k$. Suppose that $s_i \neq t_i$ for some $i$. Then $s_i \geq 1$ and $t_i \geq 1$. Let $\phi(p_i) = p_j$. Then $\alpha_i = \alpha_j$ by Corollary 5.

Claim 1: $u \neq \frac{n}{p_i}$. If possible, suppose that $u = \frac{n}{p_i}$. Then $\alpha_i = 2s_i$. By Lemma 4, we have $\phi(u) = \phi\left(\frac{n}{p_i}\right) = \frac{n}{p_i}$. This gives $s_i + t_i = \alpha_j$. Then $\alpha_j = \alpha_i = 2s_i$, implies that $s_i = t_i$, contradicting our assumption.

Claim 2: $\phi(u) \neq \frac{n}{\phi(p_i)}$. If possible, suppose that $\phi(u) = \frac{n}{\phi(p_i)}$. Then $\alpha_j = 2t_i$. Since $\phi\left(\frac{n}{p_i}\right) = \frac{n}{\phi(p_i)}$, by Lemma 4, injectivity of $\phi$ gives $u = \frac{n}{p_i}$. This implies $s_i + t_i = \alpha_i$. Then $\alpha_i = \alpha_j$ gives that $s_i = t_i$, contradicting our assumption.

Since $u \approx \frac{n}{p_i}$, we have $\phi(u) \approx \phi\left(\frac{n}{p_i}\right) = \frac{n}{\phi(p_i)}$ (Lemma 4). This implies $s_i < t_i$ (as $s_i \neq t_i$). Since $\phi(u) \approx \frac{n}{\phi(p_i)} = \phi\left(\frac{n}{p_i}\right)$, we get $u \approx \frac{n}{p_i}$. But this is not possible as $s_i < t_i$. Therefore, $s_i = t_i$ for all $i \in [k]$ and hence $\phi(u) = \phi(p_1)^{s_1} \phi(p_2)^{s_2} \cdots \phi(p_k)^{s_k}$. 

Corollary 6. If $k \geq 2$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_k \geq 1$, then $|\text{Aut}(\Upsilon_n)| = 2$ or 1 according as $n = p_1^2 p_2$ or not.

Proof. If $n = p_1^2 p_2$, then $\Upsilon_n$ is a path of length three and hence $|\text{Aut}(\Upsilon_n)| = 2$. Assume that $n \neq p_1^2 p_2$. Let $\phi$ be an automorphism of $\Upsilon_n$. Since $\alpha_1 > \alpha_2 > \cdots > \alpha_k$, $\phi$ fixes each of the pendant vertices $p_1, p_2, \ldots, p_k$ by Corollary 5. Then Proposition 11 implies that each vertex of $\Upsilon_n$ is fixed by $\phi$ and so $\phi$ is the identity map.
Corollary 7. If \( k \geq 2 \) and two automorphisms of \( \Upsilon_n \) agree on the pendant vertices, then they are equal.

Lemma 5. Let \( k \geq 2 \) and \( \alpha_{i_1} = \alpha_{i_2} = \cdots = \alpha_{i_a} \) for some subset \( A := \{i_1, i_2, \ldots, i_a\} \) of \([k]\). Given a permutation \( \tau \) of \( \{p_{i_1}, p_{i_2}, \ldots, p_{i_a}\} \), define the map \( \tau : \Upsilon_n \to \Upsilon_n \) by

\[
\tau(p_{i_1}^{s_1}p_{i_2}^{s_2} \cdots p_{i_a}^{s_a}) = \left( \prod_{i_j \in A} \tau(p_{i_j})^{s_j} \right) \left( \prod_{l \in [k] \setminus A} p_{i_l}^{s_l} \right)
\]

for \( p_{i_1}^{s_1}p_{i_2}^{s_2} \cdots p_{i_a}^{s_a} \in V(\Upsilon_n) \). Then \( \tau \) is an automorphism of \( \Upsilon_n \).

Proof. If \( \tau(p_{i_j}) = p_{i_r} \) for \( i_j, i_r \in A \), then \( s_{i_j} \leq \alpha_{i_j} = \alpha_{i_r} \) and so \( \tau \) is well-defined.

Let \( u = p_{1}^{s_1} p_{2}^{s_2} \cdots p_{k}^{s_k} \) and \( v = p_{1}^{t_1} p_{2}^{t_2} \cdots p_{k}^{t_k} \) be two vertices of \( \Upsilon_n \). If \( \tau(u) = \tau(v) \), then comparing the prime powers on both sides we get \( s_{i_j} = t_{i_j} \) for \( i_j \in A \) and \( s_l = t_l \) for \( l \in [k] \setminus A \). Hence \( s_r = t_r \) for \( r \in [k] \) and hence \( \tau \) is injective. Write \( v = \left( \prod_{i_j \in A} p_{i_j}^{t_{i_j}} \right) \left( \prod_{l \in [k] \setminus A} p_{i_l}^{t_l} \right) \) and define

\[
w := \left( \prod_{i_j \in A} \tau^{-1}(p_{i_j})^{t_{i_j}} \right) \left( \prod_{l \in [k] \setminus A} p_{i_l}^{t_l} \right).
\]

If \( \tau^{-1}(p_{i_j}) = p_{i_r} \) for some \( i_r \in A \), then \( t_{i_j} \leq \alpha_{i_j} = \alpha_{i_r} \) and so \( w \) is a vertex of \( \Upsilon_n \). It is clear that \( \tau(w) = v \), implying \( \tau \) is surjective.

Since \( \alpha_{i_1} = \cdots = \alpha_{i_a} \), it can be observed that \( \tau(u) \sim \tau(v) \) if and only if \( s_r + t_r \geq \alpha_r \) for all \( r \in [k] \). The later holds if and only if \( u \sim v \). Hence \( u \sim v \) if and only if \( \tau(u) \sim \tau(v) \). Thus \( \tau \) is an automorphism of \( \Upsilon_n \).

In the following proposition, we determine the full automorphism group of \( \Upsilon_n \) when \( k \geq 2 \) with \( n \neq p_1^2 p_2 \). Let \( \alpha_{r_1}, \alpha_{r_2}, \ldots, \alpha_{r_b} \) be the distinct integers in the list \( \alpha_1, \alpha_2, \ldots, \alpha_k \). For \( 1 \leq i \leq b \), define

\[
A_{r_i} := \{ j \in [k] : \alpha_j = \alpha_{r_i} \}
\]

and set \( |A_{r_i}| = k_i \). Then \( A_{r_1} \cup A_{r_2} \cup \cdots \cup A_{r_b} \) is a partition of \([k]\) and so \( k_1 + k_2 + \cdots + k_b = k \). For a given positive integer \( m \), \( S_m \) denotes the symmetric group defined on the set \([m]\).

Proposition 12. Let \( k \geq 2 \) with \( n \neq p_1^2 p_2 \). Then \( \text{Aut}(\Upsilon_n) \cong S_{k_1} \times S_{k_2} \times \cdots \times S_{k_b} \), where the integers \( k_1, k_2, \ldots, k_b \) are as defined above.

Proof. For \( 1 \leq i \leq b \), consider the sets \( A_{r_i} \) as defined above and let \( X_{r_i} := \{ p_j : j \in A_{r_i} \} \). Then \( X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_b} \) is a partition of the set \( X = \{ p_1, p_2, \ldots, p_k \} \). Given \( \phi \in \text{Aut}(\Upsilon_n) \), let \( \phi_{r_i} \) denote the restriction of \( \phi \) to \( X_{r_i} \). Then \( \phi_{r_i} \) is a permutation of
\(X_r\) by Corollary 5. This gives \((\phi_{r_1}, \ldots, \phi_{r_b}) \in Sym(X_{r_1}) \times \cdots \times Sym(X_{r_b})\), where \(Sym(X_r)\) is the symmetric group defined on \(X_r\). Thus the map \(f : Aut(\Upsilon_n) \to Sym(X_{r_1}) \times \cdots \times Sym(X_{r_b})\) taking \(\phi\) to \((\phi_{r_1}, \phi_{r_2}, \ldots, \phi_{r_b})\) is well defined. We prove that \(f\) is a group isomorphism.

Let \(\phi, \psi \in Aut(\Upsilon_n)\). We claim that \(f(\phi \psi) = f(\phi) \cdot f(\psi)\). It is enough to show that \((\phi \psi)_{r_i} = \phi_{r_i} \cdot \psi_{r_i}\) for \(1 \leq i \leq b\). Indeed, for \(p_j \in X_{r_i}\), we have

\[
(\phi \psi)_{r_i}(p_j) = (\phi \psi)(p_j) = \phi(\psi(p_j)) = \phi(\psi_{r_i}(p_j)) = \phi_{r_i}(\psi_{r_i}(p_j)) = (\phi_{r_i} \cdot \psi_{r_i})(p_j).
\]

Thus \(f\) is a group homomorphism. Corollary 7 implies that \(f\) is injective. Consider \((\tau_1, \ldots, \tau_b) \in Sym(X_{r_1}) \times \cdots \times Sym(X_{r_b})\). For \(1 \leq i \leq b\), let \(\tau_i\) be the automorphism of \(\Upsilon_n\) as obtained in Lemma 5. Define \(\tau := \tau_1 \tau_2 \cdots \tau_b\), the composition of \(\tau_1, \tau_2, \ldots, \tau_b\). Then \(\tau \in Aut(\Upsilon_n)\) and observe that \(\tau_{r_i} = \tau_i\) for \(1 \leq i \leq b\). This gives \(f(\tau) = (\tau_1, \tau_2, \ldots, \tau_b)\) and hence \(f\) is surjective.

Thus \(f\) is a group isomorphism and so \(Aut(\Upsilon_n) \cong Sym(X_{r_1}) \times \cdots \times Sym(X_{r_b})\). Since \(|X_{r_i}| = |A_{r_i}| = k_i\), we have \(Sym(X_{r_i}) \cong S_{k_i}\) for every \(1 \leq i \leq b\) and hence the result follows.

### 5. Graph Parameters of \(\Upsilon_n\)

In this section, we shall determine the graph parameters clique number, chromatic number, chromatic index, domination number, independence number, matching number, vertex and edge covering numbers of \(\Upsilon_n\).

#### 5.1. Clique Number \(\omega(\Upsilon_n)\)

Let \(n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}\) be the prime power factorization of \(n\) such that the integers \(\alpha_1, \ldots, \alpha_l\) are odd and \(\alpha_{l+1}, \ldots, \alpha_k\) are even for some \(l \in \{0, 1, 2, \ldots, k\}\). Consider the subsets \(A\) and \(B\) of \(V(\Upsilon_n)\) as defined below:

\[
A := \{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} : \lfloor \alpha_i/2 \rfloor \leq r_i \leq \alpha_i, 1 \leq i \leq k\} \setminus \{n\},
\]

\[
B := \left\{ \frac{n}{p_j^{\lfloor \alpha_j/2 \rfloor}} : 1 \leq j \leq l \right\},
\]

where \(B = \emptyset\) if \(l = 0\). Let \(K\) denote the induced subgraph of \(\Upsilon_n\) with vertex set \(A \cup B\). Observe that \(A\) and \(B\) are disjoint, and that any two distinct vertices in \(A \cup B\) are adjacent. Thus \(K\) is a clique in \(\Upsilon_n\) with

\[
|V(K)| = |A| + |B| = \left\lfloor \frac{\alpha_1}{2} \right\rfloor + \left\lfloor \frac{\alpha_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{\alpha_l}{2} \right\rfloor + \left\lceil \frac{\alpha_{l+1}}{2} \right\rceil + \cdots + \left\lceil \frac{\alpha_k}{2} \right\rceil + l - 1. \quad (13)
\]
Proposition 13. Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) be the prime power factorization of \( n \) such that the integers \( \alpha_1, \ldots, \alpha_l \) are odd and \( \alpha_{l+1}, \ldots, \alpha_k \) are even for some \( l \in \{0, 1, 2, \ldots, k\} \). Then

\[
\omega(\Upsilon_n) = \left\lceil \frac{\alpha_1}{2} \right\rceil \left\lceil \frac{\alpha_2}{2} \right\rceil \cdots \left\lceil \frac{\alpha_l}{2} \right\rceil \left( \frac{\alpha_{l+1}}{2} + 1 \right) \cdots \left( \frac{\alpha_k}{2} + 1 \right) + l - 1.
\]

Proof. Consider the clique \( K \) in \( \Upsilon_n \) defined above with vertex set \( A \cup B \). Let \( H \) be an arbitrary clique in \( \Upsilon_n \). We prove that \( |V(H)| \leq |V(K)| \) and then (13) would complete the proof. It is enough to show that there exists an injective map \( \phi : V(H) \to V(K) \).

Let \( y = p_1^{s_1} \cdots p_l^{s_l} p_{l+1}^{s_{l+1}} \cdots p_k^{s_k} \) be a vertex of \( H \). If \( \left\lceil \frac{\alpha_i}{2} \right\rceil \leq s_i \leq \alpha_i \) for every \( i \in [k] \), then \( y \) is a vertex of \( K \) that is contained in \( A \). In this case, we define \( \phi(y) := y \in A \).

Suppose that \( s_i < \left\lceil \frac{\alpha_i}{2} \right\rceil \) for some \( i \in [k] \). Let \( j \in [k] \) be the smallest integer such that \( s_j < \left\lceil \frac{\alpha_j}{2} \right\rceil \). For every vertex \( z = p_1^{t_1} \cdots p_l^{t_l} p_{l+1}^{t_{l+1}} \cdots p_k^{t_k} \in V(H) \setminus \{y\} \), the fact that \( y \sim z \) gives

\[
t_j \geq \begin{cases} 
\left\lfloor \frac{\alpha_j}{2} \right\rfloor & \text{if } j \leq l; \\
\left\lfloor \frac{\alpha_j}{2} \right\rfloor + 1 = \frac{\alpha_j}{2} + 1 & \text{if } j \geq l + 1.
\end{cases}
\]

Thus, \( y \) is the only vertex of \( H \) with \( s_j < \left\lceil \frac{\alpha_j}{2} \right\rceil \). If \( j \leq l \), we define

\[
\phi(y) := \frac{n}{p_j^{\alpha_j/2}} \in B.
\]

If \( j \geq l + 1 \), then there is no such vertex \( z \) of \( H \) with \( t_j = \left\lceil \frac{\alpha_j}{2} \right\rceil = \frac{\alpha_j}{2} \). In this case, we define

\[
\phi(y) := p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}} p_j^{\alpha_j/2} p_{j+1}^{\alpha_{j+1}} \cdots p_k^{\alpha_k} \in A.
\]

It follows from the construction of the map \( \phi : V(H) \to V(K) \) that \( \phi \) is well-defined and it is one-one.

As a consequence of Proposition 13, we have the following:

Corollary 8. Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \). Then \( \omega(\Upsilon_n) \geq k \).

Corollary 8 can also be seen directly as follows. Since \( \frac{n}{p_i} \sim \frac{n}{p_j} \) for \( 1 \leq i \neq j \leq k \), the induced subgraph of \( \Upsilon_n \) with vertex set \( \left\{ \frac{n}{p_i} : 1 \leq i \leq k \right\} \) is a clique in \( \Upsilon_n \).

5.2. Chromatic Number \( \chi(\Upsilon_n) \)

In the following proposition, we prove that the chromatic number and the clique number of \( \Upsilon_n \) are equal.
Proposition 14. Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) be the prime power factorization of \( n \) such that the integers \( \alpha_1, \ldots, \alpha_l \) are odd and \( \alpha_{l+1}, \ldots, \alpha_k \) are even for some \( l \in \{0, 1, 2, \ldots, k\} \). Then

\[
\chi(\Upsilon_n) = \omega(\Upsilon_n) = \left\lfloor \frac{\alpha_1}{2} \right\rfloor \left\lfloor \frac{\alpha_2}{2} \right\rfloor \cdots \left\lfloor \frac{\alpha_l}{2} \right\rfloor \left( \frac{\alpha_{l+1} + 1}{2} \right) \cdots \left( \frac{\alpha_k + 1}{2} \right) + l - 1.
\]

Proof. Consider the clique \( K \) in \( \Upsilon_n \) (defined in Section 5.1 with \( V(K) = A \cup B \)). We have \( \omega(\Upsilon_n) = |V(K)| \) by (13) and Proposition 13. Assign \( \omega(\Upsilon_n) \) distinct colors to the vertices of \( K \). Out of the \( \omega(\Upsilon_n) \) colors used so far, we shall choose \( k \) of them (possible as \( \omega(\Upsilon_n) \geq k \) by Corollary 8) and assign these \( k \) colors suitably to the remaining vertices of \( \Upsilon_n \).

For \( 1 \leq i \leq k \), set \( \gamma_i := \left\lfloor \frac{\alpha_i}{2} \right\rfloor \) and let \( c_i \) be the color assigned to the vertex \( w_i \in V(K) \), where

\[
w_i = \begin{cases} p_1^{\alpha_1} \cdots p_{l-1}^{\alpha_{l-1}} p_l^{\gamma_l-1} p_{l+1}^{\alpha_{l+1}} \cdots p_l^{\alpha_l} \cdots p_k^{\alpha_k} & \text{if } 1 \leq i \leq l; \\
p_1^{\alpha_1} \cdots p_{l-1}^{\alpha_{l-1}} p_l^{\gamma_l} p_{l+1}^{\alpha_{l+1}} \cdots p_k^{\alpha_k} & \text{if } l + 1 \leq i \leq k.
\end{cases}
\]

Let \( u \) be a vertex of \( \Upsilon_n \) outside \( K \). Then \( p_l^{\gamma_l} \) does not divide \( u \) for some \( i \in [k] \). We assign the color \( c_i \) to the vertex \( u \), where \( t \in [k] \) is the smallest integer such that \( p_t^{\gamma_t} \nmid u \). Thus \( u \) and \( w_i \) receive the same color. In this way, we color all the vertices of \( \Upsilon_n \). Note that if \( x, y \) are two distinct vertices of \( \Upsilon_n \) with the same color \( c_i \), then \( p_t^{\alpha_t} \nmid xy \) implies \( x \sim y \).

Since \( \chi(\Upsilon_n) = \omega(\Upsilon_n) \), it is natural to ask whether \( \Upsilon_n \) is perfect. In [9], the zero-divisor type graph \( \Gamma(Z_n) \) of \( Z_n \) is defined and it is proved in Theorem 4.1 that \( \Gamma(Z_n) \) is perfect if and only if the zero divisor graph \( \Gamma(Z_n) \) is perfect. Further, using the Strong Perfect Graph Theorem, the author proved that the graph \( \Gamma(Z_n) \) is perfect if and only if \( n \in \{p_1^{\alpha_1}, p_1^{\alpha_1} p_2^{\alpha_2}, p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}, p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}\} \). From the construct of \( \Gamma(Z_n) \), it can be seen that the proper divisor graph \( \Upsilon_n \) is isomorphic to \( \Gamma(Z_n) \). As a consequence, it follows that \( \Upsilon_n \) is perfect if and only if \( n \in \{p_1^{\alpha_1}, p_1^{\alpha_1} p_2^{\alpha_2}, p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}, p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}\} \).

5.3. Matching Number \( \alpha'(\Upsilon_n) \)

Proposition 15. Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) and \( \mathcal{M} \) be the collection of all edges of \( \Upsilon_n \) of the form \( \{x, \frac{n}{x}\} \) with \( x < \sqrt{n} \). Then \( \mathcal{M} \) is a matching in \( \Upsilon_n \) of maximum size and

\[
\alpha'(\Upsilon_n) = |\mathcal{M}| = \lfloor \pi(n)/2 \rfloor.
\]

Proof. Since \( |V(\Upsilon_n)| = \pi(n) \), we have \( \alpha'(\Upsilon_n) \leq \lfloor \pi(n)/2 \rfloor \). Clearly, two distinct edges contained in \( \mathcal{M} \) do not share any common vertex. So \( \mathcal{M} \) is a matching in \( \Upsilon_n \). Every vertex of \( \Upsilon_n \) is an end point of some edge contained in \( \mathcal{M} \), with the exception of the vertex \( \sqrt{n} \) when \( n \) is a perfect square (in which case \( |V(\Upsilon_n)| \) is odd). This gives \( |\mathcal{M}| = \lfloor \pi(n)/2 \rfloor \) and it follows that the matching \( \mathcal{M} \) is of maximum size. \( \square \)
Corollary 10. Let $Z = \{x \in V(\mathcal{Y}_n) : x < \sqrt{n}\}$. Then $|Z| = \lfloor \frac{\pi(n)}{2} \rfloor$.

Proof. This follows from Proposition 15 using the fact that $Z$ is in bijective correspondence with the set $\mathcal{M} := \{\{x, \frac{n}{2}\} : x \in V(\mathcal{Y}_n), x < \sqrt{n}\}$. □

Corollary 11. The edge covering number $\beta'(\mathcal{Y}_n)$ of $\mathcal{Y}_n$ is given by: $\beta'(\mathcal{Y}_n) = [\pi(n)/2]$.

Proof. Since $\mathcal{Y}_n$ has no isolated vertices, we have $\alpha'(\mathcal{Y}_n) + \beta'(\mathcal{Y}_n) = |V(\mathcal{Y}_n)| = \pi(n)$ by \cite[Theorem 3.1.22]{10}. Then Proposition 15 gives that $\beta'(\mathcal{Y}_n) = \pi(n) - [\pi(n)/2] = [\pi(n)/2]$. □

5.4. Independence Number $\alpha(\mathcal{Y}_n)$

If $n = p_1 p_2$, then $\mathcal{Y}_n \cong K_2$ and so $\alpha(\mathcal{Y}_n) = 1$. If $n \neq p_1 p_2$, then no two vertices among $p_1, p_2, \ldots, p_k$ are adjacent and hence $\alpha(\mathcal{Y}_n) \geq k$. We prove the following:

Proposition 16. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $\mathcal{I} := \{x \in V(\mathcal{Y}_n) : x \leq \sqrt{n}\}$. Then $\mathcal{I}$ is an independent set in $\mathcal{Y}_n$ of maximum size and

$$\alpha(\mathcal{Y}_n) = |\mathcal{I}| = [\pi(n)/2].$$

Proof. If $x, y \in \mathcal{I}$ with $x \neq y$, then at least one of them is less than $\sqrt{n}$ and so $xy < n$. This implies that $x$ and $y$ are not adjacent. Thus $\mathcal{I}$ is an independent set in $\mathcal{Y}_n$.

Let $\mathcal{J}$ be an independent set in $\mathcal{Y}_n$ of maximum size. Then $|\mathcal{I}| \leq |\mathcal{J}|$. We claim that $|\mathcal{I}| \geq |\mathcal{J}|$. This follows if $\mathcal{J} \setminus \mathcal{I} = \emptyset$. Assume that $\mathcal{J} \setminus \mathcal{I} \neq \emptyset$. Let $y \in \mathcal{J} \setminus \mathcal{I}$. Then $y > \sqrt{n}$. This implies $\frac{n}{y} < \sqrt{n}$ and so $\frac{n}{y} \in \mathcal{I}$. Since $y \sim \frac{n}{y}$ (as $n \neq y^2$) and $\mathcal{J}$ is an independent set, it follows that $\frac{n}{y} \notin \mathcal{J}$ and hence $\frac{n}{y} \in \mathcal{I} \setminus \mathcal{J}$. Thus $y \mapsto \frac{n}{y}$ defines an injective map from $\mathcal{J} \setminus \mathcal{I}$ to $\mathcal{I} \setminus \mathcal{J}$. Then $|\mathcal{J}| = |\mathcal{I} \cap \mathcal{J}| + |\mathcal{J} \setminus \mathcal{I}| \leq |\mathcal{I} \cap \mathcal{J}| + |\mathcal{I} \setminus \mathcal{J}| = |\mathcal{I}|$. Thus $|\mathcal{I}| = |\mathcal{J}|$ and hence the independent set $\mathcal{I}$ is of maximum size.

We have $\sqrt{n} \in \mathcal{I}$ if and only if $n$ is a perfect square if and only if $\pi(n) = |V(\mathcal{Y}_n)|$ is odd. Consider the set $Z$ defined in Corollary 10. When $n$ is a perfect square, we have $Z = \mathcal{I} \setminus \{\sqrt{n}\}$ and this gives $|\mathcal{I}| = |Z| + 1 = \lfloor \frac{\pi(n)}{2} \rfloor + 1 = \lceil \frac{\pi(n)}{2} \rceil$, otherwise $Z = \mathcal{I}$ and we get $|\mathcal{I}| = |Z| = \lfloor \frac{\pi(n)}{2} \rfloor = \lceil \frac{\pi(n)}{2} \rceil$.

Corollary 12. The vertex covering number $\beta(\mathcal{Y}_n)$ of $\mathcal{Y}_n$ is given by: $\beta(\mathcal{Y}_n) = [\pi(n)/2]$.

Proof. We have $\alpha(\mathcal{Y}_n) + \beta(\mathcal{Y}_n) = |V(\mathcal{Y}_n)| = \pi(n)$ by \cite[Lemma 3.1.21]{10}. Then Proposition 16 gives that $\beta(\mathcal{Y}_n) = \pi(n) - [\pi(n)/2] = [\pi(n)/2]$. □
5.5. Domination Number $\gamma(\Upsilon_n)$

**Proposition 17.** Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. If $n \neq p_1 p_2$, then $Y := \{ \frac{n}{p_1}, \frac{n}{p_2}, \ldots, \frac{n}{p_k} \}$ is a dominating set of minimum size and so

$$\gamma(\Upsilon_n) = \begin{cases} 1 & \text{if } n = p_1 p_2; \\ k & \text{otherwise.} \end{cases}$$

**Proof.** Every vertex of $\Upsilon_n$ is adjacent or equal to at least one of the vertices in $Y$. This implies that $Y$ is a dominating set in $\Upsilon_n$. Assume that $n \neq p_1 p_2$. Then the set consisting of the $k$ distinct edges $\{ p_i, \frac{n}{p_i} \}, 1 \leq i \leq k$, is a matching in $\Upsilon_n$. The fact that $p_1, p_2, \ldots, p_k$ are pendant vertices of $\Upsilon_n$ then implies that any dominating set in $\Upsilon_n$ must contain $p_i$ or $\frac{n}{p_i}$ for every $i \in [k]$. Thus every dominating set must contain at least $k$ vertices and hence the dominating set $Y$ is of minimum size. The rest is clear. 

5.6. Chromatic Index $\chi'(\Upsilon_n)$

Clearly, $\chi'(\Upsilon_n) \geq \Delta(\Upsilon_n)$. We shall prove that $\chi'(\Upsilon_n) = \Delta(\Upsilon_n)$. The following result proved in [1, Remark 1] is helpful for us, which was obtained as an application of Vizing’s Adjacency Lemma [11, Corollary 3.6(iii)].

**Lemma 6 ([1]).** Let $G$ be a simple graph and $D := \{ u \in V(G) : \deg(u) = \Delta(G) \}$. Suppose that, for every vertex $u \in D$, there is an edge $\{u, w\}$ of $G$ such that $\Delta(G) - \deg(w) + 2 > |D|$. Then $\chi'(G) = \Delta(G)$.

**Proposition 18.** Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Then $\chi'(\Upsilon_n) = \Delta(\Upsilon_n)$.

**Proof.** If $n = p_1^3$ or $p_1 p_2$, then $\Upsilon_n \cong K_2$ and so $\chi'(\Upsilon_n) = 1 = \Delta(\Upsilon_n)$. Assume that $n \notin \{ p_1^3, p_1 p_2 \}$. Define $u_i := \frac{n}{p_i}$ for $1 \leq i \leq k$ and $D := \{ v \in V(\Upsilon_n) : \deg(v) = \Delta(\Upsilon_n) \}$. By Proposition 6, we have $D \subseteq \{ u_1, u_2, \ldots, u_k \}$ and so $1 \leq |D| \leq k$.

Let $v \in D$. Then $v = u_i$ for some $i \in [k]$. We have $u_i \sim p_i$ and $u_i \sim u_j$ for $j \in [k] \setminus \{ i \}$. Since $n \neq p_1 p_2$, the vertex $p_i$ is different from each such $u_j$. Thus $\deg(v) = \deg(u_i) \geq k$. For the edge $\{ v, p_i \}$ of $\Upsilon_n$, we have $\Delta(\Upsilon_n) - \deg(p_i) + 2 = \deg(v) - 1 + 2 \geq k + 1 > |D|$. Hence $\chi'(\Upsilon_n) = \Delta(\Upsilon_n)$ by Lemma 6. 

5.6.1. Coloring the Edges of $\Upsilon_n$

When $n$ is a prime power or a product of distinct primes, we are able to provide algorithms in the following for a proper edge coloring of $\Upsilon_n$ using $\Delta(\Upsilon_n)$ distinct colors. It would be interesting to develop such algorithms for other values of $n$ as well.
(i) $n = p_1^{\alpha_1}$:

We have $\chi'(\Upsilon_n) = \Delta(\Upsilon_n) = \pi\left(\frac{n}{p_1}\right) = \alpha_1 - 2$ by Proposition 18 and Corollary 4. Consider $\alpha_1 - 2$ distinct colours, say $c_0, c_1, \ldots, c_{\alpha_1 - 3}$. For a given edge $\{u, v\}$ of $\Upsilon_n$, there exists a unique $j \in \{0, 1, \ldots, \alpha_1 - 3\}$ such that $\frac{uv}{n} = p_1^j$. Then assign the color $c_j$ to the edge $\{u, v\}$. If two distinct adjacent edges $\{u, v\}$ and $\{v, w\}$ of $\Upsilon_n$ receive the same color $c_t$, then $\frac{uv}{n} = \frac{vw}{n}$ gives $u = w$, a contradiction. Thus, two distinct adjacent edges receive different colors, implying a proper edge coloring of $\Upsilon_n$.

(ii) $n = p_1 p_2 \cdots p_k, k \geq 2$:

Here $\chi'(\Upsilon_n) = \Delta(\Upsilon_n) = \pi\left(\frac{n}{p_1}\right) + 1 = 2^{k-1} - 1$ by Proposition 18 and Corollary 4. Consider the colors $c_j$ for $j \in I$, where $I$ is the index set defined by

$$I := \{ j : j|n \text{ and } 1 < j < \sqrt{n} \}. $$

We thus have a total of $|I| = \pi(n)/2 = 2^{k-1} - 1$ colors (see Corollary 10). We assign colors to the edges of $\Upsilon_n$ in the following way. Let $\{u, v\}$ be an edge of $\Upsilon_n$. Set $l = \frac{uv}{n}$. Then $1 \leq l < n$ and $l|n$. We have $l \neq \sqrt{n}$ as $n$ is not a perfect square. If $l \neq 1$, then $l \in I$ if $l < \sqrt{n}$, and if $l > \sqrt{n}$.

- If $1 < l < \sqrt{n}$, then we call $\{u, v\}$ an edge of Type-I and assign the color $c_l$ to it.

- If $\sqrt{n} < l < n$, then we call $\{u, v\}$ an edge of Type-II and assign the color $c_2$ to it.

- If $l = 1$, then we call $\{u, v\}$ an edge of Type-III.

We first prove that there is no conflict of interest in the above assignment of colors to Type-I and Type-II edges, and then we shall color Type-III edges suitably.

Let $\{u, v\}$ and $\{v, w\}$ be distinct adjacent edges of $\Upsilon_n$ of Type-I/Type-II. Suppose that they have received the same color. If both $\{u, v\}$ and $\{v, w\}$ are of the same type, then it follows that $\frac{uv}{n} = \frac{vw}{n}$ which gives $u = w$, a contradiction. So assume that they are of different types. Without loss of generality, we may assume that $\{u, v\}$ is of Type-I and $\{v, w\}$ is of Type-II. Then it follows that $\frac{uw}{n} = \frac{v^2}{n}$ and this gives $uv^2w = n^3$. Since $n$ is a product of distinct primes and $v \neq n$, we have that $p_j$ does not divide $v$ for some $j \in [k]$. Then $uv^2w$ is not divisible by $p_j$. This implies that there are no such $u, v, w$ with $uv^2w = n^3$. Thus, two distinct adjacent edges of Type-I/Type-II have received different colors.

Now consider an edge $\{x, y\}$ of Type-III. Then $xy = n$. Note that any other edge that is adjacent with $\{x, y\}$ must be of Type-I or Type-II. Let $a = \deg(x) - 1$ and $b = \deg(y) - 1$. If we prove that $a + b < 2^{k-1} - 1$, then we can color the edge $\{x, y\}$
with any of the \(2^{k-1} - 1 - (a + b)\) colors which are not used for the Type-I/Type-II edges adjacent with \(\{x, y\}\).

If one of \(x\) or \(y\), say \(x\), is a pendant vertex, then \(\text{deg}(y) = \Delta(\Upsilon_n)\) by Propositions 2(iii) and 6(iii). This gives \(a = 0\) and \(b = \Delta(\Upsilon_n) - 1 = 2^{k-1} - 2\) and so the claim follows. Assume that none of \(x, y\) is a pendant vertex. Then none of \(x, y\) is a prime. Since \(n = xy\) and \(n\) is a product of distinct primes, there exist distinct primes \(p_i, p_j\) each dividing \(x\) but not \(y\), and there exist distinct primes \(p_s, p_t\) each dividing \(y\) but not \(x\). Since \(\text{deg}(x) = \pi(x) + 1 \leq 2^{k-2} - 1\), we have \(a \leq 2^{k-2} - 2\). Similarly, \(b \leq 2^{k-2} - 2\). It follows that \(a + b \leq 2^{k-1} - 4 < 2^{k-1} - 1\).

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